THE HOMOLOGY OF A FREE LOOP SPACE

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Denote by $X^{S^1}$ the space of all continuous maps from the circle into a simply connected finite CW complex, $X$. Theorem: Let $k$ be a field and suppose that either $\text{char} \ k > \dim X$ or that $X$ is $k$-formal. Then the betti numbers $b_q = \dim H_q(X^{S^1}; k)$ are uniformly bounded above if and only if the $k$-algebra $H^*(X; k)$ is generated by a single cohomology class. Corollary: If, in addition, $X$ is a smooth closed manifold and $k$ is as in the theorem, and if $H^*(X; k)$ is not generated by a single class then $X$ has infinitely many distinct closed geodesics in any Riemannian metric.

1. Introduction. In this paper (co)homology is always singular and $b_q(\cdot; k) = \dim H_q(\cdot; k)$ denotes the $q$th betti number with respect to a field $k$. The free loop space, $X^{S^1}$, of a simply connected space, $X$, is the space of all continuous maps from the circle into $X$.

The study of the homology of $X^{S^1}$ is motivated by the following result of Gromoll and Meyer:

Theorem [16]. Assume that $X$ is a simply connected, closed smooth manifold, and that for some field $k$ the betti numbers $b_q(X^{S^1}; k)$ are unbounded. Then $X$ has infinitely many distinct closed geodesics in any Riemannian metric.

(The proof in [16] is for $k = \mathbb{R}$, but the arguments work in general.)

The Gromoll-Meyer theorem raises the problem of finding simple criteria on a topological space $X$ which imply that the $b_q(X^{S^1}; k)$ are unbounded for some $k$. This problem was solved for $k = \mathbb{Q}$ by Sullivan and Vigué-Poirrier [28]. They considered simply connected spaces $X$ such that $\dim H^*(X; \mathbb{Q})$ was finite, and they showed that then the $b_q(X^{S^1}; \mathbb{Q})$ were unbounded if and only if the cohomology algebra $H^*(X; \mathbb{Q})$ was not generated by a single class. And they drew the obvious corollary following from the Gromoll-Meyer theorem.
It is generally conjectured that the same phenomenon should hold in any characteristic; explicitly:

**Conjecture.** Suppose \( X \) is simply connected and, for some field \( k \), \( H^*(X; k) \) is finite dimensional. Then the \( b_q(X^{S^1}; k) \) are unbounded if and only if the \( k \)-algebra \( H^*(X; k) \) is not generated by a single class.

One direction of the conjecture is trivial:

**Remark.** If \( H^*(X; k) \) is generated by a single class then the \( b_q(X^{S^1}; k) \) are uniformly bounded. Indeed, consider the Eilenberg-Moore spectral sequence [12], [25] for the fibre square

\[
\begin{array}{ccc}
X^{S^1} & \longrightarrow & X^I \\
\downarrow & & \downarrow \pi \\
X & \longrightarrow & X \times X
\end{array}
\]

It converges from \( \text{Tor}^{H \otimes H}(H, H) \) to \( H^*(X^{S^1}; k) \), where \( H = H^*(X; k) \) is considered as a module over \( H \otimes H \) via \( (\alpha \otimes \beta) \cdot \gamma = (-1)^{\deg \beta \deg \gamma} \alpha \beta \gamma \).

Now if \( H \) is generated by a single class then it is easy to compute \( \text{Tor}^{H \otimes H}(H, H) \) explicitly and to see that \( b_q(X^{S^1}; k) \leq 2 \), all \( q \). \( \square \)

In this paper we establish the conjecture under an additional hypothesis; in particular we prove it for any \( X \) if \( H^i(X; k) = 0 \) for all \( i > \text{char } k \). It was already known in some cases: for instance it was shown by L. Smith [26] in characteristic two when \( H^*(X; \mathbb{Z}_2) \) has the form \( \bigotimes_i \mathbb{Z}_2[x_i]/x_i^{n_i} \) and \( Sq^1 = 0 \). And McCleary and Ziller [20] and Ziller [30] have proved it for homogeneous spaces in all characteristics. Results have also been obtained by Anick [4] and Roos [24]. And McCleary [19] has established a weaker form of the conjecture: if \( \Omega X \) denotes the classical loop space of based maps \( S^1 \to X \) then the \( b_q(\Omega X; k) \) are unbounded if and only if \( H^*(X; k) \) is not generated by a single class.

