Pacific Journal of Mathematics

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Vol. 148, No. 1 March 1991

PACIFIC JOURNAL OF MATHEMATICS
Vol. 148, No. 1, 1991

DUALITY AND INVARIANTS FOR BUTLER GROUPS

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A duality is used to develop a complete set of numerical quasi-isomorphism invariants for the class of torsion-free abelian groups consisting of strongly indecomposable cokernels of diagonal embeddings $A_1 \cap \cdots \cap A_n \to A_1 \oplus \cdots \oplus A_n$ for n-tuples $(A_1, \ldots, A_n)$ of subgroups of the additive group of rational numbers.

A major theme in the theory of abelian groups is the classification of groups by numerical invariants. For the special case of torsion-free abelian groups of finite rank, one must first consider the decidedly non-trivial problem of classification up to quasi-isomorphism. To this end, we develop a contravariant duality on the quasi-homomorphism category of T-groups for a finite distributive lattice T of types.

A *Butler group* is a finite rank torsion-free abelian group that is isomorphic to a pure subgroup of a finite direct sum of subgroups of Q, the additive group of rationals. Isomorphism classes of subgroups of Q, called *types*, form an infinite distributive lattice. For a finite distributive sublattice T of types, a *T-group* is a Butler group G with each element of the *typeset of* G (the set of types of pure rank-1 subgroups of G) in T. Each Butler group is a T-group for some T, since Butler groups have finite typesets [**BU1**], but T is not, in general, unique. There are various characterizations of Butler groups, as found in [**AR2**], [**AR3**], and [**AV1**], but a complete structure theory has yet to be determined. As E. L. Lady has pointed out in [**LA1**] and [**LA2**], the theory generalizes directly to Butler modules over Dedekind domains.

Define B_T to be the category of T-groups with morphism sets $Q \otimes_Z \mathrm{Hom}_Z(G, H)$. Isomorphism in B_T is called *quasi-isomorphism* and an indecomposable in B_T is called *strongly indecomposable*. B. Jońsson in [**JO**] showed that direct sum decompositions in B_T are unique up to order and quasi-isomorphism (see [**AR1**] for the categorical version). Thus, classification of T-groups up to quasi-isomorphism depends only on the classification of strongly indecomposable T-groups.

A complete set of numerical quasi-isomorphism invariants for strongly indecomposable T-groups of the form $G = G(A_1, \ldots, A_n)$, the kernel of the map $A_1 \oplus \cdots \oplus A_n \to Q$ given by $(a_1, \ldots, a_n) \to a_1 + \cdots + a_n$ for $(A_1, \ldots, A_n)$ an n-tuple of subgroups of Q, is given in **[AV2]**. Specifically, the invariants are $\{r_G[M] | M \subseteq T\}$, where $r_G[M] = \operatorname{rank}(\bigcap\{G[\sigma] | \sigma \in M\})$.

Given an anti-isomorphism $\alpha : T \to T'$ of finite lattices of types, there is a contravariant duality $D(\alpha)$ from B_T to $B_{T'}$ (Corollary 5). The duality $D(\alpha)$ coincides with a duality on T-valuated Q-vector spaces given by F. Richman in **[RI1]** and includes, as special cases, the duality for *quotient divisible Butler groups* (all types are isomorphism classes of subrings of Q) given in **[AR5]** and by E. L. Lady in **[LA1]**, and the duality given for certain self-dual T in **[AV1]**. The search for lattices anti-isomorphic to a given lattice is simplified by an observation in **[RI1]** that each finite distributive lattice is isomorphic to a sublattice of a Boolean algebra of subrings of Q.

Groups of the form $G = G(A_1, \ldots, A_n)$ are sent by the duality $D(\alpha)$ to groups of the form $G = G[A_1, \ldots, A_n]$, the cokernel of the embedding $\bigcap\{A_i | 1 \leq i \leq n\} \to A_1 \oplus \cdots \oplus A_n$ given by $a \to (a, \ldots, a)$. This observation gives rise to an application of the duality $D(\alpha)$.

COROLLARY I. *Let T be a finite distributive lattice of types. A complete set of numerical quasi-isomorphism invariants for strongly indecomposable T-groups of the form $G = G[A_1, \ldots, A_n]$ is given by $\{r_G(M) | M$ a subset of $T\}$, where $r_G(M) = \operatorname{rank}(\Sigma\{G(\tau) | \tau \in M\})$. Each such group has quasi-endomorphism ring isomorphic to Q.*

Despite other options, we develop duality in terms of representations of finite *posets* (partially ordered sets) over an arbitrary field k. This choice is motivated by the fact that duality in this context is an easy consequence of vector space duality. Moreover, the quasi-isomorphism invariants given in Corollary I arise naturally when the groups are viewed as representations. As an added bonus, this duality is also applicable to classes of finite valuated p-groups. Specifically, given any finite poset S and prime p, there is an embedding from the category of Z/pZ-representations of S to the category of finite valuated p-groups that preserves isomorphism and indecomposability **[AR4]**. Implications of this embedding will be examined elsewhere.

Unexplained notation and terminology will be as in **[AR1]**, **[AR2]** **[AR4]**, and **[AV1]**.

If k is a field and S is a finite poset, then a *k-representation of S* is $X = (U, U_i | i \in S)$, where U is a finite dimensional k-vector space, each U_i is a subspace of U, and $i \leq j$ in S implies that

$U_i \subseteq U_j$. Let $\operatorname{Rep}(k, S)$ denote the category of k-representations of a finite poset S, where a *morphism* $f: (U, U_i|i \in S) \to (U', U_i'|i \in S)$ is a k-linear transformation $f: U \to U'$ with $f(U_i) \subseteq U_i'$ for each i. This category is a pre-abelian category (as defined in **[RIW]**) with finite direct sums defined by

$$(U, U_i|i \in S) \oplus (U', U_i'|i \in S) = (U \oplus U', U_i \oplus U_i'|i \in S).$$

Direct sum decompositions into indecomposable representations exist and are unique, up to isomorphism and order, since endomorphism rings of indecomposable representations are local. A sequence in $\operatorname{Rep}(k, S)$, $0 \to (U, U_i) \to (U', U_i') \to (U'', U_i'') \to 0$, is exact if and only if $0 \to U \to U'' \to U'' \to 0$ and $0 \to U_i \to U_i' \to U_i'' \to 0$ are exact sequences of vector spaces for each $i \in S$.

For a poset S, let S^{op} denote S with the reverse ordering.

PROPOSITION 1 **[DR]**. *Suppose that S is a finite poset. There is an exact contravariant duality* $\sigma: \operatorname{Rep}(k, S) \to \operatorname{Rep}(k, S^{\mathrm{op}})$ *defined by* $\sigma(U, U_i: i \in S) = (U^*, U_i^\perp: i \in S^{\mathrm{op}})$, *where* $U^* = \operatorname{Hom}_k(U, k)$ *and* $U_i^\perp = \{f \in U^*: f(U_i) = 0\}$.

Proof. A routine exercise in finite dimensional vector spaces, noting that if $f: X \to X'$ is a morphism of representations, then $\sigma(f) = f^*: \sigma(X') \to \sigma(X)$ is a morphism of representations and that σ^2 is naturally equivalent to the identity functor.

There are some extremal representations to be dealt with. A representation of the form $X = (U, U_i|i \in S)$ is called a *simple representation* of S if $U = k$ and $U_i = 0$ for each i, and a *co-simple representation* if $U = k = U_i$ for each i. Simple representations are indecomposable projective and co-simple representations are indecomposable injective relative to exact sequences in $\operatorname{Rep}(k, S)$. The duality σ carries simple representations into co-simple representations. It is easy to verify that a representation $X = (U, U_i|i \in S)$ has no simple summands if and only if $U = \Sigma\{U_i|i \in S\}$ and no co-simple summands if and only if $\bigcap\{U_i|i \in S\} = 0$.

Recall that types are ordered by $[X] \leq [Y]$ if and only if X is isomorphic to a subgroup of Y, where $[X]$ denotes the isomorphism class of a subgroup X of Q. The join of $[X]$ and $[Y]$ is $[X + Y]$, and the meet is $[X \cap Y]$.

Let G be a T-group and $0 \neq x \in G$. Then $\operatorname{type}_G(x)$ is the type of the pure rank-1 subgroup of G generated by x. Define $G(\tau) = \{x \in G|\operatorname{type}_G(x) \geq \tau\}$, the *$\tau$-socle of* G. Let $QG = Q \otimes_Z G$ denote the divisible hull of G, regard G as a subgroup of QG, and write $QG(\tau)$ for the Q-subspace of QG generated by $G(\tau)$.

Define $\mathrm{JI}(T)$ to be the set of *join-irreducible* elements of a finite lattice T of types. That is, $\mathrm{JI}(T) = \{\tau \in T \mid \text{if } \tau = \delta \text{ join } \gamma \text{ for } \delta, \gamma \in T, \text{ then } \tau = \gamma \text{ or } \tau = \delta\}$. The poset $\mathrm{JI}(T)^{\mathrm{op}}$ has a greatest element, namely the least element of T. In the correspondence of the following lemma, the simple indecomposables in $\mathrm{Rep}(Q, \mathrm{JI}(T)^{\mathrm{op}})$ have no non-zero group analogs. Thus, define $\mathrm{Rep}_0(Q, \mathrm{JI}(T)^{\mathrm{op}})$ to be $\mathrm{Rep}(Q, \mathrm{JI}(T)^{\mathrm{op}})$ subject to identifying a simple indecomposable representation with the indecomposable projective representation $(U, U_\tau \mid \tau \in \mathrm{JI}(T)^{\mathrm{op}})$ defined by $U = Q$, $U_\tau = Q$ if τ is the greatest element of $\mathrm{JI}(T)^{\mathrm{op}}$, and $U_\tau = 0$ otherwise. This guarantees that a simple indecomposable representation corresponds to a rank-1 group in B_T with type equal to the least element of T.

LEMMA 2 (a) **[BU2, BU3]**. *There is a category equivalence* $F_T\colon B_T \to \mathrm{Rep}_0(Q, \mathrm{JI}(T)^{\mathrm{op}})$ *given by* $F_T(G) = (QG, QG(\tau) \mid \tau \in \mathrm{JI}(T)^{\mathrm{op}})$.

(b) F_T *is an exact functor.*

Proof. (a) We observe only that the inverse of F_T sends $(U, U_\tau \mid \tau \in \mathrm{JI}(T)^{\mathrm{op}})$ to the subgroup of U generated by $\{G_\tau \mid \tau \in \mathrm{JI}(T)^{\mathrm{op}}\}$, where G_τ is a subgroup of torsion index in U_τ that is *τ-homogeneous completely decomposable* (isomorphic to a direct sum of rank-1 groups with types in τ). The proof is outlined in **[BU3]** with details in **[BU2]**.

(b) Note that B_T is also a pre-abelian category and that a sequence $0 \to G \xrightarrow{f} H \xrightarrow{g} K \to 0$ of T-groups is exact in B_T if and only if f is monic, $(\mathrm{kernel}\, g + \mathrm{image}\, f)/(\mathrm{kernel}\, g \cap \mathrm{image}\, f)$ is finite, and $(\mathrm{image}\, g + K)/(\mathrm{image}\, g \cap K)$ is finite. In particular, $0 \to QG \to QH \to QK \to 0$ is exact. Recall that, since we are working in a quasi-homomorphism category, equality in B_T is to be interpreted as *quasi-equality* of groups (G and H are quasi-equal if $QG = QH$ and there is a non-zero integer n with $nG \subseteq H$ and $nH \subseteq G$) and purity in B_T as *quasi-purity* (quasi-equal to a pure subgroup).

Let $0 \to G \xrightarrow{f} H \xrightarrow{g} K \to 0$ be an exact sequence in B_T. It is sufficient to show that if $\tau \in \mathrm{JI}(T)^{\mathrm{op}}$, then $QH(\tau) \xrightarrow{g} QK(\tau) \to 0$ is exact. In this case, $0 \to QG(\tau) \to QH(\tau) \to QK(\tau) \to 0$ is exact and $0 \to F_T(G) \to F_T(H) \to F_T(K) \to 0$ is exact in $\mathrm{Rep}(Q, \mathrm{JI}(T)^{\mathrm{op}})$.

If X is a pure rank-1 subgroup of K of type $\geq \tau$, then $g^{-1}(X)$ is generated in B_T by a finite set L of pure rank-1 subgroups of H whose types are in T **[BU1]**. Thus, $\mathrm{type}(X)$ is the join of the elements in a set L' of types of groups in L with nonzero image under g in QX. Also, τ is the join of the elements in $\{\sigma \text{ meet } \tau | \sigma \in L'\}$. But τ join irreducible in T implies that $\sigma \geq \tau$ for some $\sigma \in L'$, whence QX is in the image of $QH(\tau) \xrightarrow{g} QK(\tau)$. Consequently, $QH(\tau) \xrightarrow{g} QK(\tau) \to 0$ is exact, as desired.

At this stage, it is tempting to try to define a duality from $B_T \to B_T$, for anti-isomorphic lattices T and T' by using Lemma 2 and Proposition 1. This would require, however, that $\mathrm{JI}(T')^{\mathrm{op}}$ be lattice isomorphic to $\mathrm{JI}(T)$, a rare occurrence as $\mathrm{JI}(T')^{\mathrm{op}}$ has a greatest element but $\mathrm{JI}(T)$ need not. To overcome this difficulty, we need a functor from B_T to $\mathrm{Rep}(Q, S)$ for some other partially ordered set S. A candidate for S is the opposite of $\mathrm{MI}(T)$, the set of meet irreducible elements of T.

