

Pacific Journal of Mathematics

THE HOMOLOGY OF A FREE LOOP SPACE

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Denote by X^{S^1} the space of all continuous maps from the circle into a simply connected finite CW complex, X . **THEOREM:** Let \mathbb{k} be a field and suppose that either $\text{char } \mathbb{k} > \dim X$ or that X is \mathbb{k} -formal. Then the betti numbers $b_q = \dim H_q(X^{S^1}; \mathbb{k})$ are uniformly bounded above if and only if the \mathbb{k} -algebra $H^*(X; \mathbb{k})$ is generated by a single cohomology class. **COROLLARY:** If, in addition, X is a smooth closed manifold and \mathbb{k} is as in the theorem, and if $H^*(X; \mathbb{k})$ is not generated by a single class then X has infinitely many distinct closed geodesics in any Riemannian metric.

1. Introduction. In this paper (co)homology is always singular and $b_q(-; \mathbb{k}) = \dim H_q(-; \mathbb{k})$ denotes the q th betti number with respect to a field \mathbb{k} . The free loop space, X^{S^1} , of a simply connected space, X , is the space of all continuous maps from the circle into X .

The study of the homology of X^{S^1} is motivated by the following result of Gromoll and Meyer:

THEOREM [16]. *Assume that X is a simply connected, closed smooth manifold, and that for some field \mathbb{k} the betti numbers $b_q(X^{S^1}; \mathbb{k})$ are unbounded. Then X has infinitely many distinct closed geodesics in any Riemannian metric.*

(The proof in [16] is for $\mathbb{k} = \mathbb{R}$, but the arguments work in general.)

The Gromoll-Meyer theorem raises the problem of finding simple criteria on a topological space X which imply that the $b_q(X^{S^1}; \mathbb{k})$ are unbounded for some \mathbb{k} . This problem was solved for $\mathbb{k} = \mathbb{Q}$ by Sullivan and Vigué-Poirrier [28]. They considered simply connected spaces X such that $\dim H^*(X; \mathbb{Q})$ was finite, and they showed that then the $b_q(X^{S^1}; \mathbb{Q})$ were unbounded if and only if the cohomology algebra $H^*(X; \mathbb{Q})$ was not generated by a single class. And they drew the obvious corollary following from the Gromoll-Meyer theorem.

It is generally conjectured that the same phenomenon should hold in any characteristic; explicitly:

Conjecture. Suppose X is simply connected and, for some field \mathbb{k} , $H^*(X; \mathbb{k})$ is finite dimensional. Then the $b_q(X^{S^1}; \mathbb{k})$ are unbounded if and only if the \mathbb{k} -algebra $H^*(X; \mathbb{k})$ is not generated by a single class.

One direction of the conjecture is trivial:

REMARK. If $H^*(X; \mathbb{k})$ is generated by a single class then the $b_q(X^{S^1}; \mathbb{k})$ are uniformly bounded. Indeed, consider the Eilenberg-Moore spectral sequence [12], [25] for the fibre square

$$\begin{array}{ccc} X^{S^1} & \longrightarrow & X^I \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow{\Delta} & X \times X \end{array}, \quad \pi f = (f(0), f(1)), \quad \Delta x = (x, x).$$

It converges from $\text{Tor}^{H \otimes H}(H, H)$ to $H^*(X^{S^1}; \mathbb{k})$, where $H = H^*(X; \mathbb{k})$ is considered as a module over $H \otimes H$ via $(\alpha \otimes \beta) \cdot \gamma = (-1)^{\deg \beta \deg \gamma} \alpha \beta \gamma$.

Now if H is generated by a single class then it is easy to compute $\text{Tor}^{H \otimes H}(H, H)$ explicitly and to see that $b_q(X^{S^1}; \mathbb{k}) \leq 2$, all q . \square

In this paper we establish the conjecture under an additional hypothesis; in particular we prove it for any X if $H^i(X; \mathbb{k}) = 0$ for all $i > \text{char } \mathbb{k}$. It was already known in some cases: for instance it was shown by L. Smith [26] in characteristic two when $H^*(X; \mathbb{Z}_2)$ has the form $\bigotimes_i \mathbb{Z}_2[x_i]/x_i^{n_i}$ and $Sq^1 = 0$. And McCleary and Ziller [20] and Ziller [30] have proved it for homogeneous spaces in all characteristics. Results have also been obtained by Anick [4] and Roos [24]. And McCleary [19] has established a weaker form of the conjecture: if ΩX denotes the classical loop space of based maps $S^1 \rightarrow X$ then the $b_q(\Omega X; \mathbb{k})$ are unbounded if and only if $H^*(X; \mathbb{k})$ is not generated by a single class.

To state our theorem we first set (for a given field \mathbb{k})

$$\begin{aligned} r_X + 1 &= \inf\{i \geq 2 \mid H^i(X; \mathbb{k}) \neq 0\} \quad \text{and} \\ n_X &= \sup\{i \mid H^i(X; \mathbb{k}) \neq 0\}. \end{aligned}$$

Then we have

THEOREM I. *Let X be a simply connected space and let \mathbb{k} be a field such that $H^*(X; \mathbb{k})$ is finite dimensional. Then the conjecture holds for X and for \mathbb{k} if either:*

- (A) $\text{char } \mathbb{k} \geq n_X/r_X$ or (B) X is \mathbb{k} -formal ([3], [13]).