To state our theorem we first set (for a given field \( k \))

\[
r_X + 1 = \inf\{i \geq 2 \mid H^i(X; k) \neq 0\}
\]

and

\[
n_X = \sup\{i \mid H^i(X; k) \neq 0\}.
\]

Then we have
**THEOREM I.** Let \( X \) be a simply connected space and let \( k \) be a field such that \( H^*(X; k) \) is finite dimensional. Then the conjecture holds for \( X \) and for \( k \) if either:

(A) \( \text{char } k \geq n_X/r_X \) or (B) \( X \) is \( k \)-formal ([3], [13]).

The Gromoll-Meyer theorem then implies the

**COROLLARY.** Let \( X \) be a simply connected closed manifold and let \( p > 0 \) be a prime. If \( H^*(X; \mathbb{Z}_p) \) is not generated by a single class, and if either \( p \geq n_X/r_X \) or \( X \) is \( p \)-formal then \( X \) has infinitely many distinct closed geodesics in any Riemannian metric.

The definition of \( k \)-formal will be recalled in §3. Here we limit ourselves to giving:

**Examples of \( k \)-formal spaces.** The class of \( k \)-formal spaces includes suspensions, and those spaces \( X \) for which \( \tilde{H}_i(X; k) \) is zero if \( i \) is outside an interval of the form \([k+1, 3k+1]\), and this class is closed under products and wedges—for all this see [3]. Manifolds \( X \) are \( k \)-formal if \( \tilde{H}_i(X; k) \) is zero outside an interval of the form \([k+1, 4k+2]\) ([13]) if \( \text{char } k \neq 2, 3 \). And if \( X \) is a simply connected finite complex such that \( \tilde{H}_i(X, k) \) is zero outside an interval of the form \([k+1, 2k]\) then the boundary of a regular neighbourhood of \( X \) (embedded in a large \( \mathbb{R}^N \)) is a \( k \)-formal manifold. \( \Box \)

We turn now to the proof of Theorem I, which we shall outline here, the details following in §§2, 3, 4. We work henceforth over a fixed field \( k \) and denote \( \otimes_k \) and \( \text{Hom}_k \) simply by \( \otimes \) and \( \text{Hom} \). The tensor algebra on a vector space, \( V \), is denoted by \( T(V) \). We adopt the convention "\( V^k = V_{-k} \)" to raise and lower degrees in graded vector spaces, \( V \); in a differential graded vector space (DGV) the differential maps \( V_k \rightarrow V_{k-1} \) (and hence \( V^k \rightarrow V^{k+1} \)). Differential graded algebras are called DGA's and a DGA morphism which induces an isomorphism of cohomology is called a DGA quism and denoted by \( \sim \).

Recall now that the Hochschild homology \( \text{HH}_*(A) \) of an algebra, \( A \), is given by \( \text{HH}_*(A) = \text{Tor}^{A \otimes A^{\text{op}}}(A, A) \). If \( A \) is a DGA we shall use the same terminology:

\[ \text{HH}_*(A) = \text{Tor}^{A \otimes A^{\text{op}}}(A, A) \]
denotes the *Hochschild homology* of $A$, where now $\text{Tor}$ is the differential tor of Eilenberg-Moore [21]. When we want to emphasize that we are in the DGA case we write $\text{HH}_*(A, d)$. (Some authors call this Hochschild hyperhomology.)

The starting point for the proof of Theorem I is a result of Burghelea-Fiedorowicz [8] and Cohen [11] which asserts that

$$(1.1) \quad H_*(X^{S^1}; k) = \text{HH}_*(C_*(\Omega X; k), d),$$

where $C_*(\Omega X; k)$ is the DGA of singular chains on the Moore loop space of $X$. Thus if $(T(V), d) \xrightarrow{\sim} (C_*(\Omega X; k), d)$ is an Adams-Hilton model [2] for $X$ then we have

$$(1.2) \quad H_*(X^{S^1}; k) \cong \text{HH}_*(T(V), d),$$

because DGA quisms induce isomorphisms of Hochschild homology, as follows from the Eilenberg-Moore comparison theorem [21; Theorem 2.3].