Note that $\mathrm{MI}(T)^{\mathrm{op}}$ has a least element, the greatest element of T. Define $\mathrm{Rep}^0(Q, \mathrm{MI}(T)^{\mathrm{op}})$ to be $\mathrm{Rep}(Q, \mathrm{MI}(T)^{\mathrm{op}})$ with a co-simple indecomposable representation identified with the indecomposable injective representation $(U = Q, U_i | i \in S)$, where $U_i = 0$ if i is the least element of $\mathrm{MI}(T)^{\mathrm{op}}$ and $U_i = Q$ otherwise.

For a Butler group G and a type τ the *τ-radical of* G, $G[\tau]$, is defined to be $\bigcap\{\text{kernel } f | f: G \to Q,\ \text{type(image } f) \leq \tau\}$.

LEMMA 3 **[LA2]**. *Let T be a finite lattice of types, G a T-group, and $\tau \in T$.*

(a) $QG[\tau] = \Sigma\{QG(\gamma) | \gamma \in T, \gamma \nleq \tau\}$.
(b) $QG(\tau) = \bigcap\{QG[\gamma] | \tau \nleq \gamma \in T\}$.
(c) *If τ is the meet of γ and δ, then* $QG[\tau] = QG[\gamma] + QG[\delta]$.
(d) *If τ is the join of γ and δ, then* $QG(\tau) = QG(\gamma) \cap QG(\delta)$.

Proof. Proofs of (a) and (b) are given in **[AV1**, Proposition 1.9**]**. (c) and (d) then follow.

THEOREM 4. *Assume that T is a finite lattice of types. There is an exact category equivalence $E_T: B_T \to \mathrm{Rep}^0(Q, \mathrm{MI}(T)^{\mathrm{op}})$ given by $E_T(G) = (QG, QG[\tau] | \tau \in \mathrm{MI}(T)^{\mathrm{op}})$.*

Proof. Clearly, E_T is a functor where if $q \otimes f \in Q \otimes \mathrm{Hom}_Z(G, H)$, then $E_T(q \otimes f) = q(1 \otimes f): QG \to QH$. Also, E_T is well defined, since $\gamma \leq \tau$ in $\mathrm{MI}(T)^{\mathrm{op}}$ implies that $G[\gamma] \subseteq G[\tau]$.

The fact that $E_T\colon Q\,\mathrm{Hom}(G, H) \to \mathrm{Hom}(E_T(G), E_T(H))$ is an isomorphism is proved in **[LA2**, Theorem 1.5]. Also E_T has a well defined inverse, since G can be recovered, up to quasi-isomorphism, from $(QG, QG(\tau)|\tau \in \mathrm{JI}(T)^{\mathrm{op}})$ by Lemma 2 and the $QG(\tau)$'s can be recovered from $(QG, QG[\gamma]|\gamma \in \mathrm{MI}(T)^{\mathrm{op}})$ by Lemma 3.

It remains to show exactness of E_T. Assume that $0 \to G \to H \xrightarrow{g} K \to 0$ is exact in B_T, and let X be a pure rank-1 subgroup of K in B_T of type not less than or equal to γ. As noted in the proof of Lemma 2, $g^{-1}(X)$ is generated in B_T by a finite number of pure rank-1 subgroups of H in B_T such that type(X) is the join of the types of those groups having non-zero image under g in QX. Therefore, at least one of these types is not less than or equal to γ. It follows from Lemma 3.a that QX is contained in $g(QH[\gamma])$. Thus, $QH[\gamma] \xrightarrow{g} QK[\gamma] \to 0$ is exact, since $g(QH[\gamma]) \subseteq QK[\gamma]$ is immediate. Note that this part of the proof does not require γ to be meet irreducible.

Next, $QG \cap QH[\gamma] \supseteq QG[\gamma]$ for each γ. To show that $QG[\gamma] \supseteq QG \cap QH[\gamma]$ for $\gamma \in \mathrm{MI}(T)$, let X be a pure rank-1 subgroup of G in B_T and assume that $X \cap G[\gamma] = 0$. Then type$(X) \le \gamma$, by Lemma 3.a. As H is a pure subgroup in B_T of a finite rank completely decomposable T-group, type(X) is the meet of the elements in a subset L of types of rank-1 torsion-free quotients of H in B_T such that the image of X in each of these quotients is non-zero **[AV1]**. In view of the distributivity of T, γ is the meet of the elements in $\{\gamma \text{ join } \alpha|\alpha \in L\}$. Since γ is meet irreducible, $\alpha \le \gamma$ for some $\alpha \in L$. Hence, $X \cap H[\gamma] = 0$, as X is not in the kernel of a homomorphism from H to a rank-1 torsion-free quotient of H with type $= \alpha \le \gamma$. Consequently, if X is a pure rank-1 subgroup of $G \cap H[\gamma]$, then $X \subseteq G[\gamma]$, since $X \cap G[\gamma] = 0$ implies that $X \cap H[\gamma] = 0$, as desired.

An exact sequence $0 \to G \to H \to K \to 0$ in B_T is *balanced* if $0 \to G(\tau) \to H(\tau) \to K(\tau) \to 0$ is exact in B_T for each type $\tau \in T$ and *cobalanced* if $0 \to G/G[\tau] \to H/H[\tau] \to K/K[\tau] \to 0$ is exact in B_T for each type $\tau \in T$.

COROLLARY 5. *Let $\alpha\colon T \to T'$ be a lattice anti-isomorphism of finite distributive lattices of types. There is a contravariant exact category equivalence $D = D(\alpha)\colon B_T \to B_{T'}$ defined by $D(G) = H$, $QH = QG^* = \mathrm{Hom}_Q(QG, Q)$, and $QH[\alpha(\tau)] = QG(\tau)^{\perp}$ for each $\tau \in T$, with the following properties:*

(a) $D(\alpha^{-1})D(\alpha)$ *is naturally equivalent to the identity functor on* B_T, $\operatorname{rank}(D(G)) = \operatorname{rank}(G)$, *and* $QH(\alpha(\tau)) = QG[\tau]^{\perp}$ *for each* $\tau \in T$.

(b) $D(G(\tau))$ *is quasi-isomorphic to* $D(G)/D(G)[\alpha(\tau)]$ *and* $D(G/G(\tau))$ *is quasi-isomorphic to* $D(G)[\alpha(\tau)]$ *for each* $\tau \in T$.

(c) *If* X *is a* rank-1 *T-group with* $\operatorname{type}(X) =$ *the join of the elements in a subset* $\{\tau_1, \dots, \tau_n\}$ *of* $\mathrm{JI}(T)$, *then* $\operatorname{type}(D(X))$ *is the meet of the elements in* $\{\alpha(\tau_1), \dots, \alpha(\tau_n)\} \subset \mathrm{MI}(T')$.

(d) D *sends balanced sequences to cobalanced sequences and conversely.*

(e) $D(G(A_1, \dots, A_n))$ *is quasi-isomorphic to* $G[D(A_1), \dots, D(A_n)]$ *for each* n*-tuple* $(A_1, \dots, A_n)$ *of subgroups of* Q *with types in* T.

Proof. (a) Define $D = D(\alpha) = E_{T'}^{-1}, \sigma\alpha F_T$, where F_T and $E_{T'}$, are as defined in Lemma 2 and Theorem 4, respectively;

$$\alpha : \mathrm{Rep}_0(Q, \mathrm{JI}(T)^{\mathrm{op}}) \to \mathrm{Rep}_0(Q, \mathrm{MI}(T'))$$

is a relabelling; and

$$\sigma : \mathrm{Rep}_0(Q, \mathrm{MI}(T')) \to \mathrm{Rep}_0(Q, \mathrm{MI}(T')^{\mathrm{op}})$$

is as given in Proposition 1. Note that D is contravariant, since σ is, and that D is exact since each of the defining functors are exact. Unravelling the definition of D shows that $D(G) = H$, where $QH = (QG)^*$ and $QH[\alpha(\tau)] = (QG(\tau))^{\perp}$ for $\tau \in \mathrm{JI}(T)$. In fact, $QH[\alpha(\tau)] = QG(\tau)^{\perp}$ for each $\tau \in T$. To see this, note that τ is the join of elements in a subset M of $\mathrm{JI}(T)$. Therefore,

$$QG(\tau) = \bigcap \{QG(\delta) | \delta \in M\},$$

by Lemma 3.d, and

$$\begin{aligned} QG(\tau)^{\perp} &= \Sigma\{QG(\delta)^{\perp} | \delta \in M\} \\ &= \Sigma\{QH[\alpha(\delta)] | \delta \in M\} = QH[\alpha(\tau)], \end{aligned}$$

by Lemma 3.c, since $\alpha(\tau)$ is the meet of the elements in $\{\alpha(\delta) | \delta \in M\}$.

Now G is naturally quasi-isomorphic to $D(\alpha^{-1})D(\alpha)(G)$, via the natural vector space isomorphism $QG \to QG^{**}$, as a consequence of Lemma 3. Clearly, $\operatorname{rank}(D(G)) = \operatorname{rank}(G)$. An argument using Lemma 3, analogous to that of the preceding paragraph, shows that if $H = D(G)$, then $QH(\alpha(\tau)) = QG[\tau]^{\perp}$ for each $\tau \in T$.

(a) is now clear; (c) and (e) follow from (a) and the exactness of D; and (d) is a consequence of (b).

As for (b), observe that $QD(G/G(\tau)) = \operatorname{Hom}(QGQG(\tau), Q)$ can be identified with $QG(\tau)^{\perp} = QD(G)[\alpha(\tau)]$. Under this identification, $QD(G/G(\tau))[\alpha(\delta)] = Q(G/G(\tau))(\delta)^{\perp}$ corresponds to $QG(\tau)^{\perp}[\alpha(\delta)] = QD(G)[\alpha(\tau)][\alpha(\delta)]$ for each $\delta \in \mathrm{JI}(T)$. Therefore, $D(G/G(\tau))$ is quasi-isomorphic to $D(G)[\alpha(\tau)]$, as desired. The other part of (b) now follows from the fact that D is a contravariant exact duality.

The proof of Corollary 5 shows that if G has rank one with type τ, then $D(G)$ is rank one with type $\alpha(\tau)$. This observation, together with Corollary 5.c, shows that $D = D(\alpha)$ is the duality induced by the duality of T-valuated vector spaces given in **[RI1]**. In case T is a locally free lattice, as defined in **[AV1]**, then T' and D may be chosen with D representable as $\operatorname{Hom}_Z(*, X)$ for X a rank-1 group with type equal to the greatest element in T. This special case of Corollary 5 follows from Warfield duality **[WA]**.

As noted earlier, given a finite lattice T of types, there is a quotient divisible T' anti-isomorphic to T **[RI1]**. If, for example, T is quotient divisible, then T' and $\alpha : T \to T'$ may be chosen by $\alpha(\tau) = \tau'$, where the p-component of τ' is 0 if and only if the p-component of τ is ∞ and the p-component of τ' is ∞ if and only if the p-component of τ is 0. Thus, D induces a duality, independent of T, on the quasi-homomorphism category of quotient divisible Butler groups. This duality coincides with the duality functor A on quotient divisible Butler groups given in **[LA1]** and the restriction of the functor F given in **[AR5]** to quotient divisible Butler groups.

For a T-group G and a subset M of T, define

$$G(M) = \Sigma\{G(\tau) | \tau \in M\} \quad \text{and} \quad G[M] = \bigcap\{G[\tau] | \tau \in M\}.$$

Then $r_G(M) = \operatorname{rank}(G(M))$ and $r_G[M] = \operatorname{rank}(G[M])$, as defined in the introduction. Lemma 3 can be applied to see that the $r_G(M)$'s or the $r_G[M]$'s appear as the dimensions of associated subspaces of QG generated by $\{QG(\tau) | \tau \in \mathrm{JI}(T)^{\mathrm{op}}\}$ or $\{QG[\tau] | \tau \in \mathrm{MI}(T)^{\mathrm{op}}\}$.

Proof of Corollary I. Since T is a finite distributive lattice of types there is a (quotient divisible) lattice T' of types and an anti-isomorphism $\alpha : T \to T'$. Let $D = D(\alpha)$ be as defined in Corollary 5. If G and H are T-groups both of the form $G[B_1, \dots, B_n]$ and $r_G(M) = r_H(M)$, then $QG(M)^{\perp}$ and $QH(M)^{\perp}$ have the same Q-dimension. But $D(G)[\alpha(M)] = QG(M)^{\perp}$ and $D(H)[\alpha(M)] = QH(M)^{\perp}$ via Corollary 5 and Lemma 3. Consequently, if $r_G(M) = r_H(M)$ for each subset M of T, then $r_{D(G)}[M'] = r_{D(H)}[M']$ for each subset M' of

T'. Now $D(G)$ and $D(H)$ are both of the form $G(A_1, \dots, A_n)$, by Corollary 5.e, so that $D(G)$ and $D(H)$ are quasi-isomorphic **[AV2]**. This implies that, by applying the duality $D(\alpha^{-1})$, G and H are quasi-isomorphic as desired. Finally, each strongly indecomposable group of the form $G(A_1, \dots, A_n)$ has endomorphism ring isomorphic to Q in B_T [**AV2**], and D is a category equivalence. The last statement of the corollary follows.