The Gromoll-Meyer theorem then implies the

COROLLARY. *Let X be a simply connected closed manifold and let $p > 0$ be a prime. If $H^*(X; \mathbb{Z}_p)$ is not generated by a single class, and if either $p \geq n_X/r_X$ or X is p -formal then X has infinitely many distinct closed geodesics in any Riemannian metric.*

The definition of \mathbb{k} -formal will be recalled in §3. Here we limit ourselves to giving:

Examples of \mathbb{k} -formal spaces. The class of \mathbb{k} -formal spaces includes suspensions, and those spaces X for which $\tilde{H}_i(X; \mathbb{k})$ is zero if i is outside an interval of the form $[k + 1, 3k + 1]$, and this class is closed under products and wedges—for all this see [3]. Manifolds X are \mathbb{k} -formal if $\tilde{H}_i(X; \mathbb{k})$ is zero outside an interval of the form $[k + 1, 4k + 2]$ ([13]) if $\text{char } \mathbb{k} \neq 2, 3$. And if X is a simply connected finite complex such that $\tilde{H}_i(X, \mathbb{k})$ is zero outside an interval of the form $[k + 1, 2k]$ then the boundary of a regular neighbourhood of X (embedded in a large \mathbb{R}^N) is a \mathbb{k} -formal manifold. □

We turn now to the proof of Theorem I, which we shall outline here, the details following in §§2, 3, 4. We work henceforth over a fixed field \mathbb{k} and denote $\otimes_{\mathbb{k}}$ and $\text{Hom}_{\mathbb{k}}$ simply by \otimes and Hom . The tensor algebra on a vector space, V , is denoted by $T(V)$. We adopt the convention “ $V^k = V_{-k}$ ” to raise and lower degrees in graded vector spaces, V ; in a *differential graded vector space* (DGV) the differential maps $V_k \rightarrow V_{k-1}$ (and hence $V^k \rightarrow V^{k+1}$). *Differential graded algebras* are called DGA’s and a DGA morphism which induces an isomorphism of cohomology is called a DGA *quism* and denoted by $\xrightarrow{\cong}$.

Recall now that the *Hochschild homology* $\text{HH}_*(A)$ of an algebra, A , is given by $\text{HH}_*(A) = \text{Tor}^{A \otimes A^{\text{opp}}}(A, A)$. If A is a DGA we shall use the same terminology:

$$\text{HH}_*(A) = \text{Tor}^{A \otimes A^{\text{opp}}}(A, A)$$

denotes the *Hochschild homology* of A , where now Tor is the differential tor of Eilenberg-Moore [21]. When we want to emphasize that we are in the DGA case we write $\text{HH}_*(A, d)$. (Some authors call this Hochschild hyperhomology.)

The starting point for the proof of Theorem I is a result of Burghel-Fiedorowicz [8] and Cohen [11] which asserts that

$$(1.1) \quad H_*(X^{S^1}; \mathbb{k}) = \text{HH}_*(C_*(\Omega X; \mathbb{k}), d),$$

where $C_*(\Omega X; \mathbb{k})$ is the DGA of singular chains on the Moore loop space of X . Thus if $(T(V), d) \xrightarrow{\cong} (C_*(\Omega X; \mathbb{k}), d)$ is an Adams-Hilton model [2] for X then we have

$$(1.2) \quad H_*(X^{S^1}; \mathbb{k}) \cong \text{HH}_*(T(V), d),$$

because DGA quisms induce isomorphisms of Hochschild homology, as follows from the Eilenberg-Moore comparison theorem [21; Theorem 2.3].

Let (Ω^*, d) be the DGA obtained by dualizing the bar construction on $(T(V), d)$ —we recall the definition in §2. The main result (Theorem II) of §2 will show that

$$(1.3) \quad \text{HH}^*(\Omega^*, d) \cong \text{Hom}(\text{HH}_*(T(V), d), \mathbb{k}).$$

In §3, on the other hand, we observe that either of conditions (A) and (B) gives a DGA quism $(\Omega^*, d) \xrightarrow{\cong} (A, d)$, where (A, d) is a commutative differential graded algebra (CDGA). In the case of condition (A) this follows from a deep theorem of Anick [4]; in the case of condition (B) it is a consequence of one of the equivalent definitions of \mathbb{k} -formal ([3], [13]). In either case we again apply the comparison theorem of [21] to obtain

$$(1.4) \quad \text{HH}^*(\Omega^*, d) \cong \text{HH}^*(A, d).$$

The isomorphisms (1.1), (1.2), (1.3) and (1.4) combine to yield

$$(1.5) \quad H^*(X^{S^1}; \mathbb{k}) \cong \text{HH}^*(A, d).$$

As we note in §3, the CDGA (A, d) satisfies $H(A) = H^*(X; \mathbb{k})$. Indeed when X is \mathbb{k} -formal $(A, d) = (H^*(X; \mathbb{k}), 0)$ and so (1.5) becomes

$$H^*(X^{S^1}; \mathbb{k}) \cong \text{HH}^*(H^*(X; \mathbb{k})),$$

in this case. This answers a question of Anick [3] in positive characteristic; in characteristic zero it has been proved by Vigué-Poirrier [29] and Anick [3].