Let $(\Omega^*, d)$ be the DGA obtained by dualizing the bar construction on $(T(V), d)$—we recall the definition in §2. The main result (Theorem II) of §2 will show that

$$(1.3) \quad \text{HH}^*(\Omega^*, d) \cong \text{Hom}(\text{HH}_*(T(V), d), k).$$

In §3, on the other hand, we observe that either of conditions (A) and (B) gives a DGA quism $(\Omega^*, d) \xrightarrow{\sim} (A, d)$, where $(A, d)$ is a commutative differential graded algebra (CDGA). In the case of condition (A) this follows from a deep theorem of Anick [4]; in the case of condition (B) it is a consequence of one of the equivalent definitions of $k$-formal ([3], [13]). In either case we again apply the comparison theorem of [21] to obtain

$$(1.4) \quad \text{HH}^*(\Omega^*, d) \cong \text{HH}^*(A, d).$$

The isomorphisms (1.1), (1.2), (1.3) and (1.4) combine to yield

$$(1.5) \quad H^*(X^{S^1}; k) \cong \text{HH}^*(A, d).$$

As we note in §3, the CDGA $(A, d)$ satisfies $H(A) = H^*(X; k)$. Indeed when $X$ is $k$-formal $(A, d) = (H^*(X; k), 0)$ and so (1.5) becomes

$$H^*(X^{S^1}; k) \cong \text{HH}^*(H^*(X; k)), $$

in this case. This answers a question of Anick [3] in positive characteristic; in characteristic zero it has been proved by Vigué-Poirrier [29] and Anick [3].
The last step in the proof of Theorem I is the proof, in §4 of

**Theorem III.** Let \((A, d)\) be a CDGA such that \(H^{<0}(A) = 0\), \(H^0(A) = k\), \(H^1(A) = 0\) and \(H(A)\) is finite dimensional. Then the integers \(b_q = \dim HH^q(A, d)\) are unbounded if and only if \(H(A)\) is not generated by a single class.

The proof of Theorem III follows the lines of the proof in [28] when \(k = \mathbb{Q}\) via the construction of a Sullivan model for \((A, d)\), but with additions and modifications to cover the problems caused by positive characteristic.

2. Hochschild homology. In this section we prove a result which implies (1.3), namely

**Theorem II.** Suppose \((R, d)\) is an augmented DGA such that \(H_{<0}(R) = 0\), \(H_0(R) = k\) and each \(H_i(R)\) is finite dimensional. If \((\Omega^*(R), d)\) is the DGA dual to the bar construction on \((R, d)\) then

\[
HH^*(\Omega^*(R), d) \cong \text{Hom}(HH_*(R, d), k).
\]

Before starting the proof, however, we recall some definitions and facts from or about:

(a) differential homological algebra, (b) the opposite of a DGA, (c) differential coalgebras and comodules and (d) bar constructions.

(a) Differential homological algebra ([21], [5], [14]). An \((R, d)\)-module is a DGV, \((V, d)\), together with an \(R\)-module structure on \(V\) such that \(d(r \cdot v) = dr \cdot v + (-1)^{\deg r} r \cdot dv\). It is semi-free if it is the increasing union of submodules \(V(0) \subset V(1) \subset \cdots\) such that \(V(0)\) and each \(V(i+1)/V(i)\) is \(R\)-free on a basis of cycles. For any \((R, d)\)-module, \((M, d)\) there is a morphism \(\phi: (V, d) \to (M, d)\) from a semi-free module \((V, d)\) such that \(H(\phi)\) is an isomorphism; such a morphism is called a semi-free resolution of \((M, d)\). Given any such resolution and any second \((R, d)\)-module, \((N, d)\), we have

\[
\text{Tor}^R(M, N) = H(V \otimes_R N).
\]

(b) The opposite DGA. The opposite DGA, \((R^{\text{opp}}, d)\), has the same underlying differential graded vector space as \((R, d)\), but the product "\(\circ\)" is given by: \(r \circ r' = (-1)^{\deg r \deg r'} r'r\). The enveloping DGA \((R^e, d)\), is then defined by \((R^e, d) = (R, d) \otimes (R^{\text{opp}}, d)\) so that

\[
(r_1 \otimes r_2)(r_3 \otimes r_4) = (-1)^{\deg r_1 \deg r_3 + \deg r_4} r_1 r_3 \otimes r_4 r_2.
\]

Notice that multiplication makes \((R, d)\) into a left \((R^e, d)\)-module: \((r_1 \otimes r_2) \cdot r = (-1)^{\deg r_1 \deg r_2} r_1 r_2 r_2\); similarly we can make \((R, d)\) into a right \((R^e, d)\)-module.
(c) Differential comodules [21, §6]. A comodule over a differential graded coalgebra (DGC), \((C, d)\), is a DGV, \((W, d)\), together with a DGV morphism \(\gamma: (W, d) \rightarrow (W, d) \otimes (C, d)\) which makes \(W\) into a graded \(C\)-comodule. If \((W, d)\) is also an \((R, d)\)-module via \(\alpha: (R, d) \otimes (W, d) \rightarrow (W, d)\) then these structures are compatible if \(\gamma\) is an \(R\)-module map (equivalently \(\alpha\) is a \(C\)-comodule map).