Corollary I includes a complete set of quasi-isomorphism invariants for the proper-subclass, co-CT-groups, of T-groups of the form $G[A_1, \dots, A_n]$ studied by W. Y. Lee in [**LE**].

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Received September 13, 1988. The first author's research was supported in part by NSF grant DMS-8802062. The second author's research was supported in part by NSF grant DMS-8802833.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 148, No. 1, 1991

OBSTRUCTION TO PRESCRIBED POSITIVE RICCI CURVATURE

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Obstruction to positive curvature is a phenomenon currently explored in global Riemannian geometry; the strongest results bear of course on the scalar curvature. Hereafter we consider the Ricci curvature and we adapt DeTurck and Koiso's device to non-compact manifolds. We also record a simple non-existence result on Kähler manifolds.

1. Statement of results. Let X be a connected non-compact C^3 n-manifold, $n > 2$, and $\mathbf{h}$ be a fixed C^2 Riemannian metric on X. We are interested in finding conditions on $\mathbf{h}$ which prevent it from being the Ricci tensor of any Riemannian metric on X. Following **[5]** we consider the largest eigenvalue $\lambda(\mathbf{h})$ of the curvature operator acting on covariant symmetric 2-tensors (see **[1]**). Given any C^2 metric $\mathbf{g}$ on X, we let $e(\mathbf{g})$ denote the energy density of the identity map from $(X, \mathbf{g})$ to $(X, \mathbf{h})$.

THEOREM 1. *Assume $\lambda(\mathbf{h}) \leq 1 - \varepsilon$ on X, for some positive real ε. Then there is no complete C^2 metric $\mathbf{g}$ on X which admits $\mathbf{h}$ as Ricci curvature.*

THEOREM 2. *Assume $\lambda(\mathbf{h}) \leq 1$ on X and $\mathbf{h}$ complete. Then there is no C^2 metric $\mathbf{g}$ on X, with $e(\mathbf{g})$ assuming a local maximum, which admits $\mathbf{h}$ as Ricci curvature.*

THEOREM 3. *Assume $\lambda(\mathbf{h}) \leq 1$ on X. Then there is no C^2 metric $\mathbf{g}$ on X, with $e(\mathbf{g})$ vanishing at infinity, which admits $\mathbf{h}$ as Ricci curvature.*

2. Remarks and examples. Our results and methods of proof extend **[5]** from compact to non-compact manifolds. Related, though weaker, results, obtained by different techniques, are those of **[0]** (a reference kindly pointed out to us by the referee).

Theorem 1 may be viewed as the "true" extension of [**5**, Theorem 3.2-b]. Interestingly, Theorem 2 looks somewhat stronger than

[5, Theorem 3.2-b] due to the non-compactness of X; an example here for $(X, \mathbf{h})$ is the Poincaré disk, since constant curvature -1 implies at once $\lambda(\mathbf{h}) \equiv 1$ by [1, Proposition 4.3]. Theorem 3 typically applies when $(X, \mathbf{g})$ is asymptotically flat; as such, it generalizes [8].

It is not possible to drop the completeness of both metrics and just assume $\lambda(\mathbf{h}) \leq 1$, as the following example shows: X is the euclidean n-space, $\mathbf{h}$ the conformal metric $4(n-1)\sigma^{-4}E$, E denoting the standard euclidean metric and $\sigma := \sqrt{1+|\mathbf{x}|^2}$. $\mathbf{h}$ satisfies $\lambda(\mathbf{h}) \equiv 1$ and $\mathrm{Ricci}(\mathbf{h}) = \mathbf{h}$ because it is constructed in the following way: start with the round n-sphere $(S^n, \mathbf{g}_0)$ of radius $r = \sqrt{n-1}$ so that $\mathrm{Ricci}(\mathbf{g}_0) = \mathbf{g}_0$. By [1, Proposition 4.3] we see at once that $\lambda(\mathbf{g}_0) \equiv 1$. Now $\mathbf{h}$ is obtained as the pull-forward of $\mathbf{g}_0$ by a stereographic projection composed with the dilation of ratio $1/r$.

From the identity $\lambda(c\mathbf{h}) = \frac{1}{c}\lambda(\mathbf{h})$ valid for any positive constant c, one would like to infer that, given *any* C^2 metric $\mathbf{h}$ on X, the preceding theorems hold with $c\mathbf{h}$ for suitable $c \gg 1$. This is what DeTurck and Koiso do on compact X. However, this cannot be done on non-compact X without *assuming* that $\lambda(\mathbf{h})$ is uniformly bounded from above (a mistake to be corrected in [8]). Keeping this in mind, one can formulate in an obvious way corollaries of our three theorems analogous to those of [5].

3. Proofs. For each theorem we argue by contradiction and assume the existence of a metric $\mathbf{g}$ with the asserted properties. As observed in [5], the Bianchi identity thus satisfied by $\mathbf{h}$ with respect to the metric $\mathbf{g}$ means that the identity map from $(X, \mathbf{g})$ to $(X, \mathbf{h})$ is harmonic. Hence the energy density $e(\mathbf{g})$ satisfies on X the elliptic differential inequality

$$\Delta[e(\mathbf{g})] \leq -2\|T\|^2 - [1-\lambda(\mathbf{h})]|\mathbf{h}|^2 \tag{1}$$

deduced in [5] from an identity discovered by R. Hamilton [6]. Here Δ stands for the Laplacian (with *negative* symbol) of $\mathbf{g}$, T for the $\binom{1}{2}$-tensor difference between the Christoffel symbols of $\mathbf{g}$ and $\mathbf{h}$, $|\cdot|$ for the norm in the metric $\mathbf{g}$, $\|\cdot\|$ for another norm (see [5]). Under the assumption $\lambda(\mathbf{h}) \leq 1$, made in all three theorems, $e(\mathbf{g})$ is thus C^2 positive subharmonic on $(X, \mathbf{g})$.

Proof of Theorem 1. By Schwarz inequality $e(\mathbf{g}) \leq \sqrt{n}|\mathbf{h}|$; so (1) implies that $e(\mathbf{g})$ solves on X the inequality

$$\Delta u \leq -f(u) \tag{2}$$

where

$$f(t) := (2\varepsilon/n)t^2.$$

The function f is positive strictly increasing on $(0, \infty)$ and it readily satisfies the following condition: for all $a < b$ in $(0, \infty)$,

$$\int_b^\infty \left(\int_a^s f(t)\, dt\right)^{-1/2} ds < \infty. \tag{3}$$

Assume provisionally that $(X, \mathbf{g})$ is of class C^3. Since $\mathbf{h} = \mathrm{Ricci}(\mathbf{g})$ is non-negative, $(X, \mathbf{g})$ and f fulfill all the conditions required for the proof of Calabi's extension of Hopf's maximum principle **[2]** (Theorem 4). Fixing $a \in (0, \min_X[e(\mathbf{g})])$ in (3) and arguing as in **[2]** yields an impossibility for $e(\mathbf{g})$ to satisfy (2) on X. So we get the desired contradiction.

We are left with the C^3 regularity of $(X, \mathbf{g})$. It follows basically from local elliptic regularity, as a repeated use of **[4]** now shows. Fix α in $(0, 1)$. Since $\mathbf{g}$ is $C^{1,\alpha}$, X admits a $C^{2,\alpha}$ atlas of coordinates harmonic for $\mathbf{g}$ **[4]** (Lemma 1.2). Being $C^{1,\alpha}$ in the original atlas, $\mathbf{h}$ remains so in the harmonic atlas **[4]** (Corollary 1.4). Since $\mathrm{Ricci}(\mathbf{g}) = \mathbf{h}$, $\mathbf{g}$ is $C^{3,\alpha}$ in the harmonic atlas **[4]** (Theorem 4.5-b) and the atlas itself actually is $C^{4,\alpha}$ **[4]** (Lemma 1.2). □

Proof of Theorem 2. By Hopf's maximum principle [7], $e(\mathbf{g})$ is necessarily constant on X. It follows from (1) that $T \equiv 0$ hence $\mathrm{Ricci}(\mathbf{h}) = \mathbf{h}$ on X. Moreover, the regularity argument above, now applied to $\mathbf{h}$, combined with a bootstrap argument, provides a harmonic atlas in which $(X, \mathbf{h})$ is a C^∞ Riemannian manifold. So Myers' theorem **[10]** holds for $(X, \mathbf{h})$, contradicting the noncompactness of X. □

Proof of Theorem 3. Since $e(\mathbf{g})$ vanishes at infinity, it assumes a positive global maximum M. Fix μ in $(0, M)$ and let K be a compact subdomain of X outside which $e(\mathbf{g}) \leq \mu$. Hopf's maximum principle [7] applied to $e(\mathbf{g})$ inside K implies that either $e(\mathbf{g})$ is constant on K, or $e(\mathbf{g}) \leq \mu$ on K. In both cases it contradicts $\mu < M$. □

4. A non-existence result on Kähler manifolds. Let X be a connected complex manifold, of complex dimension $n \geq 1$, admitting a C^2 Kähler metric $\mathbf{h}$. Denote by $|\mathbf{h}|$ the Riemannian density of $\mathbf{h}$.

THEOREM 4. *Assume that the scalar curvature of* $\mathbf{h}$ *is bounded above by* n, *but not identical to* n. *Then there exists no* C^2 *Kähler metric* $\mathbf{g}$

on X, with relative density $|\mathbf{g}|/|\mathbf{h}|$ assuming a local minimum, which admits $\mathbf{h}$ as Ricci curvature.

Proof. Again by contradiction; let $\mathbf{g}$ be such a metric. Then the C^2 function $f := \mathrm{Log}(|\mathbf{g}|/|\mathbf{h}|)$ satisfies on X the equation $\Delta f = n - S$, S standing for the scalar curvature of $\mathbf{h}$, Δ for its (complex) Laplacian. From the assumption, f is superharmonic on $(X, \mathbf{h})$; moreover, it assumes a local minimum, so it must be *constant* according to Hopf's maximum principle [7]. It implies that $S \equiv n$, contradicting the assumption. □

For non-compact X, Theorem 4 typically applies when $(X, \mathbf{g})$ is Kähler asymptotically $\mathbb{C}^n$ [3]. For compact X, recalling that $S(c\mathbf{h}) = S(\mathbf{h})/c$ for any positive constant c, we obtain a simple proof of the following

COROLLARY. *Let $(X, \mathbf{h})$ be a C^2 compact Kähler manifold. Then there exists a positive real $c(\mathbf{h})$ such that, for any real $c > c(\mathbf{h})$, no C^2 Kähler metric on X admits $c\mathbf{h}$ as Ricci curvature.*

Of course, as emphasized by J.-P. Bourguignon (in a letter to us), the classical cohomological constraint bearing on Ricci tensors of compact Kähler manifolds makes Theorem 4 rather relevant for *non-compact* simply connected X.

Acknowledgment. This work originated from a question posed to me by Albert Jeune, about the contradiction between **[3]** and Jeune's Corollary 1 in **[8]**; as pointed out in §2, the latter turns out to be incorrect without a boundedness assumption on $\lambda(\mathbf{h})$.

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Received June 28, 1989 and in revised form October 31, 1989.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 148, No. 1, 1991

NONPOSITIVELY CURVED HOMOGENEOUS SPACES OF DIMENSION FIVE

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In this paper we classify, in terms of the rank, the simply connected homogeneous spaces of nonpositive curvature and dimension five. In particular, an affirmative answer is given to the conjecture "An irreducible homogeneous space of nonpositive curvature and rank $k \geq 2$ is a symmetric space of rank k".

We exhibit examples in dimension five of rank one homogeneous spaces of nonpositive curvature having totally geodesic two-flats isometrically imbedded. Moreover, these examples show that the rank in a Lie group is not invariant under the change of left invariant metrics of nonpositive curvature

Introduction. In this paper we study, in terms of the rank, the simply connected Lie groups G of dimension five with left invariant metrics of nonpositive curvature $(K \leq 0)$. The results obtained are then used to get a classification of the simply connected homogeneous spaces of nonpositive curvature of rank two and dimension five. We exhibit on G, the Lie group of 3×3 upper triangular real matrices of determinant one, many different left invariant metrics of $K \leq 0$ and rank one. We remark that G also has a unique, up to a positive constant factor, left invariant metric of $K \leq 0$ and rank two which turns it into a symmetric space. Thus we obtain examples of rank one homogeneous spaces of nonpositive curvature having two-flats isometrically embedded. Moreover, we show that a Lie group (of dimension five) may admit different left invariant metrics of nonpositive curvature of different ranks.

In §1 we classify the simply connected five-dimensional homogeneous spaces H of nonpositive curvature with no flat de Rham factor and rank two. We show that, either $H = H^2 \times T^3$ where H^2 is a two-dimensional space of constant negative curvature and T^3 is a rank one homogeneous space of $K \leq 0$, or $H = \mathrm{SL}(3, \mathbf{R})/\mathrm{SO}(3)$ the irreducible symmetric space of noncompact type and rank two, provided that we multiply the metric by a suitable positive constant.

Section 2 is an auxiliary section needed to complete the classification given in §1. Here, we study a particular example in dimension five that corresponds to studying all the left invariant metrics of $K \leq 0$ on the group G of 3×3-upper triangular real matrices of determinant one.

In §3 we exhibit many different metrics turning G into rank one homogeneous spaces having 2-flats isometrically imbedded. Furthermore, a comparison result between the symmetric metric on G and non-symmetric ones is obtained.