The last step in the proof of Theorem I is the proof, in §4 of

THEOREM III. *Let (A, d) be a CDGA such that $H^{<0}(A) = 0$, $H^0(A) = \mathbb{k}$, $H^1(A) \doteq 0$ and $H(A)$ is finite dimensional. Then the integers $b_q = \dim \text{HH}^q(A, d)$ are unbounded if and only if $H(A)$ is not generated by a single class.*

The proof of Theorem III follows the lines of the proof in [28] when $\mathbb{k} = \mathbb{Q}$ via the construction of a Sullivan model for (A, d) , but with additions and modifications to cover the problems caused by positive characteristic.

2. Hochschild homology. In this section we prove a result which implies (1.3), namely

THEOREM II. *Suppose (R, d) is an augmented DGA such that $H_{<0}(R) = 0$, $H_0(R) = \mathbb{k}$ and each $H_i(R)$ is finite dimensional. If $(\Omega^*(R), d)$ is the DGA dual to the bar construction on (R, d) then*

$$\text{HH}^*(\Omega^*(R), d) \cong \text{Hom}(\text{HH}_*(R, d), \mathbb{k}).$$

Before starting the proof, however, we recall some definitions and facts from or about:

(a) differential homological algebra, (b), the opposite of a DGA, (c) differential coalgebras and comodules and (d) bar constructions.

(a) *Differential homological algebra* ([21], [5], [14]). An (R, d) -module is a DGV, (V, d) , together with an R -module structure on V such that $d(r \cdot v) = dr \cdot v + (-1)^{\deg r} r \cdot dv$. It is *semi-free* if it is the increasing union of submodules $V(0) \subset V(1) \subset \dots$ such that $V(0)$ and each $V(i+1)/V(i)$ is R -free on a basis of cycles. For any (R, d) -module, (M, d) there is a morphism $\phi: (V, d) \rightarrow (M, d)$ from a semi-free module (V, d) such that $H(\phi)$ is an isomorphism; such a morphism is called a *semi-free resolution* of (M, d) . Given any such resolution and any second (R, d) -module, (N, d) , we have

$$\text{Tor}^R(M, N) = H(V \otimes_R N).$$

(b) *The opposite DGA.* The opposite DGA, (R^{opp}, d) , has the same underlying differential graded vector space as (R, d) , but the product “ \circ ” is given by: $r \circ r' = (-1)^{\deg r \deg r'} r' r$. The *enveloping DGA* (R^e, d) , is then defined by $(R^e, d) = (R, d) \otimes (R^{\text{opp}}, d)$ so that

$$(r_1 \otimes r_2)(r_3 \otimes r_4) = (-1)^{\deg r_2(\deg r_3 + \deg r_4)} r_1 r_3 \otimes r_4 r_2.$$

Notice that multiplication makes (R, d) into a left (R^e, d) -module: $(r_1 \otimes r_2) \cdot r = (-1)^{\deg r \deg r_2} r_1 r r_2$; similarly we can make (R, d) into a right (R^e, d) -module.

(c) *Differential comodules* [21, §6]. A *comodule* over a differential graded coalgebra (DGC), (C, d) is a DGV, (W, d) , together with a DGV morphism $(W, d) \xrightarrow{\gamma} (W, d) \otimes (C, d)$ which makes W into a graded C -comodule. If (W, d) is also an (R, d) -module via $\alpha: (R, d) \otimes (W, d) \rightarrow (W, d)$ then these structures are *compatible* if γ is an R -module map (equivalently α is a C -comodule map).

If M and N are respectively a right and left (C, d) -comodule then their *cotensor product*, $M \square_C N$ is the kernel of the DGV morphism $\gamma_M \otimes 1 - 1 \otimes \gamma_N: M \otimes N \rightarrow M \otimes C \otimes N$. If M has a compatible left (R, d) -module structure and if Q is any right (R, d) -module then a natural DGV map

$$(2.1) \quad \omega: Q \otimes_R (M \square_C N) \rightarrow (Q \otimes_R M) \square_C N$$

is constructed as follows:

Observe that $M \square_C N$ is a sub (R, d) -module of $M \otimes N$, so that the inclusion induces $\phi: Q \otimes_R (M \square_C N) \rightarrow Q \otimes_R (M \otimes N)$. Since clearly $\gamma_{Q \otimes_R M} \otimes 1 - 1 \otimes \gamma_N$ vanishes on $\text{Im } \phi$, we have $\text{Im } \phi \subset (Q \otimes_R M) \square_C N$, and so (2.2) is defined by ϕ .