If \(M\) and \(N\) are respectively a right and left \((C, d)\)-comodule then their cotensor product, \(M \otimes_C N\) is the kernel of the DGV morphism \(\gamma_M \otimes 1 - 1 \otimes \gamma_N: M \otimes N \rightarrow M \otimes C \otimes N\). If \(M\) has a compatible left \((R, d)\)-module structure and if \(Q\) is any right \((R, d)\)-module then a natural DGV map

\[
(2.1) \quad \omega: Q \otimes_R (M \otimes_C N) \rightarrow (Q \otimes_R M) \otimes_C N
\]

is constructed as follows:

Observe that \(M \otimes_C N\) is a sub \((R, d)\)-module of \(M \otimes N\), so that the inclusion induces \(\phi: Q \otimes_R (M \otimes_C N) \rightarrow Q \otimes_R (M \otimes N)\). Since clearly \(\gamma_Q \otimes R \otimes (M \otimes_C N)\) vanishes on \(\text{Im } \phi\), we have \(\text{Im } \phi \subset (Q \otimes_R M) \otimes_C N\), and so (2.2) is defined by \(\phi\).

(d) Bar constructions. Denote the augmentation ideal of \(R\) by \(\overline{R}\) and define a graded vector space \(s\overline{R}\) by \((s\overline{R})_n = \overline{R}_{n-1}\). The bar construction ([21], [29]) on \((R, d)\), denoted by \((BR, \delta)\), is the DGC defined (modulo signs) by: \(BR\) is the tensor coalgebra on \(s\overline{R}\) (as usual \(sr_1 \otimes \cdots \otimes sr_n\) is written \([sr_1| \cdots |sr_n]\)) and

\[
\delta[sr_1| \cdots |sr_n] = \sum_{i=1}^n \pm[sr_1| \cdots |sdr_i| \cdots |sr_n]
\]

\[
\quad + \sum_{i=1}^{n-1} \pm[sr_1| \cdots |s(r_ir_{i+1})| \cdots |sr_n].
\]

The dual DGA, \(\text{Hom}((BR, \delta); k)\), is denoted by \((\Omega^*(R), d)\).

From the bar construction one builds the classic acyclic construction \((R \otimes BR, \nabla)\) given by \(\nabla = d \otimes 1 + 1 \otimes \delta + \tau\) with

\[
\tau(r \otimes [sr_1| \cdots |sr_n]) = \pm rr_1 \otimes [sr_2| \cdots |sr_n].
\]

It is in an obvious way a left \((R, d)\)-module and a right \((BR, \delta)\)-comodule. Finally, we have the two-sided bar construction \((R \otimes BR \otimes R^{opp}, D)\) with \(D = d \otimes 1 \otimes 1 + 1 \otimes \delta \otimes 1 + 1 \otimes 1 \otimes d + \theta\), and

\[
\theta(r \otimes [sr_1| \cdots |sr_n] \otimes r') = \pm rr_1 \otimes [sr_2| \cdots |sr_n] \otimes r'
\]

\[
\quad + r \otimes [sr_1| \cdots |sr_{n-1}] \otimes r_n r'.
\]
It is straightforward ([21; §6]) that the augmentation \( e: BR \to k \), together with the multiplication map \( R \otimes R^{\text{opp}} \to R \) defines an \((R^e, d)\)-semi-free resolution \( (R \otimes BR \otimes R^{\text{opp}}, D) \to (R, d) \). Thus

\[
H(R \otimes R^e (R \otimes BR \otimes R^{\text{opp}})) = \text{Tor}_{R}^{R^{\text{opp}}}(R, R) = \text{HH}_*(R, d),
\]

and indeed this was the original definition of Hochschild homology.

These constructions may also be applied to \((R^{\text{opp}}, d)\) to yield the DGC \((B(R^{\text{opp}}, d), \delta)\) and the acyclic construction \((R^{\text{opp}} \otimes B(R^{\text{opp}}), \nabla)\). Moreover a DGC isomorphism, \( \omega: (B(R^{\text{opp}}), \delta) \to ((BR)^{\text{opp}}, \delta) \), onto the opposite DGC is defined by

\[
\omega[s_{r_1} | \cdots | s_{r_n}] = (-1)^k [s_{r_1} | \cdots | s_{r_1}], \quad k = \sum_{i < j} (\deg s_{r_i})(\deg s_{r_j}).
\]

Thus \( 1 \otimes \omega \) converts \((R^{\text{opp}} \otimes B(R^{\text{opp}}), \nabla)\) into a DGV, \((R^{\text{opp}} \otimes (BR)^{\text{opp}}, \nabla')\), which is both an \((R^{\text{opp}}, d)\)-module and an \(((BR)^{\text{opp}}, \delta)\)-comodule.