Preliminaries. Let H be a complete simply connected Riemannian manifold of nonpositive curvature ($K \leq 0$). If γ is a unit speed geodesic in H, $\operatorname{rank}(\gamma)$ is defined to be the dimension of the vector space of all parallel Jacobi fields along γ. The minimum of $\operatorname{rank}(\gamma)$ over all geodesics γ of H is called *rank of* H and denoted by $\operatorname{rank}(H)$. This definition was introduced in [**3**] and coincides with the usual one if H is a symmetric space.

Assume that H is a homogeneous space. Then $\operatorname{rank}(H)$ is the minimum of $\operatorname{rank}(\gamma)$ over all geodesics γ of H such that $\gamma(0) = p$ for some p in H. In this case, H admits a simply transitive and solvable group of isometries (see [**1**]) and hence, H can be represented as a solvable Lie group G with a left invariant metric of nonpositive curvature.

Given a Lie group G with Lie algebra $\mathfrak{g}$ and left invariant metric $\langle\ ,\ \rangle$, we recall that if $X, Y, Z \in \mathfrak{g}$ then the Riemannian connection ∇ is given by

$$2\langle \nabla_X Y, Z\rangle = \langle [X, Y], Z\rangle - \langle [Y, Z]\rangle + \langle [Z, X], Y\rangle.$$

If $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ is the curvature tensor associated to ∇, the sectional curvature K is given by

$$\begin{aligned} |X \wedge Y|^2 K(X, Y) &= \langle R(X, Y)Y, X\rangle \\ &= \tfrac{1}{4}|U(X, Y)|^2 - \tfrac{1}{4}\langle U(X, X), U(Y, Y)\rangle - \tfrac{3}{4}|[X, Y]|^2 \\ &\quad - \tfrac{1}{2}\langle [[X, Y], Y], X\rangle - \tfrac{1}{2}\langle [[Y, X], X], Y\rangle \end{aligned}$$

where $U(X, Y) = (\operatorname{ad}_X)^* Y + (\operatorname{ad}_Y)^* X$, and $(\operatorname{ad}_X)^*$ denotes the adjoint of ad_X.

Let G be a solvable simply connected Lie group with a left invariant metric of nonpositive curvature. If $\mathfrak{a}$ is the orthogonal complement of $[\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{g}$ with respect to the metric, it follows from [**1**, Theorem 5.2] that it is an abelian subalgebra of $\mathfrak{g}$ which is also totally geodesic ($\nabla_X Y \in \mathfrak{a}$ for all $X, Y \in \mathfrak{a}$). Moreover, $A = \exp \mathfrak{a}$, the connected Lie subgroup of G with Lie algebra $\mathfrak{a}$, is a $\dim \mathfrak{a}$-flat in G.

In general, a k-flat in H is defined to be the image of a totally geodesic isometric imbedding of R^k into H.

1. Homogeneous spaces of $K \leq 0$ and dimension five. In this section we characterize, in terms of rank, the simply connected homogeneous spaces of nonpositive curvature ($K \leq 0$) and dimension five.

Let G be a solvable and simply connected Lie group with a left invariant metric of nonpositive curvature. If $\mathfrak{g}$ is the Lie algebra of G, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{a}$ where $\mathfrak{a}$, the orthogonal complement of $[\mathfrak{g}, \mathfrak{g}]$ with respect to the metric, is an abelian subalgebra of $\mathfrak{g}$.

If $\mathfrak{g}'^{c}$ is the complexification of $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ then we have a direct sum decomposition $\mathfrak{g}'^{c} = \sum_\lambda \mathfrak{g}'^{c}_\lambda$, where

$$\mathfrak{g}'^{c}_\lambda = \{U \in \mathfrak{g}'^{c} : (\mathrm{ad}_H - \lambda(H)I)^k U = 0 \quad \text{for some } k \geq 1 \text{ and for all } H \in \mathfrak{a}\}$$

is the associated root space for the root $\lambda \in (\mathfrak{a}^*)^c$ under the abelian action of $\mathfrak{a}$ on $\mathfrak{g}'$. If $\lambda = \alpha \pm i\beta$ is a root of $\mathfrak{a}$ in $\mathfrak{g}'$ (that is, $\mathfrak{g}'^{c}_\lambda \neq 0$), the generalized root space is defined by $\mathfrak{g}'_{\alpha, \beta} = \mathfrak{g}'_{\alpha, -\beta} = \mathfrak{g}' \cap (\mathfrak{g}'^{c}_\lambda \oplus \mathfrak{g}'^{c}_{\bar{\lambda}})$ and $\mathfrak{g}'$ is the direct sum of the $\mathrm{ad}_{\mathfrak{a}}$-invariant subspaces $\mathfrak{g}'_{\alpha, \beta}$.

We assume that G has no de Rham flat factor. Then, it follows from [**2**, Theorem 4.6] that the above condition is equivalent to $\mathfrak{g}'_0 = \sum \mathfrak{g}'_{0, \beta}$ and $\mathfrak{a}_0 = \{H \in \mathfrak{a} : \alpha(H) = 0 \text{ for all roots } \alpha + i\beta\}$ are zero.

The following formulas about sectional curvatures will be used frequently; we include the proofs for the sake of completeness. In the sequel, if $H \in \mathfrak{a}$ we will denote by D_H and S_H the symmetric and skew-symmetric part of ad_H respectively with respect to the metric $\langle\ ,\ \rangle$.

LEMMA 1.1. *Assume $\mathfrak{g}'$ abelian.*

(i) *Let $\{H_i\}_{i=1}^k$ be an orthonormal basis for $\mathfrak{a}$ and set $D_i = D_{H_i}$, $i = 1, \ldots, k$. Then,*

$$\langle R(X, Y)Y, X\rangle = \sum_{i=1}^{k} (\langle D_i X, Y\rangle^2 - \langle D_i X, X\rangle\langle D_i Y, Y\rangle) \quad \textit{for all } X, Y \in \mathfrak{g}'.$$

(ii) $\langle R(X, Y+H)(Y+H), X\rangle = \langle R(X, Y)Y, X\rangle + \langle R(X, H)H, X\rangle$ *for all $X, Y \in \mathfrak{g}'$ and $H \in \mathfrak{a}$.*

*In general, we have $\langle R(X, H)H, X\rangle = |S_H X|^2 - |[H, X]|^2$ ([**1**, Lemma 3.4]).*

Proof. Let $X, Y \in \mathfrak{g}'$ and $H \in \mathfrak{a}$.

(i) We note that since $\mathfrak{g}'$ is abelian, $U(X, Y) \in \mathfrak{a}$ and $\langle U(X, Y), H\rangle = -2\langle D_H X, Y\rangle$. Hence, $U(X, Y) = -2\sum_{i=1}^{k}\langle D_i X, Y\rangle H_i$; the assertion follows from the curvature formula.

(ii) Since $\nabla_H X \in \mathfrak{g}'$ and $\nabla_X Y \in \mathfrak{a}$ we have $R(X, Y)H \in \mathfrak{a}$; from this (ii) follows easily.

REMARK 1.2. If there exists an orthonormal basis $\{H_i\}_{i=1}^{k}$ of $\mathfrak{a}$ such that D_i $(i = 1, \dots, k)$ are all positive semidefinite, we have $K(X, Y) \leq 0$ for all X, Y independent in $\mathfrak{g}'$. Moreover, we get $K(X, Y) < 0$ if for some $j = 1, \dots, k$, D_i is positive definite.

THEOREM 1.3. *Let H be a simply connected homogeneous space of nonpositive curvature and* $\dim H = 5$. *If H has no de Rham flat factor then, either* $\operatorname{rank}(H) = 1$ *or* $\operatorname{rank}(H) = 2$ *and it is one of the following spaces*

(i) $H = H^2 \times T^3$, *where H^2 is a two-dimensional space of constant negative curvature and T^3 is a rank one homogeneous space of nonpositive curvature.*

(ii) $H = \mathrm{SL}(3, \mathbf{R})/\mathrm{SO}(3)$, *the irreducible symmetric space of noncompact type and rank two, up to multiplying the metric by a positive constant.*

We recall that in a three dimensional homogeneous space of nonpositive curvature, rank one and the visibility axiom are equivalent. These spaces were completely characterized in **[6]** (see Corollary 2.5 and Remark 4.3).

Proof. Let G be a solvable Lie group that acts simply and transitively on H. Then, we may assume that $H = G$ is a solvable and simply connected Lie group of dimension five with a left invariant metric of $K \leq 0$ with no flat de Rham factor.

Let $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{a}$, $\mathfrak{a}$ the orthogonal complement of $\mathfrak{g}'$ with respect to the metric $\langle\ ,\ \rangle$. We only need to consider the case $\dim \mathfrak{a} = 2$. In fact, in the case $\dim \mathfrak{a} = 1$ it follows from [7, Theorem 1.5] that G has rank one. If $\dim \mathfrak{a} = 3$, there exist at most two roots of $\mathfrak{a}$ in $\mathfrak{g}'$ $(\dim \mathfrak{g}' = 2)$ and consequently we may choose $H \in \mathfrak{a}$ satisfying $\alpha(H) = 0$ for all α with $\alpha + i\beta$ root; this implies that G has de Rham flat factor (see the remark at the beginning of this section). If $\dim \mathfrak{a} = 4$, $\mathfrak{g}$ is the example given in [**6**, Example 3.4] and G is isometric to $R^3 \times H^2$.

Henceforth we assume that $\dim \mathfrak{a} = 2$. Note that counting according to multiplicities, there are three roots of $\mathfrak{a}$ on $\mathfrak{g}'$. Their real parts span the dual space $\mathfrak{a}^*$ (otherwise $\mathfrak{a}_0$ would be nonzero). Thus there are two cases: either

(1) two real parts are proportional and the third is independent of them, or

(2) the three real parts (necessarily roots) are pairwise independent.

We first show the following lemma.

LEMMA. *If $\mathfrak{g}'$ is not abelian, then $\mathfrak{a}$ has three real roots λ_1, λ_2 and λ_3 on $\mathfrak{g}'$ such that λ_1 and λ_2 are independent and $\lambda_3 = \lambda_1 + \lambda_2$. Moreover the center $\mathfrak{z}$ of $\mathfrak{g}'$ is the root space of λ_3.*

Proof. Note that $\mathfrak{z} \neq 0$ because $\mathfrak{g}$ is solvable and hence $\mathfrak{g}'$ is nilpotent. Since $\mathfrak{z}$ is one-dimensional and $\mathrm{ad}_{\mathfrak{a}}$-invariant we have $\mathfrak{z} = \mathfrak{g}'_\lambda$, the root space associated to a nonzero real root λ ($\mathfrak{g}'_0 = 0$). We observe that there is no complex root $\gamma = \alpha + i\beta$, $\alpha \neq 0$; if this is the case, $\mathfrak{g}'^c = \mathfrak{g}'^c_\lambda \oplus \mathfrak{g}'^c_{\bar{\gamma}} \oplus \mathfrak{g}'^c_\lambda$ with $0 \neq [\mathfrak{g}'^c_\gamma, \mathfrak{g}'^c_{\bar{\gamma}}] \subset \mathfrak{g}'^c_{\gamma+\bar{\gamma}} = \mathfrak{g}'^c_{2\alpha}$. Thus $\lambda = 2\alpha$, implying that G has de Rham flat factor. Hence, since $\mathfrak{g}'$ is not abelian we have real roots λ_1, λ_2 and $\lambda_1 + \lambda_2$ ($0 \neq [\mathfrak{g}'_{\lambda_1}, \mathfrak{g}'_{\lambda_2}] \subset \mathfrak{g}'_{\lambda_1+\lambda_2}$) where λ_1 and λ_2 are independent.

Case 1. The lemma shows that $\mathfrak{g}'$ is abelian. It follows from the direct sum decomposition of $\mathfrak{g}'$ in generalized root spaces that there is an $\mathrm{ad}_{\mathfrak{a}}$ invariant orthogonal direct sum decomposition $\mathfrak{g}' = \mathfrak{g}'_1 \oplus \mathfrak{g}'_2$ (see [**1**, §5.3]) in which

(i) $\mathfrak{g}'_i$ has dimension i ($i = 1, 2$).

(ii) There is a basis $\{\gamma, \alpha\}$ of $\mathfrak{a}^*$ such that γ is the (necessarily real) root of $\mathfrak{a}$ on $\mathfrak{g}'_1$ and the real part of every root of $\mathfrak{a}$ on $\mathfrak{g}'_2$ is proportional to α.