(d) *Bar constructions*. Denote the augmentation ideal of R by \bar{R} and define a graded vector space $s\bar{R}$ by $(s\bar{R})_n = \bar{R}_{n-1}$. The *bar construction* ([21], [29]) on (R, d) , denoted by (BR, δ) , is the DGC defined (modulo signs) by: BR is the tensor coalgebra on $s\bar{R}$ (as usual $sr_1 \otimes \cdots \otimes sr_n$ is written $[sr_1 | \cdots | sr_n]$) and

$$\begin{aligned} \delta[sr_1 | \cdots | sr_n] &= \sum_{i=1}^n \pm [sr_1 | \cdots | sdr_i | \cdots | sr_n] \\ &\quad + \sum_{i=1}^{n-1} \pm [sr_1 | \cdots | s(r_i r_{i+1}) | \cdots | sr_n]. \end{aligned}$$

The dual DGA, $\text{Hom}((BR, \delta); \mathbb{k})$, is denoted by $(\Omega^*(R), d)$.

From the bar construction one builds the classic acyclic construction $(R \otimes BR, \nabla)$ given by $\nabla = d \otimes 1 + 1 \otimes \delta + \tau$ with

$$\tau(r \otimes [sr_1 | \cdots | sr_n]) = \pm rr_1 \otimes [sr_2 | \cdots | sr_n].$$

It is in an obvious way a left (R, d) -module and a right (BR, δ) -comodule. Finally, we have the two-sided bar construction $(R \otimes BR \bar{\otimes} R^{\text{opp}}, D)$ with $D = d \otimes 1 \otimes 1 + 1 \otimes \delta \otimes 1 + 1 \otimes 1 \otimes d + \theta$, and

$$\begin{aligned} \theta(r \otimes [sr_1 | \cdots | sr_n] \otimes r') &= \pm rr_1 \otimes [sr_2 | \cdots | sr_n] \otimes r' \\ &\quad \pm r \otimes [sr_1 | \cdots | sr_{n-1}] \otimes r_n r'. \end{aligned}$$

It is straightforward ([21; §6]) that the augmentation $\varepsilon: BR \rightarrow \mathbb{k}$, together with the multiplication map $R \otimes R^{\text{opp}} \rightarrow R$ defines an (R^e, d) -semi-free resolution $(R \otimes BR \otimes R^{\text{opp}}, D) \rightarrow (R, d)$. Thus

$$(2.2) \quad H(R \otimes_{R^e} (R \otimes BR \otimes R^{\text{opp}})) = \text{Tor}^{R \otimes R^{\text{opp}}}(R, R) = \text{HH}_*(R, d),$$

and indeed this was the original definition of Hochschild homology.

These constructions may also be applied to (R^{opp}, d) to yield the DGC $(B(R^{\text{opp}}, d), \delta)$ and the acyclic construction $(R^{\text{opp}} \otimes B(R^{\text{opp}}), \nabla)$. Moreover a DGC isomorphism, $\omega: (B(R^{\text{opp}}), \delta) \rightarrow ((BR)^{\text{opp}}, \delta)$, onto the opposite DGC is defined by

$$\omega[sr_1 | \cdots | sr_n] = (-1)^k [sr_n | \cdots | sr_1], \quad k = \sum_{i < j} (\deg sr_i)(\deg sr_j).$$

Thus $1 \otimes \omega$ converts $(R^{\text{opp}} \otimes B(R^{\text{opp}}), \nabla)$ into a DGV, $(R^{\text{opp}} \otimes (BR)^{\text{opp}}, \nabla')$, which is both an (R^{opp}, d) -module and an $((BR)^{\text{opp}}, \delta)$ -comodule.

We come now to the

Proof of Theorem II. As in [6] there is DGA quism of the form $(T(V), d) \xrightarrow{\cong} (R, d)$ with $v_i = 0, i \leq 0$, and each V_i finite dimensional. By the Eilenberg-Moore comparison theorem [21; Theorem 2.3] Ω^* preserves quisms and HH^* converts quisms to isomorphisms. We may thus replace (R, d) by $(T(V), d)$ and assume that

$$(2.3) \quad R = R_{\geq 0}, R_0 = \mathbb{k} \text{ and each } R_i \text{ is finite dimensional.}$$

Now let $((BR)^e, \delta)$ denote the DGC $(BR, \delta) \otimes ((BR)^{\text{opp}}, \delta)$ and set

$$M(R) = (R \otimes BR \otimes R^{\text{opp}} \otimes (BR)^{\text{opp}}, \nabla \otimes 1 + 1 \otimes \nabla').$$

Evidently $M(R)$ has compatible left (R^e, d) -module and right $((BR)^e, \delta)$ -comodule structures. Moreover, we have

LEMMA 2.4. *For any right (R^e, d) -module, Q , and any left $((BR)^e, \delta)$ -comodule N the natural DGV map*

$$\omega: Q \otimes_{R^e} (M(R) \square_{(BR)^e} N) \rightarrow (Q \otimes_{R^e} M(R)) \square_{(BR)^e} N$$

is an isomorphism.