We come now to the Proof of Theorem II. As in [6] there is DGA quism of the form \((T(V), d) \xrightarrow{\cong} (R, d)\) with \( v_i = 0 \), \( i \leq 0 \), and each \( V_i \) finite dimensional. By the Eilenberg-Moore comparison theorem [21; Theorem 2.3] \( \Omega^* \) preserves quisms and \( \text{HH}^* \) converts quisms to isomorphisms. We may thus replace \((R, d)\) by \((T(V), d)\) and assume that

\[
R = R_{\geq 0}, \quad R_0 = k \quad \text{and each} \quad R_i \quad \text{is finite dimensional}.
\]

Now let \(((BR)^e, \delta)\) denote the DGC \((BR, \delta) \otimes ((BR)^{\text{opp}}, \delta)\) and set

\[
M(R) = (R \otimes BR \otimes R^{\text{opp}} \otimes (BR)^{\text{opp}}, \nabla \otimes 1 + 1 \otimes \nabla').
\]

Evidently \( M(R) \) has compatible left \((R^e, d)\)-module and right \(((BR)^e, \delta)\)-comodule structures. Moreover, we have

**Lemma 2.4.** For any right \((R^e, d)\)-module, \( Q \), and any left \(((BR)^e, \delta)\)-comodule \( N \) the natural DGV map

\[
\omega: Q \otimes_{R^e} (M(R) \square_{(BR)^e} N) \to (Q \otimes_{R^e} M(R)) \square_{(BR)^e} N
\]

is an isomorphism.

**Proof of (2.4).** We may ignore differentials and write \( M(R) = R^e \otimes (BR)^e \). The standard isomorphism \((BR)^e \square_{(BR)^e} N \cong N\) gives an isomorphism

\[
M(R) \square_{(BR)^e} N \cong R^e \otimes N
\]
of $R^e$ modules. Analogously, we have a $(BR)^e$-comodule isomorphism

\[(2.6)\quad Q \otimes_{R^e} M(R) \cong Q \otimes (BR)^e.\]

Using (2.5) and (2.6) one easily identifies $\omega$ with the identity of $Q \otimes N$. \(\Box\)

We apply Lemma 2.4 with $Q = (R, d)$ and $N = (BR, \delta)$, the module (resp., comodule) structures being defined by multiplication (resp., comultiplication) as described in (b) above. Notice that (2.5) becomes

\[M(R) \boxtimes_{(BR)^e} BR \cong R^e \otimes BR \cong R \otimes BR \otimes R^{opp};\]

according to [17; Lemma 2.01] the differential induced thereby in $R \otimes BR \otimes R^{opp}$ is that of the two-sided bar construction. Thus (cf. (2.2))

\[H(R \otimes R^e (M(R) \boxtimes_{(BR)^e} BR)) \cong \text{Tor}^{R \otimes R^{opp}}(R, R).\]

For simplicity denote the graded dual of a graded vector space by $V^* = \text{Hom}(V, k)$. Thus $(\Omega^*(R), d) = (BR, \delta)^\#$. Because of our assumption (2.3) both $R$ and $BR$ are concentrated in degrees $\geq 0$, and are finite dimensional in each degree. For such spaces $\#$ commutes with $\otimes$ so that, for instance, $([\Omega^*(R)]^e, d) = ((BR)^e, \delta)^\#$. Thus we deduce from Lemma 2.4 that

\[(2.7)\quad \text{HH}_*(R, d)^\# \cong H^*\{[(R \otimes R^e M(R)) \boxtimes_{(BR)^e} BR]^\#\}.\]

Write $Y = [R \otimes R^e M(R)]^\#$. We shall show that $Y$ is an $(\Omega^*(R)^e, d)$ semi-free resolution of $\Omega^*(R)$. Since

\[[(R \otimes R^e M(R)) \boxtimes_{(BR)^e} BR]^\# = Y \otimes_{\Omega^*(R)^e} \Omega^*(R),\]

it will then follow from (2.7) that $\text{HH}_*(R, d)^\# = \text{HH}^*(\Omega^*(R), d)$, as desired.