We define $H_1, H_2 \in \mathfrak{a}$ by $\gamma(H) = \langle H, H_1\rangle$ and $\alpha(H) = \langle H, H_2\rangle$ for all $H \in \mathfrak{a}$. It follows from Lemma 5.4 (iv) of [**1**] that $\langle H_1, H_2\rangle \geq 0$. Thus, there are two cases to consider: either

$$\langle H_1, H_2\rangle = 0 \tag{1.1}$$

or

$$\langle H_1, H_2\rangle > 0. \tag{1.2}$$

Case 1.1. In this case it turns out that G is isometric to a Riemannian product. Let $\mathfrak{t} = \mathfrak{g}'_2 \oplus \mathbf{R}H_2$ and $\mathfrak{h} = \mathfrak{g}'_1 \oplus \mathbf{R}H_1$. Then $\mathfrak{t}$ is an ideal of $\mathfrak{g}$, $\mathfrak{h}$ is a subalgebra, and $\mathfrak{g}$ is the orthogonal direct sum of $\mathfrak{t}$ and $\mathfrak{g}$. Note that $\mathrm{ad}_{H_1|\mathfrak{g}_2'}$ is almost normal and has purely imaginary eigenvalues because $\alpha(H_1) = \langle H_1, H_2\rangle = 0$. It follows from Lemma 4.4 of [**1**] that $\mathrm{ad}_{H_1|\mathfrak{g}_2'}$ is skew symmetric. Since $\mathfrak{g}'$ and $\mathfrak{a}$ are abelian, it now follows that $\mathrm{ad}_{X|\mathfrak{t}}$ is skew symmetric for every $X \in \mathfrak{h}$. Hence, G is isometric to the Riemannian product $T^3 \times H^2$ where T^3 and H^2 are the connected Lie subgroups of G with Lie algebras $\mathfrak{t}$ and $\mathfrak{h}$ respectively, and left invariant metric induced by the one of $\mathfrak{g}$ (see [**6**, Lemma 4.1]). Moreover, H^2 has sectional curvature $K = K(e_1, H_1) = -|H_1|^2$ (e_1 is a unit vector in $\mathfrak{g}_1'$) and T^3 is a rank one homogeneous space of $K \le 0$ since it has no flat de Rham factor (see [**7**, Theorem 1.5]).

Case 1.2. In this case it turns out that G has rank one. We will prove this in the two following steps:

(1) $\langle R(X, Y)Y, X\rangle < 0$ whenever $X, Y \in \mathfrak{g}'$ are independent.

(2) There is $X \in \mathfrak{g}'$ with $\langle R(X, H)H, X\rangle < 0$ for all nonzero $H \in \mathfrak{a}$.

Hence, applying Lemma 1.1-(ii) we get $K(X, Y + H) < 0$ for all Y independent of X in $\mathfrak{g}'$ and all $H \in \mathfrak{a}$; consequently the geodesic γ in G satisfying $\gamma(0) = e$, $\gamma'(0) = X$ has rank one and therefore $\mathrm{rank}(G) = 1$.

Step 1. This will be done by showing that D_{H_2} is positive definite and the unit vector $H_0 \in \mathfrak{a}$ with $\langle H_0, H_2\rangle = 0$ and $\langle H_0, H_1\rangle > 0$ gives D_{H_0} positive semidefinite. Then by applying Remark 1.2, assertion (1) follows.

Note that the choice of H_0 means that ad_{H_0} has a positive eigenvalue on the one-dimensional space $\mathfrak{g}_1'$ and has purely imaginary eigenvalues on $\mathfrak{g}_2'$ ($\gamma(H_0) = \langle H_0, H_1\rangle > 0$ and $\alpha(H_0) = \langle H_0, H_1\rangle = 0$). By the argument explained above in Case 1.1, one sees that ad_{H_0} is skew symmetric on $\mathfrak{g}_2'$. Thus, D_{H_0} vanishes on $\mathfrak{g}_2'$ and hence it is positive semidefinite on $\mathfrak{g}'$.

Since $\langle H_1, H_2\rangle > 0$, it follows that D_{H_2} is positive definite on $\mathfrak{g}_1'$. It remains to show that D_{H_2} is positive definite on $\mathfrak{g}_2'$. We observe first that if $c\alpha$ is the real part of a root of $\mathfrak{a}$ on $\mathfrak{g}'$ it follows from Lemma 5.4 (iv) of [**1**] that $c > 0$ ($\mathfrak{g}'$ is abelian). Hence, both eigenvalues of $\mathrm{ad}_{H_2|\mathfrak{g}_2'}$ have positive real part and since $\mathrm{Tr}(D_{H_2|\mathfrak{g}_2'}) = \mathrm{Tr}(\mathrm{ad}_{H_2|\mathfrak{g}_2'}) > 0$, we have that $D_{H_2|\mathfrak{g}_2'}$ cannot be negative definite. Thus, it suffices to prove that $D_{H_2|\mathfrak{g}_2'}$ is definite. If this is not the case, then there is

$X \in \mathfrak{g}_2'$ with $D_{H_2}X = 0$. Since D_{H_0} vanishes on $\mathfrak{g}_2'$, it follows that $D_H X = 0$ for all $H \in \mathfrak{a}$, which is impossible because the only one-parameter subgroups which are geodesics are $\exp tH$, $H \in \mathfrak{a}$ (see [**8**, Theorem 3.6]).

Step 2. Since D_{H_2} is positive definite, we can choose a nonzero vector $e_2 \in \mathfrak{g}_2'$ such that $D_{H_2}e_2$ is a nonzero multiple of e_2 ($D_{H_2|\mathfrak{g}_2'}$ is symmetric). Let e_1 be a nonzero vector in $\mathfrak{g}_1'$ and let $X = e_1 + e_2$. For any $H \in \mathfrak{a}$, $D_H e_1$ and $D_H e_2$ are orthogonal, and $D_H e_1 = 0$, $D_H e_2 = 0$ if and only if H is orthogonal to H_1, and H is a multiple of H_0 respectively (H_0 is the same as in Step 1). Since $\langle H_0, H_2 \rangle = 0$ and H_1, H_2 are independent, it follows that $D_H X \neq 0$ for all nonzero $H \in \mathfrak{a}$.

Now, we observe that $\langle D_H X, S_H X \rangle = 0$ for all $H \in \mathfrak{a}$ ($S_H e_1 = 0$, $D_{H_0|\mathfrak{g}_2'} = 0$, $D_{H_2}e_2$ is a multiple of e_2).

Hence $K(X, H) = |S_H X|^2 - |[H, X]|^2 = -|D_H X|^2 < 0$ for all nonzero $H \in \mathfrak{a}$.

Case 2. We will show that either G has rank one or G is an irreducible symmetric space of rank two.

Case 2.1. $\mathfrak{g}'$ abelian with three pairwise independent real roots λ_1, λ_2 and λ_3.

We prove next that G has rank one. By permuting λ_1, λ_2 and λ_3, one can assume that $\lambda_3 = a\lambda_1 + b\lambda_2$ with both a and b positive. In fact, we define $H_i \in \mathfrak{a}$ by $\lambda_i(H) = \langle H, H_i \rangle$ ($i = 1, 2, 3$) for all $H \in \mathfrak{a}$. Then the H_i's are three nonzero vectors in the two-dimensional space $\mathfrak{a}$ and since $\langle H_i, H_j \rangle \geq 0$ (see [**1**, Lemma 5.4(iv)]) the angle between any two of them is at most $\pi/2$. We assign the indices so that H_1 and H_2 are the two outer vectors and H_3 lies in between.

Since $\mathfrak{g}_{\lambda_i}'$ ($i = 1, 2, 3$), the root space associated to λ_i, is one-dimensional and the roots λ_i are pairwise independent, we have an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{g}'$ (see [**1**, §5.3 (iii)]) such that:

$$[H, e_1] = \lambda_1(H)e_1, \quad [H, e_2] = \lambda_2(H)e_2, \quad [H, e_3] = \lambda_3(H)e_3$$

for all $H \in \mathfrak{a}$. Hence ad_H is symmetric for all $H \in \mathfrak{a}$ and its matrix with respect to the basis $\{e_1, e_2, e_3\}$ is given by

$$\mathrm{ad}_H = \begin{bmatrix} \langle H, H_1 \rangle & 0 & 0 \\ 0 & \langle H, H_2 \rangle & 0 \\ 0 & 0 & \langle H, aH_1 + bH_2 \rangle \end{bmatrix}.$$

Let H_0 be a unit vector in $\mathfrak{a}$ such that $\langle H_0, H_1\rangle = 0$ and $\langle H_0, H_2\rangle > 0$. Observe that $D_0 = \mathrm{ad}_{H_0}$ is positive semidefinite and restricted to $\mathfrak{g}'_{2,3} = \mathfrak{g}'_{\lambda_2} \oplus \mathfrak{g}'_{\lambda_3}$ is positive definite. Also, $D_1 = \mathrm{ad}_{H_1}$ is positive semidefinite and restricted to $\mathfrak{g}'_{1,3} = \mathfrak{g}'_{\lambda_1} \oplus \mathfrak{g}'_{\lambda_3}$ is positive definite. Hence, if $X = ce_1 + de_2 + ee_3$ is a unit vector and $Y \in \mathfrak{g}'$, it follows from the curvature formula given in Lemma 1.1-(i) that, $\langle R(X, Y)Y, X\rangle = 0$ if and only if $P|_{\mathfrak{g}'_{2,3}} Y$ is proportional to de_2+ee_3 and $p|_{\mathfrak{g}'_{1,3}} Y$ is proportional to ce_1+ee_3, where p denotes the orthogonal projection onto the indicated subspaces.

By a simple computation we deduce that if $e \neq 0$, $\langle R(X, Y)Y, X\rangle = 0$ if and only if Y is proportional to X. Hence, choosing $d \neq 0$, $e \neq 0$ (or $c \neq 0$) for any Y independent of X in $\mathfrak{g}'$ we get $\langle R(X, Y)Y, X\rangle < 0$. Moreover, for any nonzero vector $H \in \mathfrak{a}$,

$$\begin{aligned}\langle R(X, H)H, X\rangle &= -|[H, X]|^2\\ &= -c^2\lambda_1(H)^2 - d^2\lambda_2(H)^2 - e^2\lambda_3(H)^2 < 0\end{aligned}$$

since $\lambda_2(H)$ and $\lambda_3(H)$ (or $\lambda_1(H)$) cannot be simultaneously zero. Therefore, if γ is the geodesic in G with $\gamma(0) = e$, $\gamma'(0) = X$, γ has rank one and hence $\mathrm{rank}(G) = 1$.

Case 2.2. Assume $\mathfrak{g}'$ nonabelian. It turns out that either G has rank one or G is an irreducible symmetric space of rank two.

It follows from the lemma that there are three real roots λ_1, λ_2 and $\lambda_3 = \lambda_1 + \lambda_2$ with λ_1 and λ_2 independent. Moreover, $\mathfrak{z}$ is the eigenspace associated to λ_3. By the same argument as in Case 2.1 we get an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{g}'$ such that $[H, e_i] = \lambda_i(H)e_i$ $(i = 1, 2, 3)$ for all $H \in \mathfrak{a}$.

Let H_i be defined by $\lambda_i(H) = \langle H, H_i\rangle$ $(i = 1, 2)$, $H \in \mathfrak{a}$. We consider a unit vector H_0 in $\mathfrak{a}$ such that $\langle H_0, H_1 + H_2\rangle = 0$ and $\langle H_0, H_1\rangle > 0$. If $\widetilde{H} = (H_1+H_2)/|H_1+H_2|$, the matrices of ad_{H_0} and $\mathrm{ad}_{\widetilde{H}}$ with respect to the orthonormal basis $\{e_1, e_2, e_3\}$ are given by

$$\mathrm{ad}_{H_0} = \begin{bmatrix} \langle H_0, H_1\rangle & & 0\\ 0 & \langle H_0, H_2\rangle & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$\mathrm{ad}_{\widetilde{H}} = \begin{bmatrix} \langle \widetilde{H}, H_1\rangle & 0 & 0\\ 0 & \langle \widetilde{H}, H_2\rangle & 0\\ 0 & 0 & \langle \widetilde{H}, H_1 + H_2\rangle\end{bmatrix}.$$

Since $\mathfrak{g}'$ is nonabelian, $[e_1, e_2] = \varepsilon e_e$ ($[\mathfrak{g}'_{\lambda_1}, \mathfrak{g}'_{\lambda_2}] \subset \mathfrak{g}'_{\lambda_1+\lambda_2} = \mathfrak{z}$) and we may assume that $\varepsilon > 0$ (otherwise we change e_3 to $-e_3$). Set $e_4 = H_0$, $e_5 = \widetilde{H}$, $\alpha = \langle \widetilde{H}, H_1 \rangle$, $\beta = \langle \widetilde{H}, H_2 \rangle$ and $\gamma = \langle H_0, H_1 \rangle > 0$. Then, $\{e_1, e_2, e_3, e_4, e_5\}$ is an orthonormal basis of $\mathfrak{g}$ satisfying:

$$\begin{array}{lll} [e_1, e_2] = \varepsilon e_3, & [e_1, e_3] = 0 = [e_2, e_3], & \\ [e_4, e_1] = \gamma e_1, & [e_4, e_2] = -\gamma e_2, & [e_4, e_3] = 0 = [e_4, e_5], \\ [e_5, e_1] = \alpha e_1, & [e_5, e_2] = \beta e_2, & [e_5, e_3] = (\alpha + \beta) e_3 \end{array}$$

with $\varepsilon > 0$, $\gamma > 0$ and $\alpha + \beta > 0$. Moreover, $\alpha > 0$ and $\beta > 0$ since $K(e_1, e_3) = \frac{1}{4}\varepsilon^2 - \alpha(\alpha + \beta)$, $K(e_2, e_3) = \frac{1}{4}\varepsilon^2 - \beta(\alpha + \beta)$ (see §2, (3)) and the sectional curvature $K \leq 0$. This special case will be studied in detail in §2. As we will see, G is isomorphic to the Lie group of 3×3 upper triangular real matrices of determinant one, and it follows from Corollary 2.8 that G has rank one or two. In the latter case, provided that one multiplies the metric by a suitable positive constant, G is isometric to the irreducible symmetric space of noncompact type and rank two $\mathrm{SL}(3, \mathbf{R})/\mathrm{SO}(3)$ (see Remark 2.8).