Proof of (2.4). We may ignore differentials and write $M(R) = R^e \otimes (BR)^e$. The standard isomorphism $(BR)^e \square_{(BR)^e} N \cong N$ gives an isomorphism

$$(2.5) \quad M(R) \square_{(BR)^e} N \cong R^e \otimes N$$

of R^e modules. Analogously, we have a $(BR)^e$ -comodule isomorphism

$$(2.6) \quad Q \otimes_{R^e} M(R) \cong Q \otimes (BR)^e .$$

Using (2.5) and (2.6) one easily identifies ω with the identity of $Q \otimes N$. □

We apply Lemma 2.4 with $Q = (R, d)$ and $N = (BR, \delta)$, the module (resp., comodule) structures being defined by multiplication (resp., comultiplication) as described in (b) above. Notice that (2.5) becomes

$$M(R) \square_{(BR)^e} BR \cong R^e \otimes BR \cong R \otimes BR \otimes R^{opp} ;$$

according to [17; Lemma 2.01] the differential induced thereby in $R \otimes BR \otimes R^{opp}$ is that of the two-sided bar construction. Thus (cf. (2.2))

$$H(R \otimes_{R^e} (M(R) \square_{(BR)^e} BR)) \cong \text{Tor}^{R \otimes R^{opp}}(R, R) .$$

For simplicity denote the graded dual of a graded vector space by $V^\# = \text{Hom}(V, \mathbb{k})$. Thus $(\Omega^*(R), d) = (BR, \delta)^\#$. Because of our assumption (2.3) both R and BR are concentrated in degrees ≥ 0 , and are finite dimensional in each degree. For such spaces $\#$ commutes with \otimes so that, for instance, $([\Omega^*(R)]^e, d) = ((BR)^e, \delta)^\#$. Thus we deduce from Lemma 2.4 that

$$(2.7) \quad \text{HH}_*(R, d)^\# \cong H^* \{ [(R \otimes_{R^e} M(R)) \square_{(BR)^e} BR]^\# \} .$$

Write $Y = [R \otimes_{R^e} M(R)]^\#$. We shall show that Y is an $(\Omega^*(R)^e, d)$ semi-free resolution of $\Omega^*(R)$. Since

$$[(R \otimes_{R^e} M(R)) \square_{(BR)^e} BR]^\# = Y \otimes_{\Omega^*(R)^e} \Omega^*(R) ,$$

it will then follow from (2.7) that $\text{HH}_*(R, d)^\# = \text{HH}^*(\Omega^*(R), d)$, as desired.

That Y is $(\Omega^*(R)^e, d)$ -semi-free can be seen by filtering it by the spaces F_j of functions vanishing on $[R_{\geq j} + d(R_j)] \otimes_{R^e} M(R)$. And a homology isomorphism $Y \rightarrow \Omega^*(R)$ of $(\Omega^*(R)^e, d)$ -modules is defined by dualizing the diagonal $BR \rightarrow BR \otimes BR$, regarded as a map

$$BR \rightarrow 1 \otimes (BR)^e \subset R \otimes_{R^e} M(R) .$$

3. Reduction to the commutative case. Let

$$(T(V), d) \xrightarrow{\simeq} (C_*(\Omega X; \mathbb{k}), d)$$

be an Adams-Hilton model [2] for a space X satisfying the conditions of the conjecture, and denote the dual of the bar construction on

$(T(V), d)$ by (Ω^*, d) . In this section we prove

PROPOSITION 3.1. *If X satisfies condition (A) or condition (B) of Theorem I then there is a DGA quism $(\Omega^*, d) \xrightarrow{\cong} (A, d)$ with (A, d) a CDGA and $H(A) \cong H^*(X; \mathbb{k})$. If condition (B) holds, $(A, d) = (H^*(X; \mathbb{k}), 0)$.*

Proof. The main result of [4] asserts that if (A) holds then the differential in the Adams-Hilton model may be chosen so as to map V into the sub Lie algebra $L \subset T(V)$ generated by V . This identifies $(T(V), d)$ as the universal enveloping algebra, $U(L, d)$ of the DGL (differential graded Lie algebra) (L, d) .

Recall that the bar construction is a tensor coalgebra, and in particular contains the sub-coalgebra, S , of symmetric (in the graded sense) tensors. In particular, we have $S(sL) \subset S(s(UL_+)) \subset B(UL)$. As in the case of characteristic zero ([23; Appendix B], [10]), $S(sL)$ is a sub DGC of $B(UL)$ and the inclusion $S(sL) \rightarrow B(UL)$ is a homology isomorphism [22]. Dualizing this gives a quism from (Ω^*, d) to the CDGA $S(sL)^\#$. On the other hand [1] $(C_*(\Omega X; \mathbb{k}), d)$ is connected by DGA quisms to the cobar construction on $(C_*(X; \mathbb{k}), d)$, and hence to $\Omega^*(C^*(X; \mathbb{k}), d)$. Thus $\Omega^*(T(V), d)$ is connected by quisms to $\Omega^*\Omega^*(C^*(X; \mathbb{k}), d)$, and so by [21; Theorem 6.2] we have $H(A) \cong H(\Omega^*(T(V), d)) \cong H^*(X; \mathbb{k})$.

Now suppose X satisfies condition (B); i.e., X is \mathbb{k} -formal. One of the equivalent definitions of this is ([3], [13]) that X have an Adams-Hilton model which is the dual of the bar construction on $H^*(X; \mathbb{k})$: $(T(V), d) = \Omega^*(H^*(X; \mathbb{k}), 0)$. Thus $(\Omega^*, d) = \Omega^*(\Omega^*(H^*(X; \mathbb{k}), 0))$ and by [21; Theorem 6.2] this maps by a quism to $(H^*(X; \mathbb{k}), 0)$: $(\Omega^*, d) \xrightarrow{\cong} (H^*(X; \mathbb{k}), 0)$.