That $Y$ is $(\Omega^*(R)^e, d)$-semi-free can be seen by filtering it by the spaces $F_j$ of functions vanishing on $[R_{\geq j} + d(R_j)] \otimes R^e M(R)$. And a homology isomorphism $Y \to \Omega^*(R)$ of $(\Omega^*(R)^e, d)$-modules is defined by dualizing the diagonal $BR \to BR \otimes BR$, regarded as a map

\[BR \to 1 \otimes (BR)^e \subset R \otimes R^e M(R).\]

3. Reduction to the commutative case. Let

\[(T(V), d) \cong (C_*(\Omega X; k), d)\]

be an Adams-Hilton model [2] for a space $X$ satisfying the conditions of the conjecture, and denote the dual of the bar construction on
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(T(V), d) by (Ω*, d). In this section we prove

**Proposition 3.1.** If X satisfies condition (A) or condition (B) of Theorem I then there is a DGA quism (Ω*, d) \( \cong \) (A, d) with (A, d) a CDGA and \( H(A) \cong H^*(X; k) \). If condition (B) holds, (A, d) = (\( H^*(X; k) \), 0).

Proof. The main result of [4] asserts that if (A) holds then the differential in the Adams-Hilton model may be chosen so as to map \( V \) into the sub Lie algebra \( L \subset T(V) \) generated by \( V \). This identifies \( (T(V), d) \) as the universal enveloping algebra, \( U(L, d) \) of the DGL (differential graded Lie algebra) \( (L, d) \).

Recall that the bar construction is a tensor coalgebra, and in particular contains the sub-coalgebra, \( S \), of symmetric (in the graded sense) tensors. In particular, we have \( S(sL) \subset S(s(UL_+)) \subset B(UL) \). As in the case of characteristic zero ([23; Appendix B], [10]), \( S(sL) \) is a sub DGC of \( B(UL) \) and the inclusion \( S(sL) \rightarrow B(UL) \) is a homology isomorphism [22]. Dualizing this gives a quism from \( (\Omega^*, d) \) to the CDGA \( S(sL)^\# \). On the other hand [1] (\( C_*(\Omega X; k) \), d) is connected by DGA quisms to the cobar construction on \( (C_*(X; k), d) \), and hence to \( \Omega^*(C^*(X; k), d) \). Thus \( \Omega^*(T(V), d) \) is connected by quism to \( \Omega^*\Omega^*(C^*(X; k), d) \), and so by [21; Theorem 6.2] we have \( H(A) \cong H(\Omega^*(T(V), d)) \cong H^*(X; k) \).

Now suppose X satisfies condition (B); i.e., X is \( k \)-formal. One of the equivalent definitions of this is ([3], [13]) that X have an Adams-Hilton model which is the dual of the bar construction on \( H^*(X; k) \): \( (T(V), d) = \Omega^*(H^*(X; k), 0) \). Thus \( (\Omega^*, d) = \Omega^*(\Omega^*(H^*(X; k), 0)) \) and by [21; Theorem 6.2] this maps by a quism to \( (H^*(X; k), 0): (\Omega^*, d) \cong (H^*(X; k), 0) \).

4. The commutative case. In this section we prove

**Theorem III.** Let (A, d) be a CDGA such that \( H^{<0}(A) = 0 \), \( H^0(A) = k \), \( H^1(A) = 0 \) and \( H(A) \) is finite dimensional. Then the integers \( b_q = \dim HH^q(A, d) \) are unbounded if and only if \( H(A) \) is not generated by a single class.

Proof. As in the rational case ([27], [7], [18]) it is straightforward to construct a DGA quism of the form

\( (\Lambda V, d) \cong (A, D) \)
in which: \( V = V^{\geq 2} \) is a graded vector space, \( \Lambda V = \text{exterior algebra} \ (V^{\text{odd}}) \otimes \text{symmetric algebra} \ (V^{\text{even}}) \) and \( \text{Im} \, d \subset (\Lambda V)^{+} \cdot (\Lambda V)^{+} \). Using the Eilenberg-Moore comparison theorem [21; Theorem 2.3] we replace \((A, d)\) by \((\Lambda V, d)\).

The same argument as given in [28] for \( k = \mathbb{Q} \) now establishes

**Lemma 4.1.** The algebra \( H(\Lambda V) \) is generated by a single class if and only if \( \dim V^{\text{odd}} \leq 1 \).

If, moreover, \( H(\Lambda V) \) is generated by a single class then the hypothesis \( \dim H(\Lambda V) < \infty \) implies, in view of (4.1) that the only possibilities for \((\lambda V, d)\) are: \( V = 0 \), \( V = (x) \) with \( \deg x \) odd, or \( V = (x, y) \) with \( dy = x^{k} \) and \( \deg y \) odd. In all these cases there is an obvious quism \((\Lambda V, d) \xrightarrow{\sim} (H(\Lambda V), 0)\), which induces an isomorphism of Hochschild homology. Now a direct calculation shows \( \dim HH^{q}(H(\Lambda V), 0) \leq 2 \) for all \( q \).