By examining all the cases, Theorem 1.3 follows. Note that G satisfies visibility or not depending on whether $\dim \mathfrak{a} = 1$ or 2.

COROLLARY 1.4. *The simply connected homogeneous spaces H of nonpositive curvature, with no flat de Rham factor, with* $\dim(H) \leq 5$ *and* $\operatorname{rank}(H) = 2$ *are $H^2 \times T^2$, $H^2 \times T^3$ or H an irreducible symmetric space of noncompact type.*

Proof. It is immediate by Theorem 1.3 and Corollary 4.4 of **[6]**. H^2, T^2 and T^3 are as in the statement of Theorem 1.3.

2. Example. Let $\mathfrak{g}$ be the Lie algebra of dimension five generated by $\{e_i\}_{i=1}^5$ and Lie bracket given by

$$\begin{array}{lll} [e_1, e_2] = \varepsilon e_3, & [e_1, e_3] = 0 = [e_2, e_3], & \\ [e_4, e_1] = \gamma e_1, & [e_4, e_2] = -\gamma e_2, & [e_4, e_3] = 0 = [e_4, e_5], \\ [e_5, e_1] = \alpha e_1, & [e_5, e_2] = \beta e_2, & [e_5, e_3] = (\alpha + \beta) e_3 \end{array}$$

where α, β, γ, ε are positive real numbers. (Note that $\mathfrak{g}'$ is spanned by $\{e_1, e_2, e_3\}$.) We will say that such a $\mathfrak{g}$ is associated to $(\alpha, \beta, \gamma, \varepsilon)$.

Let $\langle\ ,\ \rangle$ be the inner product in $\mathfrak{g}$ with respect to which $\{e_i\}_{i=1}^5$ is an orthonormal basis of $\mathfrak{g}$, and let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$ and left invariant metric associated to $\langle\ ,\ \rangle$. By a straightforward computation, using the connection formula and the definitions of R, K we get:

(1)
$$\begin{array}{lll} \nabla_{e_1}e_1 = \gamma e_4 + \alpha e_5, & \nabla_{e_1}e_2 = \frac{1}{2}\varepsilon e_3, & \nabla_{e_1}e_3 = -\frac{1}{2}\varepsilon e_2, \\ \nabla_{e_2}e_2 = -\gamma e_4 + \beta e_5, & \nabla_{e_2}e_3 = \frac{1}{2}\varepsilon e_1, & \nabla_{e_3}e_3 = (\alpha+\beta)e_5, \\ \nabla_{e_1}e_4 = -\gamma e_1, & \nabla_{e_1}e_5 = -\alpha e_1, & \nabla_{e_2}e_4 = \gamma e_2, \\ \nabla_{e_2}e_5 = -\beta e_2, & \nabla_{e_3}e_4 = 0, & \nabla_{e_3}e_5 = -(\alpha+\beta)e_3. \end{array}$$

(2)
$$\begin{array}{ll} R(e_1, e_2)e_1 = \left(\frac{3}{4}\varepsilon^2 + \alpha\beta - \gamma^2\right)e_2, & R(e_1, e_2)e_2 = \left(\gamma^2 - \alpha\beta - \frac{3}{4}\varepsilon^2\right)e_1, \\ R(e_1, e_2)e_3 = -\frac{1}{2}\varepsilon(\alpha+\beta)e_5, & R(e_2, e_3)e_1 = \frac{1}{2}\varepsilon(\gamma e_4 + \alpha e_5), \\ R(e_2, e_3)e_2 = \left(-\frac{1}{4}\varepsilon^2 + \beta(\alpha+\beta)\right)e_3, & R(e_2, e_3)e_3 = \left(\frac{1}{4}\varepsilon^2 - \beta(\alpha+\beta)\right)e_2, \\ R(e_1, e_3)e_1 = \left(-\frac{1}{4}\varepsilon^2 + \alpha(\alpha+\beta)\right)e_3, & R(e_1, e_3)e_2 = \frac{1}{2}\varepsilon(\gamma e_4 - \beta e_5), \\ R(e_1 e_3)e_3 = \left(\frac{1}{4}\varepsilon^2 - \alpha(\alpha+\beta)\right)e_1. & \end{array}$$

(3)
$$\begin{array}{ll} K(e_1, e_2) = -\frac{3}{4}\varepsilon^2 + \gamma^2 - \alpha\beta, & K(e_1, e_3) = \frac{1}{4}\varepsilon^2 - \alpha(\alpha+\beta), \\ K(e_2, e_3) = \frac{1}{4}\varepsilon^2 - \beta(\alpha+\beta), & K(e_4, e_2) = K(e_4, e_1) = -\gamma^2, \\ K(e_4, e_3) = 0, & K(e_5, e_1) = -\alpha^2, \\ K(e_5, e_2) = -\beta^2, & K(e_5, e_3) = -(\alpha+\beta)^2. \end{array}$$

We note that in all computations above, α, β, γ and ε may be arbitrary.

(4) We remark that it will be shown in §3.1 that if $\alpha = \beta = \varepsilon/2 = \gamma/\sqrt{3}$ then G is a symmetric space.

Conversely, assuming G symmetric (i.e., $\nabla R = 0$) we get $\alpha = \beta = \varepsilon/2 = \gamma/\sqrt{3}$. This follows by a straightforward computation of $\nabla_{e_1}(R(e_1, e_2)e_1)$, $\nabla_{e_2}(R(e_1, e_2)e_3)$ and $\nabla_{e_1}(R(e_1, e_2)e_4)$ using $\nabla R = 0$ and (1) and (2) above.

The following lemma is proved in [7]. We state it here since it is applied in Lemma 2.2 to obtain an expression for the sectional curvature that will be used repeatedly.

LEMMA 2.1. *Let $\mathfrak{g}$ be a solvable Lie algebra with an inner product $\langle\ ,\ \rangle$ such that $\mathfrak{a}$, the orthogonal complement of $\mathfrak{g}'$ is abelian. If $\mathrm{ad}_H|_{\mathfrak{g}'}$ is symmetric with respect to $\langle\ ,\ \rangle$ for all $H \in \mathfrak{a}$, then*

$$\begin{aligned} \langle R(X+H, Y+T)(Y+T), X+H\rangle &= \langle R(X, Y)Y, X\rangle - |[H, Y] - [T, X]|^2 \\ &\quad - \langle [H, Y] - [T, X], [X, Y]\rangle + \langle [[H, Y] - [T, X], X], Y\rangle \\ &\quad - \langle [[H, Y] - [T, X], Y], X\rangle \end{aligned}$$

for all $X, Y \in \mathfrak{g}'$ and $H, T \in \mathfrak{a}$.

LEMMA 2.2. *Let* a, b, c, r, s, t *be real numbers and* H, T *elements in* $\mathfrak{a}$; *then*

$$\begin{aligned}
&\langle R(ae_1 + be_2 + ce_3 + H, re_1 + se_2 + te_3 + T)(re_1 + se_2 + te_3 + T), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ae_1 + be_2 + ce_3 + H\rangle \\
&\quad = [(cs - bt)^2 K(e_2, e_3) + \varepsilon(cs - bt)(a\lambda_1(T) - r\lambda_1(H)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - (a\lambda_1(T) - r\lambda_1(H))^2] \\
&\qquad + [(at - cr)^2 K(e_1, e_3) + \varepsilon(at - cr)(b\lambda_2(T) - s\lambda_2(H)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - (b\lambda_2(T) - s\lambda_2(H))^2] \\
&\qquad + [(as - br)^2 K(e_1, e_2) + \varepsilon(as - br)(c\lambda_3(T) - t\lambda_3(H)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - (c\lambda_3(T) - t\lambda_3(H))^2]
\end{aligned}$$

where λ_i $(i = 1, 2, 3)$ *are defined by*

$$\lambda_1(U) = \langle U, \gamma e_4 + \alpha e_5\rangle, \quad \lambda_2(U) = \langle U, -\gamma e_4 + \beta e_5\rangle \quad \textit{and}$$
$$\lambda_3(U) = (\lambda_1 + \lambda_2)(U) = (\alpha + \beta)\langle U, e_5\rangle \quad \textit{for all } U \in \mathfrak{a}.$$

Proof. First of all we show that,

$$\begin{aligned}
&\langle R(ae_1 + be_2 + ce_3, re_1 + se_2 + te_3)(re_1 + se_2 + te_3), ae_1 + be_2 + ce_3\rangle \\
&\qquad = (as - br)^2 K(e_1, e_2) + t^2(a^2 K(e_1, e_3) + b^2 K(e_2, e_3)) \\
&\qquad\quad + c^2(r^2 K(e_1, e_3) + s^2 K(e_2, e_3)) \\
&\qquad\quad - 2ct(arK(e_1, e_3) + bsK(e_2, e_3)).
\end{aligned}$$

Let $X = ae_1 + be_2$ and $Y = re_1 + se_2$. Applying the linearity of R and using that $R(X, Y)e_3$ is an element in $\mathfrak{a}$ (see (2)) we have,

$$\begin{aligned}
&\langle R(X + ce_3, Y + te_3)(Y + te_3), X + ce_3\rangle \\
&\qquad = \langle R(X, Y)Y, X\rangle + 2ct\langle R(X, e_3)Y, e_3\rangle + t^2\langle R(X, e_3)e_3, X\rangle \\
&\qquad\quad + c^2\langle R(e_3, Y)Y, e_3\rangle.
\end{aligned}$$

Now, since $R(e_1, e_3)e_3$ is a multiple of e_1 (see (2)), an easy calculation shows that

(ii) $\langle R(X, e_3)e_3, Y\rangle = arK(e_1, e_3) + bsK(e_2, e_3)$.

Hence, (i) is deduced from (ii) and the equality

$$\langle R(X, Y)Y, X\rangle = |X \wedge Y|^2 K(e_1, e_2) = (as - br)^2 K(e_1, e_2).$$

Now, the formula stated in the lemma follows by a straightforward computation using Lemma 2.1.

Next, in the two propositions below we find necessary and sufficient conditions for G to have nonpositive curvature in terms of $\alpha, \beta, \gamma, \varepsilon$.

PROPOSITION 2.3. *If G has sectional curvature $K \le 0$ then the following relations among α, β, γ, ε hold: $\varepsilon^2 \le 2\beta(\alpha+\beta)$, $\varepsilon^2 \le 2\alpha(\alpha+\beta)$, $\gamma^2 \le \frac{1}{2}\varepsilon^2 + \alpha\beta$. In particular, $K(e_2, e_3)$, $K(e_1, e_3)$ and $K(e_1, e_2)$ are all strictly negative.*

Proof. We first show that if $\varepsilon^2 - 2\beta(\alpha+\beta) > 0$ (or $\varepsilon^2 - 2\alpha(\alpha+\beta) > 0$) then there exists a plane π in $\mathfrak{g}$ with sectional curvature $K(\pi) > 0$. In fact, if we take $H = 0$, $T = \lambda e_4$ we have $\lambda_3 = \lambda_3(T) = 0$ and $\lambda_1 = \lambda_1(T) = -\gamma\lambda$ with $\lambda_1 \ne 0$ for any nonzero real λ. Hence, by applying the curvature formula given by Lemma 2.2, we get

$$\begin{aligned}\langle R(ae_1 + ce_3, e_2 + \lambda e_4)(e_2 + \lambda e_4), ae_1 + ce_3\rangle \\ = c^2 K(e_2, e_3) + a^2(K(e_1, e_2) - \lambda_1^2) - \varepsilon\lambda_1 ac,\end{aligned}$$

for any real numbers a, c. If we consider this expression as a polynomial of second degree in a $(K \le 0, \lambda_1 \ne 0)$ its discriminant Δ is given by

$$\Delta = c^2(\lambda_1^2(\varepsilon^2 + 4K(e_2, e_3)) - 4K(e_1, e_2)K(e_2, e_3)).$$

Note that $\varepsilon^2 + 4K(e_2, e_3) = 2(\varepsilon^2 - 2\beta(\alpha+\beta))$. Thus, by choosing λ so that

$$\lambda^2\gamma^2 = \lambda_1^2 > \frac{4K(e_2, e_3)K(e_1, e_2)}{2(\varepsilon^2 - 2\beta(\alpha+\beta))}$$

we get Δ strictly positive for any nonzero real c. For this λ and nonzero c, a real number a can be chosen satisfying

$$K(ae_1 + ce_3, e_2 + \lambda e_4) > 0.$$

The other statement follows in the same way by interchanging the roles of e_1 and e_2. Hence, the first two inequalities follow.

Now we prove the last one. In the same way as above, if we take $T = \lambda(-\beta e_4 + \gamma e_5)$ with $\lambda \ne 0$ (hence, $\lambda_2 = \lambda_2(T) = 0$ and $\lambda_3 = \lambda_3(T) = \lambda\gamma(\alpha+\beta) \ne 0$) and applying the curvature formula again, we have

$$\begin{aligned}\langle R(be_2 + ce_3, e_1 + T)(e_1 + T), be_2 + ce_3\rangle \\ = b^2 K(e_1, e_2) + c^2(K(e_1, e_3) - \lambda_3^2) - \varepsilon\lambda_3 bc,\end{aligned}$$

which considered as a polynomial (of second degree) in c has discriminant

$$\Delta = b^2(\lambda_3^2(\varepsilon^2 + 4K(e_1, e_2)) - 4K(e_1, e_2)K(e_1, e_3)).$$

Note firstly that $\varepsilon^2 + 4K(e_1, e_2) = 2(-\varepsilon^2 + 2(\gamma^2 - \alpha\beta))$. Thus, if we assume $2(\gamma^2 - \alpha\beta) - \varepsilon^2 > 0$ (or $\gamma^2 > \varepsilon^2/2 + \alpha\beta$), taking λ in such a way that

$$\lambda^2\gamma^2(\alpha + \beta)^2 = \lambda_3^2 > \frac{4K(e_1, e_2)K(e_1, e_3)}{\varepsilon^2 + 4K(e_1, e_2)},$$

for any nonzero real b we get $\Delta > 0$. Hence, a real c can be chosen such that $K(be_2+ce_3, e_1+T) > 0$. The assertion follows since $K \leq 0$.