4. The commutative case. In this section we prove

THEOREM III. *Let (A, d) be a CDGA such that $H^{<0}(A) = 0$, $H^0(A) = \mathbb{k}$, $H^1(A) = 0$ and $H(A)$ is finite dimensional. Then the integers $b_q = \dim HH^q(A, d)$ are unbounded if and only if $H(A)$ is not generated by a single class.*

Proof. As in the rational case ([27], [7], [18]) it is straightforward to construct a DGA quism of the form

$$(\Lambda V, d) \xrightarrow{\cong} (A, D)$$

in which: $V = V^{\geq 2}$ is a graded vector space, $\Lambda V =$ exterior algebra $(V^{\text{odd}}) \otimes$ symmetric algebra (V^{even}) and $\text{Im } d \subset (\Lambda V)^+ \cdot (\Lambda V)^+$. Using the Eilenberg-Moore comparison theorem [21; Theorem 2.3] we replace (A, d) by $(\Lambda V, d)$.

The same argument as given in [28] for $\mathbb{k} = \mathbb{Q}$ now establishes

LEMMA 4.1. *The algebra $H(\Lambda V)$ is generated by a single class if and only if $\dim V^{\text{odd}} \leq 1$.*

If, moreover, $H(\Lambda V)$ is generated by a single class then the hypothesis $\dim H(\Lambda V) < \infty$ implies, in view of (4.1) that the only possibilities for $(\lambda V, d)$ are: $V = 0$, $V = (x)$ with $\deg x$ odd, or $V = (x, y)$ with $dy = x^k$ and $\deg y$ odd. In all these cases there is an obvious quism $(\Lambda V, d) \xrightarrow{\cong} (H(\Lambda V), 0)$, which induces an isomorphism of Hochschild homology. Now a direct calculation shows $\dim \text{HH}^q(H(\Lambda V), 0) \leq 2$ for all q .

It remains to show that the $\text{HH}^q(\Lambda V, d)$ have unbounded dimensions if $\dim V^{\text{odd}} \geq 2$. Recall that sV is the graded space given by $(sV)_{k+1} = V_k$; thus $(sV)^k = V^{k+1}$. Denote by $\Gamma(sV)$ the free divided powers algebra on sV , [9], and denote the i th divided power of sx by $\gamma_i(sx)$.

Consider the multiplication homomorphism,

$$\phi: (\Lambda V, d) \otimes (\Lambda V, d) \rightarrow (\Lambda V, d).$$

According to [15; Proposition 1.9], ϕ extends to a DGA quism of the form

$$(4.2) \quad \phi: (\Lambda V \otimes \Lambda V \otimes \Gamma(sV), D) \xrightarrow{\cong} (\Lambda V, d)$$

in which

$$(4.3) \quad \phi(\Gamma(sV)^+) = 0,$$

$$(4.4) \quad \text{Im } D \subset (\Lambda V \otimes \Lambda V)^+ \otimes \Gamma(sV) \quad \text{and}$$

$$(4.5) \quad D(\gamma_i(sx)) = D(sx) \cdot \gamma_{i-1}(sx).$$

For ease of notation denote the algebra $\Lambda V \otimes \Lambda V \otimes \Gamma(sV)$ by $\Sigma(V)$, and for $\Phi \in \Lambda V$ write $\Phi' = \Phi \otimes 1 \otimes 1$ and $\Phi'' = 1 \otimes \Phi \otimes 1$. Then the model (4.2) also satisfies:

$$(4.6) \quad \text{For } x \in V^n, \quad Dsx - (x' - x'') \in \Sigma(V^{<n}).$$

Now choose a basis $x_1, x_2, \dots, x_m, y, x_{m+1}, \dots, x_i, \dots$ in which $\deg x_1 \leq \dots \leq \deg x_m \leq \deg y \leq \dots \leq \deg x_i \leq \dots$, and y is the first basis element of odd degree. (All other basis elements are denoted by x_j , some j .)

LEMMA 4.7. *The differential D in $\Sigma(V)$ can be chosen so that $Dsy - (y' - y'') \in \Sigma(x_1, \dots, x_m)$ and for all i , $Dsx_i - (x'_i - x''_i)$ is in the ideal generated by the x'_j, x''_j and $\Gamma(sx_j)^+, j < i$.*

Proof. D is constructed inductively on n ; if it has already been defined in $s(V^{\leq n})$ then there is always a linear map of degree zero,

$$f: V^{n+1} \rightarrow \Sigma(V^{\leq n}) \cap \ker \phi$$

such that $dv' - dv'' - Df(v) \equiv 0$ and given any such f , D may be extended to $\Sigma(V^{\leq n+1})$ by setting

$$D(sv) = v' - v'' - f(v), \quad v \in V^{n+1}.$$

Now notice that because $V^1 = 0$ and $\text{Im } d \subset (\Lambda V)^+ \cdot (\Lambda V)^+$ it follows that $dy \in \Lambda(x_1, \dots, x_m)$ and dx_i is in the ideal generated by the $x_j, j < i$. Moreover, that $Dsy - (y' - y'') \in \Sigma(x_1, \dots, x_m)$ is immediate from (4.6) as is $Dsx_i - (x'_i - x''_i) \in \Sigma(x_1, \dots, x_{i-1})$ for $i \leq m$.