It remains to show that the \( HH^{q}(\Lambda V, d) \) have unbounded dimensions if \( \dim V^{\text{odd}} \geq 2 \). Recall that \( sV \) is the graded space given by \((sV)_{k+1} = V_{k}\); thus \((sV)^{k} = V^{k+1}\). Denote by \( \Gamma(sV) \) the free divided powers algebra on \( sV \), [9], and denote the \( i \)-th divided power of \( sx \) by \( \gamma_{i}(sx) \).

Consider the multiplication homomorphism,

\[
\phi: (\Lambda V, d) \otimes (\Lambda V, d) \to (\Lambda V, d).
\]

According to [15; Proposition 1.9], \( \phi \) extends to a DGA quism of the form

\[
\phi: (\Lambda V \otimes \Lambda V \otimes \Gamma(sV), D) \xrightarrow{\sim} (\Lambda V, d)
\]

in which

\[
\phi(\Gamma(sV)^{+}) = 0,
\]

\[
\text{Im} \, D \subset (\Lambda V \otimes \Lambda V)^{+} \otimes \Gamma(sV) \quad \text{and}
\]

\[
D(\gamma_{i}(sx)) = D(sx) \cdot \gamma_{i-1}(sx).
\]

For ease of notation denote the algebra \( \Lambda V \otimes \Lambda V \otimes \Gamma(sV) \) by \( \Sigma(V) \), and for \( \Phi \in \Lambda V \) write \( \Phi' = \Phi \otimes 1 \otimes 1 \) and \( \Phi'' = 1 \otimes \Phi \otimes 1 \). Then the model (4.2) also satisfies:

\[
\text{For } x \in V^{n}, \quad Dsx - (x' - x'') \in \Sigma(V^{<n}).
\]

Now choose a basis \( x_{1}, x_{2}, \ldots, x_{m}, y, x_{m+1}, \ldots, x_{i}, \ldots \) in which \( \deg x_{1} \leq \cdots \leq \deg x_{m} \leq \deg y \leq \cdots \leq \deg x_{i} \leq \cdots \), and \( y \) is the first basis element of odd degree. (All other basis elements are denoted by \( x_{j} \), some \( j \).)
**Lemma 4.7.** The differential $D$ in $\Sigma(V)$ can be chosen so that $Dsy - (y' - y'') \in \Sigma(x_1, \ldots, x_m)$ and for all $i$, $Dsx_i - (x'_i - x''_i)$ is in the ideal generated by the $x'_j$, $x''_j$ and $\Gamma(sx_j)^+$, $j < i$.

**Proof.** $D$ is constructed inductively on $n$; if it has already been defined in $s(V^\leq n)$ then there is always a linear map of degree zero, $f: V^{n+1} \to \Sigma(V^\leq n) \cap \ker \phi$ such that $dv' - dv'' - Df(v) \equiv 0$ and given any such $f$, $D$ may be extended to $\Sigma(V^\leq n+1)$ by setting $D(sv) = v' - v'' - f(v)$, $v \in V^{n+1}$.

Now notice that because $V^1 = 0$ and $\text{Im} \; d \subset (\Lambda V)^+ \cdot (\Lambda V)^+$ it follows that $dy \in \Lambda(x_1, \ldots, x_m)$ and $dx_i$ is in the ideal generated by the $x_j$, $j < i$. Moreover, that $Dsy - (y' - y'') \in \Sigma(x_1, \ldots, x_m)$ is immediate from (4.6) as is $Dsx_i - (x'_i - x''_i) \in \Sigma(x_1, \ldots, x_{i-1})$ for $i \leq m$.

Suppose then that the lemma is proved for some $x_1, \ldots, x_i$, $i \geq m$. Let $I \subset \Sigma(x_1, \ldots, x_m, y, \ldots, x_i)$ be the ideal generated by the $\Sigma(x_j)^+$, $j \leq i$. Since $dx_j \in \Lambda^+(x_1, y, \ldots, x_{j-1}) \cdot \Lambda^+(x_1, \ldots, y, \ldots, x_{j-1})$

it follows from our induction hypothesis on $Dsx_j$ and from (4.5) that $D$ maps $I$ to itself. Dividing by $I$ gives us a CDGA of the form $(\Sigma(y), \overline{D})$ and a commutative diagram of CDGA morphisms

$$
\begin{array}{ccc}
(\Sigma(x_1, \ldots, y, \ldots, x_i), D) & \xrightarrow{\phi} & (\Lambda(x_1, \ldots, y, \ldots, x_i), d) \\
\downarrow \rho & & \downarrow \rho \\
(\Sigma y, \overline{D}) & \xrightarrow{\phi} & (\Lambda y, 0)
\end{array}
$$

in which

$$
\phi(y') = \phi(y'') = y, \; \phi(\gamma_i(sy)) = 0 \quad \text{and} \quad \overline{D}(\gamma_i(sy)) = (y' - y'')\gamma_{i-1}(sy).
$$