PROPOSITION 2.4. *The conditions $\varepsilon^2 \leq 2\beta(\alpha+\beta)$, $\varepsilon^2 \leq 2\alpha(\alpha+\beta)$, $\gamma^2 \leq \frac{1}{2}\varepsilon^2 + \alpha\beta$ are sufficient for G to have sectional curvature $K \leq 0$.*

Proof. We note from the curvature formula given in Lemma 2.2 that each term in between brackets is a polynomial of second degree ($K(e_1, e_3)$, $K(e_2, e_3)$ and $K(e_1, e_2)$ are negative) in $(cs - bt)$, $(at - cr)$ and $(as - br)$ respectively, with discriminant

$$\begin{aligned}
&(a\lambda_1(T) - r\lambda_1(H))^2 \quad (\varepsilon^2 + 4K(e_2, e_3)),\\
&(b\lambda_2(T) - s\lambda_2(H))^2 \quad (\varepsilon^2 + 4K(e_1, e_3)), \quad \text{and}\\
&(c\lambda_3(T) - t\lambda_3(H))^2 \quad (\varepsilon^2 + 4K(e_1, e_2)).
\end{aligned}$$

Under our assumption, $\varepsilon^2 \leq 2\beta(\alpha+\beta)$, $\varepsilon^2 \leq 2\alpha(\alpha+\beta)$ and $\gamma^2 \leq \frac{1}{2}\varepsilon^2 + \alpha\beta$, these discriminants are nonpositive and therefore each polynomial is also nonpositive. Thus,

$$K(ae_1 + be_2 + ce_3 + H, re_1 + se_2 + te_3 + T) \leq 0$$

for any real a, b, c, r, s, t and $H, T \in \mathfrak{a}$. Hence, $K \leq 0$.

Next, under the assumption $K \leq 0$, we will get some conditions for G to have rank one.

PROPOSITION 2.5. *The real number ε must satisfy $\varepsilon \leq \alpha+\beta$. Moreover, G has rank one if $\varepsilon < \alpha + \beta$.*

Proof. The condition $\varepsilon \leq \alpha + \beta$ follows immediately from the first two inequalities of Proposition 2.3. We note that $(\alpha+\beta)^2 \leq 2\alpha(\alpha+\beta)$ or $(\alpha + \beta)^2 \leq 2\beta(\alpha + \beta)$ depending on whether $\beta \leq \alpha$ or $\alpha \leq \beta$ respectively. Consequently, $\varepsilon < \alpha + \beta$ if and only if $\varepsilon^2 < 2\alpha(\alpha + \beta)$ or $\varepsilon^2 < 2\beta(\alpha + \beta)$.

Next we check the last statement. Using Lemma 2.2, for each $Y \in \mathfrak{g}'$ orthogonal to e_3 and $T \in \mathfrak{a}$, we have

$$\begin{aligned}&\langle R(e_1 + e_2, Y + te_3 + T)(Y + te_3 + T), e_1 + e_2\rangle \\ &\quad = t^2(K(e_1, e_3) + K(e_2, e_3)) - t\varepsilon(\lambda_1 - \lambda_2) \\ &\qquad - \lambda_1^2 - \lambda_2^2 + |(e_1 + e_2) \wedge Y|^2 K(e_1, e_2).\end{aligned}$$

This expression is a polynomial $p(t)$ of degree two in t whose discriminant Δ is given by

$$\begin{aligned}\Delta &= \varepsilon^2(\lambda_1 - \lambda_2)^2 \\ &\quad + 4(K(e_1, e_3) + K(e_2, e_3))(\lambda_1^2 + \lambda_2^2 - |(e_1 + e_2) \wedge Y|^2 K(e_1, e_2)).\end{aligned}$$

Now, we assume $\varepsilon < \alpha + \beta$. Since $K \leq 0$ we have

$$\Delta \leq \varepsilon^2(\lambda_1 - \lambda_2)^2 + 4(K(e_1, e_3) + K(e_2, e_3))(\lambda_1^2 + \lambda_2^2).$$

If we substitute the expressions for $K(e_1, e_3)$ and $K(e_2, e_3)$ into the expression above, we get

$$\begin{aligned}\Delta &\leq \varepsilon^2(\lambda_1 - \lambda_2)^2 + 4\left(\tfrac{1}{2}\varepsilon^2 - (\alpha + \beta)^2\right)(\lambda_1^2 + \lambda_2^2) \\ &= \varepsilon^2(\lambda_1 - \lambda_2)^2 + 2\varepsilon^2(\lambda_1^2 + \lambda_2^2) - 4(\alpha + \beta)^2(\lambda_1^2 + \lambda_2^2) \\ &= \varepsilon^2(3\lambda_1^2 + 3\lambda_2^2 - 2\lambda_1\lambda_2) - 4(\alpha + \beta)^2(\lambda_1^2 + \lambda_2^2).\end{aligned}$$

Now, we consider the two cases, $T \neq 0$ and $T = 0$. If $T \neq 0$, since $\lambda_1(T)$ and $\lambda_2(T)$ are not simultaneously zero, $3\lambda_1^2 + 3\lambda_2^2 - 2\lambda_1\lambda_2 > (\lambda_1 - \lambda_2)^2 \geq 0$. Hence, if $\varepsilon < \alpha + \beta$, we get

$$\begin{aligned}\Delta &< (\alpha + \beta)^2(3\lambda_1^2 + 3\lambda_2^2 - 2\lambda_1\lambda_2) - 4(\alpha + \beta)^2(\lambda_1^2 + \lambda_2^2) \\ &= -(\alpha + \beta)^2(\lambda_1 + \lambda_2)^2 \leq 0,\end{aligned}$$

and then $p(t) < 0$ for all real t, $T \neq 0$ in $\mathfrak{a}$ and Y in $\mathfrak{g}'$ orthogonal to e_3.

If $T = 0$, $p(t) = t^2(K(e_1, e_3) + K(e_2, e_3)) + |(e_1 + e_2) \wedge Y|^2 K(e_1, e_2) < 0$ whenever $t \neq 0$ or Y, orthogonal to e_3, is independent of $e_1 + e_2$. (Note that $K(e_1 + e_2, Y) = K(e_1, e_2) < 0$.)

Therefore, $K(e_1 + e_2, Y + te_3 + T) < 0$ for all real number t, $T \in \mathfrak{a}$, $Y \in \mathfrak{g}'$ orthogonal to e_3 and independent of $e_1 + e_2$. Thus, the geodesic γ in G satisfying $\gamma(0) = e$ and $\gamma'(0) = e_1 + e_2$ has rank one.

PROPOSITION 2.6. *The numbers α, β, γ satisfy the inequalities $\gamma^2 - 2\alpha\beta - \beta^2 \leq 0$ and $\gamma^2 - 2\alpha\beta - \alpha^2 \leq 0$. Moreover, if $\gamma^2 - 2\alpha\beta - \beta^2 < 0$ or $\gamma^2 - 2\alpha\beta - \alpha^2 < 0$, G has rank one.*

Proof. The first two inequalities follow immediately from Proposition 2.3 $(\gamma^2 - \alpha\beta \leq \varepsilon^2/2)$.

Now, we will show the last assertion. Applying Lemma 2.2, for each $T \in \mathfrak{a}$ and Y in $\mathfrak{g}'$ orthogonal to e_2, we have

$$\begin{aligned}\langle R(e_1+e_3\,,\,Y+se_2+T)(Y+se_2+T)\,,\,e_1+e_3\rangle \\ = s^2(K(e_1\,,\,e_2)+K(e_2\,,\,e_3))+s\varepsilon(\lambda_1+\lambda_3)-\lambda_1^2-\lambda_3^2 \\ +|(e_1+e_3)\wedge Y|^2K(e_1\,,\,e_3)=p(s)\,,\end{aligned}$$

where $\lambda_i=\lambda_i(T)$ $(i=1\,,\,3)$ are defined as in Lemma 2.2.

Note that $p(s)$ is a polynomial of degree two in s whose discriminant Δ is given by

$$\begin{aligned}\Delta = \varepsilon^2(\lambda_1+\lambda_3)^2+4(K(e_1\,,\,e_2)+K(e_2\,,\,e_3)) \\ \times(\lambda_1^2+\lambda_3^2-|(e_1+e_3)\wedge Y|^2K(e_1\,,\,e_3)).\end{aligned}$$

Substituting $K(e_1\,,\,e_2)$ and $K(e_2\,,\,e_3)$ for its expressions, and since $K\leq 0$ we get,

$$\begin{aligned}\Delta &\leq \varepsilon^2(\lambda_1+\lambda_3)^2+4\left(-\tfrac{1}{2}\varepsilon^2+\gamma^2-2\alpha\beta-\beta^2\right)(\lambda_1^2+\lambda_3^2) \\ &= \varepsilon^2(\lambda_1+\lambda_3)^2-2\varepsilon^2(\lambda_1^2+\lambda_3^2)+4(\gamma^2-2\alpha\beta-\beta^2)(\lambda_1^2+\lambda_3^2) \\ &= -\varepsilon^2(\lambda_1-\lambda_3)^2+4(\gamma^2-2\alpha\beta-\beta^2)(\lambda_1^2+\lambda_3^2).\end{aligned}$$

To prove that G has rank one we will see that if $\gamma^2-2\alpha\beta-\beta^2<0$ then $K(e_1+e_3\,,\,Y+se_2+T)<0$ for all $s\,,\,T$ in $\mathfrak{a}$, $Y\in\mathfrak{g}'$ orthogonal to e_2 and independent of e_1+e_3. We first consider the case $T\neq 0$; since $\lambda_1(T)\neq 0$ we have $\Delta<-\varepsilon^2(\lambda_1-\lambda_2)^2\leq 0$ and hence, the polynomial p satisfies $p(s)<0$ for all $s\,,\,T\neq 0$ in $\mathfrak{a}$ and $Y\in\mathfrak{g}'$ orthogonal to e_3. If $T=0$,

$$p(s)=s^2(K(e_1\,,\,e_2)+K(e_2\,,\,e_3))+|(e_1+e_3)\wedge Y|^2K(e_1\,,\,e_3)<0$$

whenever $s\neq 0$ or $Y\in\mathfrak{g}'$, orthogonal to e_3, is independent of e_1+e_3 $(K(e_1\,,\,e_3)<0$ and $K(e_2\,,\,e_3)<0)$. Therefore, the assertion is proved and consequently, the geodesic γ in G such that $\gamma(0)=e$ and $\gamma'(0)=e_1+e_3$ has rank one.

If $\gamma^2-2\alpha\beta-\alpha^2<0$, interchanging the roles of e_1 and e_2, we also obtain that G has rank one.

We summarize the preceding results in the following:

THEOREM 2.7. *Let G be the simply connected Lie group with Lie algebra associated to $(\alpha\,,\,\beta\,,\,\gamma\,,\,\varepsilon)$ and left invariant metric as defined above. Then G has sectional curvature $K\leq 0$ if and only if*

$$\varepsilon^2\leq 2\alpha(\alpha+\beta)\,,\quad \varepsilon^2\leq 2\beta(\alpha+\beta)\quad \textit{and}\quad \gamma^2\leq\frac{\varepsilon^2}{2}+\alpha\beta.$$

Moreover, G has rank one if any of the following conditions hold:

$$\varepsilon < \alpha + \beta\,, \quad \gamma^2 - 2\alpha\beta - \alpha^2 < 0\,, \quad \gamma^2 - 2\alpha\beta - \beta^2 < 0.$$

COROLLARY 2.8. *If G has nonpositive curvature, then G has rank one or two and in the latter case, $\alpha = \beta = \varepsilon/2 = \gamma/\sqrt{3}$.*

Proof. We note first that the roots of $\mathfrak{a}$ in $\mathfrak{g}'$ are given by $\lambda_1(H) = \langle H\,, \gamma e_4 + \alpha e_5\rangle$, $\lambda_2(H) = \langle H\,, -\gamma e_4 + \beta e_5\rangle$, $\lambda_3 = \lambda_1 + \lambda_2$ for all $H \in \mathfrak{a}$, where λ_1 and λ_2 are independent with associated root spaces $\mathfrak{g}'_{\lambda_i} = \mathbf{R}e_i$ $(i = 1\,, 2\,, 3)$. Thus, $\mathfrak{g}'_0 = 0 = \mathfrak{a}_0$ and hence G has no de Rham flat factor. Then, it follows from Theorem 1.3 of [7] that G has rank one or two. If $\operatorname{rank}(G) = 2$, Theorem 2.7 implies that $\varepsilon = \alpha + \beta$ and $\gamma^2 - 2\alpha\beta - \beta^2 = 0 = \gamma^2 - 2\alpha\beta - \alpha^2$. Hence, $\alpha = \beta = \varepsilon/2 = \gamma/\sqrt{3}$.