Suppose then that the lemma is proved for some $x_1, \dots, x_i, i \geq m$. Let $I \subset \Sigma(x_1, \dots, x_m, y, \dots, x_i)$ be the ideal generated by the $\Sigma(x_j)^+, j \leq i$. Since

$$dx_j \in \Lambda^+(x_1, y, \dots, x_{j-1}) \cdot \Lambda^+(x_1, \dots, y, \dots, x_{j-1})$$

it follows from our induction hypothesis on Dsx_j and from (4.5) that D maps I to itself. Dividing by I gives us a CDGA of the form $(\Sigma(y), \bar{D})$ and a commutative diagram of CDGA morphisms

$$\begin{array}{ccc} (\Sigma(x_1, \dots, y, \dots, x_i), D) & \xrightarrow[\simeq]{\phi} & (\Lambda(x_1, \dots, y, \dots, x_i), d) \\ \downarrow \rho & & \downarrow \rho \\ (\Sigma y, \bar{D}) & \xrightarrow[\phi]{\simeq} & (\Lambda y, 0) \end{array}$$

in which

$$\begin{aligned} \phi(y') = \phi(y'') = y, \quad \phi(\gamma_i(sy)) = 0 \quad \text{and} \\ \bar{D}(\gamma_i(sy)) = (y' - y'')\gamma_{i-1}(sy). \end{aligned}$$

As described at the start of the proof, there is always an element $w \in \Sigma(x_1, \dots, y, \dots, x_i) \cap \ker \phi$ such that $dx'_{i+1} - dx''_{i+1} - Dw = 0$, and D may be extended to $\Sigma(x_1, \dots, x_{i+1})$ by setting $Dsx_{i+1} = x'_{i+1} - x''_{i+1} - w$. And for any such w ,

$$\bar{D}\rho w = \rho Dw = \rho(dx'_{i+1} - dx''_{i+2}) = 0,$$

since dx'_{i+1} and dx''_{i+1} are in I . Moreover $\phi\rho w = \rho\phi w = 0$. Since $\phi: (\Sigma y, \bar{D}) \rightarrow (\Lambda y, 0)$ is a surjective quism it follows that $\rho w = \bar{D}u$, some $u \in \ker \phi \cap \Sigma y$.

Regard u as an element of $\ker \phi \cap \Sigma(x_1, \dots, y, x_i)$ via the inclusion of Σy . Then $\rho(w - Du) = 0$, $\phi(w - Du) = D\phi u = 0$ and so we may define $Dsx_{i+1} = x'_{i+1} - x''_{i+1} - w + Du$. Now we have $\rho(w - Du) = \rho w - \bar{D}u = 0$ and so $Dsx_{i+1} - (x'_{i+1} - x''_{i+1}) \in I$, as desired. \square

We now return to the proof of Theorem III. It follows from (4.5) and (4.6) that the quism $\phi: (\Lambda V \otimes \Lambda V \otimes \Gamma(sV), D) \rightarrow (\Lambda V, d)$ is a $(\Lambda V, d) \otimes (\Lambda V, d)$ -semi-free resolution. Hence

$$\mathrm{HH}^*(\Lambda V, d) = H(\Lambda V \otimes_{\Lambda V \otimes \Lambda V} (\Lambda V \otimes \Lambda V \otimes \Gamma(sV))) = H(\Lambda V \otimes \Gamma(sV)).$$

Denote the differential in $\Lambda V \otimes \Gamma(sV)$ by δ . Lemma 4.7 shows that $\delta(sx_i)$ is in the ideal generated by the x_j and $\Gamma(sx_j)$, $j < i$. Let $z = x_{n+1}$ ($n \geq m$) be the first x_i of odd degree and divide $\Lambda V \otimes \Gamma(sV)$ by the ideal generated by the x_j , $j \leq n$.

This produces a CDGA of the form $(\Lambda(y, z, x_{n+2}, \dots) \otimes \Gamma(sV), \bar{\delta})$. The same argument as given in [28] shows that if this CDGA has unbounded betti numbers then so does $(\Lambda V \otimes \Gamma(sV), \delta)$, as desired. But by Lemma 4.7, $\bar{\delta}sx_i$ is in the ideal generated by sx_1, \dots, sx_{i-1} , for $i \leq n$. Moreover $\Gamma(sx_i) =$ the exterior algebra $\Lambda(sx_i)$ because $\deg sx_i$ is odd. Hence $sx_1 \wedge \dots \wedge sx_n$ is a cycle.