As described at the start of the proof, there is always an element $w \in \Sigma(x_1, \ldots, y, \ldots, x_i) \cap \ker \phi$ such that $dx'_{i+1} - dx''_{i+1} - Dw = 0$, and $D$ may be extended to $\Sigma(x_1, \ldots, x_{i+1})$ by setting $Dsx_{i+1} = x'_{i+1} - x''_{i+1} - w$. And for any such $w$,

$$
\overline{D} \rho w = \rho Dw = \rho(dx'_{i+1} - dx''_{i+2}) = 0,
$$
since $dx'_{i+1}$ and $dx''_{i+1}$ are in $I$. Moreover $\phi \rho w = \rho \phi w = 0$. Since $\phi: (\Sigma y, D) \to (\Lambda y, 0)$ is a surjective quism it follows that $\rho w = \overline{Du}$, some $u \in \ker \phi \cap \Sigma y$.

Regard $u$ as an element of $\ker \phi \cap \Sigma (x_1, \ldots, y, x_i)$ via the inclusion of $\Sigma y$. Then $\rho (w - Du) = 0$, $\phi (w - Du) = D \phi u = 0$ and so we may define $Dsx_{i+1} = x'_{i+1} - x''_{i+1} - w + Du$. Now we have $\rho (w - Du) = \rho w - \overline{Du} = 0$ and so $Dsx_{i+1} - (x'_{i+1} - x''_{i+1}) \in I$, as desired. □

We now return to the proof of Theorem III. It follows from (4.5) and (4.6) that the quism $\phi: (\Lambda V \otimes \Lambda V \otimes \Gamma (sV), D) \to (\Lambda V, d)$ is a $(\Lambda V, d) \otimes (\Lambda V, d)$-semi-free resolution. Hence

$$\text{HH}^*(\Lambda V, d) = H(\Lambda V \otimes_{\Lambda V \otimes \Lambda V} (\Lambda V \otimes \Lambda V \otimes \Gamma (sV))) = H(\Lambda V \otimes \Gamma (sV)).$$

Denote the differential in $\Lambda V \otimes \Gamma (sV)$ by $\delta$. Lemma 4.7 shows that $\delta (sx_i)$ is in the ideal generated by the $x_j$ and $\Gamma (sx_j)$, $j < i$. Let $z = x_{n+1}$ ($n \geq m$) be the first $x_i$ of odd degree and divide $\Lambda V \otimes \Gamma (sV)$ by the ideal generated by the $x_j$, $j \leq n$.

This produces a CDGA of the form $(\Lambda (y, z, x_{n+2}, \ldots) \otimes \Gamma (sV), \partial)$. The same argument as given in [28] shows that if this CDGA has unbounded betti numbers then so does $(\Lambda V \otimes \Gamma (sV), \delta)$, as desired. But by Lemma 4.7, $\partial sx_i$ is in the ideal generated by $sx_1, \ldots, sx_{i-1}$, for $i \leq n$. Moreover $\Gamma (sx_i) = the exterior algebra $\Lambda (sx_i)$ because $\deg sx_i$ is odd. Hence $sx_1 \wedge \cdots \wedge sx_n$ is a cycle.

And since $\delta (sv)$ and $\delta (sz)$ are also in the ideal generated by $sx_1, \ldots, sx_n$ it follows from (4.5) that the elements $sx_1 \wedge \cdots \wedge sx_n \wedge \gamma_i (sv) \wedge \gamma_j (sz)$ are all $\partial$-cycles. Under the projection $\Lambda V \otimes \Gamma (sV) \to \Gamma (sV)$ these elements map to linearly independent homology classes, since the differential included in $\Gamma (sV)$ is zero, by (4.4). Thus they represent linearly independent classes in $H(\Lambda (y, z, \ldots) \otimes \Gamma (sV), \partial)$, and hence the betti numbers of $(\Lambda (y, z, \ldots) \otimes \Gamma (sV), \partial)$ are indeed unbounded. □

References


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