And since $\delta(sy)$ and $\delta(sz)$ are also in the ideal generated by sx_1, \dots, sx_n it follows from (4.5) that the elements $sx_1 \wedge \dots \wedge sx_n \wedge \gamma_i(sy) \wedge \gamma_j(sz)$ are all $\bar{\delta}$ -cycles. Under the projection $\Lambda V \otimes \Gamma(sV) \rightarrow \Gamma(sV)$ these elements map to linearly independent homology classes, since the differential included in $\Gamma(sV)$ is zero, by (4.4). Thus they represent linearly independent classes in $H(\Lambda(y, z, \dots) \otimes \Gamma(sV), \bar{\delta})$, and hence the betti numbers of $(\Lambda(y, z, \dots) \otimes \Gamma(sV), \bar{\delta})$ are indeed unbounded. \square

REFERENCES

- [1] J. F. Adams, *On the cobar construction*, Proc. Nat. Acad. Sci. U.S.A., **42** (1956), 409–412.
- [2] J. F. Adams and P. J. Hilton, *On the chain algebra of a loop space*, Comment. Math. Helv., **20** (1955), 305–330.
- [3] D. Anick, *A model of Adams Hilton type for fibre squares*, Illinois J. Math., **29** (1985), 463–502.
- [4] ———, *Hopf algebras up to homotopy*, J. Amer. Math. Soc., **2** (1989), 417–453.

- [5] L. Avramov and S. Halperin, *Through the looking glass, a dictionary between rational homotopy theory and local algebra*, pp. 3–27 in *Algebra, Algebraic Topology and their Iterations*, Lecture Notes in Mathematics 1183, Springer Verlag, Berlin, 1986.
- [6] H. Baues and J.-M. Lemaire, *Minimal models in homotopy theory*, *Math. Ann.*, **225** (1977), 219–242.
- [7] A. K. Bousfield and V. K. A. M. Gugenheim, *On PL de Rham theory and rational homotopy type*, *Memoirs Amer. Math. Soc.*, **179** (1976).
- [8] D. Burghlea and Z. Fiedorowicz, *Cyclic homology and algebraic K-theory of spaces*, *Topology*, **25** (1986), 303–317.
- [9] H. Cartan, *Algèbres d'Eilenberg-MacLane*, Séminaire Cartan 54/55 exposé 7, *Oeuvres v. III*, 1309-1394, Springer Verlag, Berlin.
- [10] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton Univ. Press, Princeton, N.J., 1956.
- [11] R. L. Cohen, *Pseudo-isotopies, K-theory and homotopy theory*, London Math. Soc., Lecture Note Series 117, Cambridge Univ. Press (1987), 35–71.
- [12] S. Eilenberg and J. C. Moore, *Homology and fibrations*, *Comment. Math. Helv.*, **40** (1966), 199–236.
- [13] M. El Haouari, *P-formalité des espaces*, Rapport no. 124, Université de Louvain, 1987.
- [14] Y. Felix, S. Halperin and J. C. Thomas, *Gorenstein spaces*, *Adv. in Math.*, **71** (1988), 92–112.
- [15] Y. Felix, S. Halperin, C. Jacobsson, C. Löfwall and J.-C. Thomas, *The radical of the homotopy Lie algebra*, *Amer. J. Math.*, **110** (1988), 301–332.
- [16] D. Gromoll and W. Meyer, *Periodic geodesics on compact Riemannian manifolds*, *J. Differential Geom.*, **3** (1969), 493–510.
- [17] V. K. A. M. Gugenheim and H. J. Munkholm, *On the extended functoriality of Tor and Cotor*, *J. Pure Appl. Algebra*, **4** (1974), 9–29.
- [18] S. Halperin, *Lectures on minimal models*, *Memoirs Soc. Math. France*, **9/10** (1984).
- [19] J. McCleary, *On the mod p Betti numbers of loop spaces*, *Invent. Math.*, **87** (1987), 643–654.
- [20] J. McCleary and W. Ziller, *On the free loop space of homogeneous spaces*, *Amer. J. Math.*, **109** (1987), 765–781.
- [21] J. C. Moore, *Algèbre homologique et homologie des espaces classifiants*, Séminaire Cartan, 59/60, exposé no. 7.
- [22] ———, *Differential homological algebra*, *Actes, Congrès Internat. Math.*, (1970), tome I, 335–339.
- [23] D. Quillen, *Rational homotopy theory*, *Annals of Math.*, **90** (1969), 205–295.
- [24] J. E. Roos, *Homology of free loop spaces, cyclic homology and rational Poincaré-Betti series*, preprint no. 39 (1987), Univ. of Stockholm.
- [25] L. Smith, *Homological algebra and the Eilenberg-Moore spectral sequence*, *Trans. Amer. Math. Soc.*, **129** (1967), 58–93.
- [26] L. Smith, *The EMSS and the mod 2 cohomology of certain free loop spaces*, *Illinois J. Math.*, **28** (1984), 516–522.
- [27] D. Sullivan, *Infinitesimal computations in topology*, *Publ. Math. I.H.E.S.* (1978), 269–331.
- [28] D. Sullivan and M. Vigué-Poirrier, *The homology theory of the closed geodesic problem*, *J. Differential Geom.*, **11** (1976), 633–644.
- [29] M. Vigué-Poirrier, *Réalisation de morphismes donnés en cohomologie et suite spectrale d'Eilenberg-Moore*, *Trans. Amer. Math. Soc.*, **265** (1981), 447–484.

- [30] W. Ziller, *The free loop space on globally symmetric spaces*, *Invent. Math.* **41** (1977), 1–22.

Received March 6, 1989. The research of the first author was partially supported by an NSERC grant and that of the second author by the UA au CNRS 751.

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