REMARK 2.8. It will be shown in §3 (3.1) that when $\alpha = \beta = \varepsilon/2 = \gamma/\sqrt{3}$, G coincides with the symmetric space $\mathrm{SL}(3\,, \mathbf{R})/\mathrm{SO}(3)$, provided we multiply the metric by a suitable positive constant.

3. The group of 3×3 upper triangular real matrices of determinant one.

3.1. Let G be the solvable simply connected Lie group of 3×3-upper triangular real matrices of determinant one. Its Lie algebra $\mathfrak{g}$ consists of the 3×3-upper triangular real matrices having trace zero and has a basis $\{E_i\}_{i=1}^5$ given by

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_5 = \tfrac{1}{3}(E_5^1 + E_5^2), \quad \text{where}$$

$$E_5^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{and} \quad E_5^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Let $\alpha\,, \beta\,, \gamma\,, \varepsilon$ be any positive real numbers. Setting $e_1 = 2\alpha E_1$, $e_2 = 2\beta E_2$, $e_3 = (4\alpha\beta/\varepsilon)E_3$, $e_4 = \gamma/3E_4$ and $e_5 = \frac{1}{3}(\beta E_5^1 + \alpha E_5^2)$, we obtain a basis $\{e_i\}_{i=1}^5$ of $\mathfrak{g}$ satisfying:

$$\begin{array}{lll} [e_1, e_2] = \varepsilon e_3, & [e_1, e_3] = 0 = [e_2, e_3], & \\ [e_4, e_1] = \gamma e_1, & [e_4, e_2] = -\gamma e_2, & [e_4, e_3] = 0 = [e_4, e_5], \\ [e_5, e_1] = \alpha e_1, & [e_5, e_2] = \beta e_2, & [e_5, e_3] = (\alpha + \beta) e_3. \end{array}$$

That is, $\mathfrak{g}$ is isomorphic, as a Lie algebra of matrices, to the Lie algebra associated to $(\alpha, \beta, \gamma, \varepsilon)$ which was studied in §2. I learned of this realization from **[9]**. Thus, considering on $\mathfrak{g}$ the inner product $\langle\ ,\ \rangle$ such that $\{e_i\}_{i=1}^5$ is an orthonormal basis of $\mathfrak{g}$, we see that any choice of $(\alpha, \beta, \gamma, \varepsilon)$ gives us a left invariant metric on G. Moreover, almost all these metrics are not isometric. Note, since $\mathfrak{g}'$ is nonabelian, it is deduced from the proof of Theorem 1.3 that any left invariant metric on G of $K \leq 0$ is, up to an isometry, the metric associated to some $(\alpha, \beta, \gamma, \varepsilon)$.

In the case $\alpha = \beta = \varepsilon/2$ and $\gamma = (\sqrt{3}/2)\varepsilon$, provided that we multiply the metric by a suitable positive constant, G is isometric to the irreducible symmetric space of noncompact type and rank two $H = \mathrm{SL}(3, \mathbf{R})/\mathrm{SO}(3)$. In fact, $G = NA$ where $N = \exp \mathfrak{n}$, $\mathfrak{n}$ is the Lie algebra of 3×3-strictly upper triangular real matrices and A is the group of diagonal real matrices of determinant one. Since $\mathrm{SL}(3, \mathbf{R}) = \mathrm{SO}(3)NA$ is an Iwasawa decomposition for $\mathrm{SL}(3, \mathbf{R})$, it is well known (see **[1**, Lemma 2.4] and **[10]**) that G acts simply transitively on H. Now, if $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{so}(3)$ in $\mathfrak{sl}(3, \mathbf{R})$ with respect to the Killing form B on $\mathfrak{sl}(3, \mathbf{R})$ $(B(X, Y) = 6\,\mathrm{tr}(X, Y))$, $\mathfrak{p}$ may be identified with the tangent space to H at $o = \mathrm{ISO}(3)$, and the metric on $T_0 H$ corresponds to the restriction of the Killing form to $\mathfrak{p}$. If θ is the Cartan involution in $\mathfrak{sl}(3, \mathbf{R})$ relative to $\mathfrak{so}(3)$ $(\theta(X) = -X^t)$ then the inner product in $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a}$, where $\mathfrak{a}$ is the Lie algebra of A, obtained from the metric on $\mathfrak{p}$ is given by

$$(X+H, Y+T) = -\tfrac{1}{2}B(X, \theta Y) + B(H, T) \quad \text{for } X, Y \in \mathfrak{n},\ H, T \in \mathfrak{a}.$$

It is a straightforward computation to see that the metric given by $\alpha = \beta$, $\varepsilon = 2\alpha$ and $\gamma^2 = 3\alpha^2$ (that is, $\langle E_i, E_j \rangle = 0, i \neq j$, $|E_1|^2 = |E_2|^2 = |E_3|^2 = 1/4\alpha^2$, $|E_4|^2 = 3/\alpha^2$, $|E_5|^2 = 1/\alpha^2$) is a multiple of the metric $(\ ,\ )$. Moreover, $(\ ,\ ) = 12\alpha^2 \langle\ ,\ \rangle$.

3.2. Next we will obtain a comparison result between the symmetric metric on G and nonsymmetric metrics. The idea is to compare the curvature associated to the 4-tuples $(\alpha, \beta, \gamma, \varepsilon)$ and $(\alpha_0, \alpha_0, \sqrt{3}\alpha_0, 2\alpha_0)$ where the last one corresponds to the symmetric case.

Let $\alpha, \beta, \gamma, \varepsilon$ be positive real numbers and let $\{E_i\}_{i=1}^5$ and $\{e_i\}_{i=1}^5$ be as in (3.1). We consider the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ such that $\{e_i\}_{i=1}^5$ is an orthonormal basis of $\mathfrak{g}$. Then we have:

$$\langle E_i, E_j\rangle = 0, \qquad i \neq j,$$
$$|E_1|^2 = \frac{1}{4\alpha^2}, \quad |E_2|^2 = \frac{1}{4\beta^2}, \quad |E_3|^2 = \frac{\varepsilon^2}{16\alpha^2\beta^2},$$
$$|E_4|^2 = \frac{9}{\gamma^2}, \qquad |\beta E_5^1 + \alpha E_5^2|^2 = 9.$$

In order to compare the metrics associated to different $(\alpha, \beta, \gamma, \varepsilon)$ it is convenient to multiply the metric $\langle\ \rangle$ by the factor $4\alpha^2\beta^2/\varepsilon^2$. Then the orthonormal basis with respect to the new metric, that we also denote by $\{e_i\}$ and $\langle\ ,\ \rangle$ is given by

$$e_1 = \frac{\varepsilon}{\beta}E_1, \qquad e_2 = \frac{\varepsilon}{\alpha}E_2, \quad e_3 = 2E_3,$$
$$e_4 = \frac{\varepsilon\gamma}{6\alpha\beta}E_4, \qquad e_5 = \frac{\varepsilon}{6\alpha\beta}(\alpha E_5^2 + \beta E_5^1).$$

Now, observe that the metric on $\mathfrak{z} = \mathbf{R}E_3$, the center of $\mathfrak{g}'$, does not depend on $(\alpha, \beta, \gamma, \varepsilon)$; that is, if $Z_1, Z_2 \in \mathfrak{z}$ then $\langle Z_1, Z_2\rangle = \langle Z_1, Z_2\rangle_0$ where $\langle\ ,\ \rangle_0$ is the metric associated to $(\alpha_0, \beta_0, \gamma_0, \varepsilon_0)$. Therefore, since $[\mathfrak{g}', \mathfrak{g}'] = [\mathfrak{z}^\perp, \mathfrak{z}^\perp] \subset \mathfrak{z}$ ($\mathfrak{z}^\perp$ is the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{g}'$), for $X, Y \in \mathfrak{g}'$ and $H, T \in \mathfrak{a}$, the curvature formula given in Lemma 2.1 tells us that the last three terms of its expression do not depend on $(\alpha, \beta, \gamma, \varepsilon)$.

Let $X = aE_1 + bE_2$ and $Y = Y' + dE_3$ with $Y' \in \mathfrak{z}^\perp$. Then, from (i) in the proof of Lemma 2.2, we get

$$\langle R(X, Y)Y, X\rangle = |X \wedge Y'|^2 K(e_1, e_2) + \frac{d^2}{4}\left(a^2\frac{\beta^2}{\varepsilon^2}K(e_1, e_3) + b^2\frac{\alpha^2}{\varepsilon^2}K(e_2, e_3)\right).$$

Substituting for $K(e_1, e_2)$, $K(e_1, e_3)$ and $K(e_2, e_3)$ and taking into account that the metric was multiplied by $4\alpha^2\beta^2/\varepsilon^2$, we get

$$\langle R(X, Y)Y, X\rangle = \frac{\Delta^2}{4\varepsilon^2}\left(-\tfrac{3}{4}\varepsilon^2 + \gamma^2 - \alpha\beta\right) + \frac{d^2}{16}\left[\frac{a^2}{\alpha^2}\left(\tfrac{1}{4}\varepsilon^2 - \alpha(\alpha+\beta)\right) + \frac{b^2}{\beta^2}\left(\tfrac{1}{4}\varepsilon^2 - \beta(\alpha+\beta)\right)\right]$$

where Δ^2, defined by the expression $|X \wedge Y'|^2 = (\alpha^2\beta^2/\varepsilon^4)\Delta^2$ does not depend on $(\alpha, \beta, \gamma, \varepsilon)$.

If we write $[H, Y] - [T, X] = rE_1 + sE_2 + tE_3$, we have

$$|[H, Y] - [T, X]|^2 = r^2\frac{\beta^2}{\varepsilon^2} + s^2\frac{\alpha^2}{\varepsilon^2} + \frac{t^2}{4}.$$

Therefore, if R_0 denotes the curvature tensor associated to the metric $\langle\ ,\ \rangle_0$, we get

$$(*)\qquad \begin{aligned} &\langle R(X+H, Y+T)(Y+T), X+H\rangle \\ &\quad - \langle R_0(X+H, Y+T)(Y+T), X+H\rangle_0 \\ &= \langle R(X, Y)Y, X\rangle - \langle R_0(X, Y)Y, X\rangle_0 \\ &\quad - |[H, Y] - [T, X]|^2 + |[H, Y] - [T, X]|_0^2 \\ &= \frac{\Delta^2}{4}\left(\frac{\gamma^2 - \alpha\beta}{\varepsilon^2} - \frac{\gamma_0^2 - \alpha_0\beta_0}{\varepsilon_0^2}\right) \\ &\quad + \frac{d^2}{16}\left[\frac{a^2}{4}\left(\frac{\varepsilon^2}{\alpha^2} - \frac{\varepsilon_0^2}{\alpha_0^2}\right) + \frac{b^2}{4}\left(\frac{\varepsilon^2}{\beta^2} - \frac{\varepsilon_0^2}{\beta_0^2}\right)\right. \\ &\qquad\qquad \left. - a^2\left(\frac{\beta}{\alpha} - \frac{\beta_0}{\alpha_0}\right) - b^2\left(\frac{\alpha}{\beta} - \frac{\alpha_0}{\beta_0}\right)\right] \\ &\quad + r^2\left(\frac{\beta_0^2}{\varepsilon_0^2} - \frac{\beta^2}{\varepsilon^2}\right) + s^2\left(\frac{\alpha_0^2}{\varepsilon_0^2} - \frac{\alpha^2}{\varepsilon^2}\right). \end{aligned}$$

Now, if we choose $\alpha_0 = \beta_0$, $\varepsilon_0 = 2\alpha_0$ and $\gamma_0^2 = 3\alpha_0^2$, the right hand side of $(*)$ becomes

$$\begin{aligned} &= \frac{\Delta^2}{4}\left(\frac{\gamma^2 - \alpha\beta}{\varepsilon^2} - \frac{1}{2}\right) + \frac{d^2}{16}\left(\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2}\right)\left(\frac{\varepsilon^2}{4} - \alpha\beta\right) \\ &\quad + r^2\left(\frac{1}{4} - \frac{\beta^2}{\varepsilon^2}\right) + s^2\left(\frac{1}{4} - \frac{\alpha^2}{\varepsilon^2}\right). \end{aligned}$$

Hence, if $(\alpha, \beta, \gamma, \varepsilon)$ satisfies the conditions $\varepsilon \leq 2\alpha$, $\varepsilon \leq 2\beta$ and $\gamma^2 \leq \varepsilon^2/2 + \alpha\beta$, it follows that

$$\begin{aligned} &\langle R(X+H, Y+T)(Y+T), X+H\rangle \\ &\quad - \langle R_0(X+H, Y+T)(Y+T), X+H\rangle_0 \leq 0. \end{aligned}$$

If $R(\pi) = \langle R(X+H, Y+T)(Y+T), X+H\rangle$, where π is the plane spanned by $\{X+H, Y+T\}$, we get $K \leq 0$ and the stronger condition $R(\pi) \leq R_0(\pi)$ for every plane $\pi \subset \mathfrak{g}$.

Conversely, if $R(\pi) \leq R_0(\pi)$ for all plane $\pi \subset \mathfrak{g}$, considering the planes spanned by $\{X, Y\}$ $(X, Y \in \mathfrak{z}^{\perp})$, $\{e_1 + T, e_1 + H\}$ and

$\{e_2 + T, e_2 + H\}$ (T and H such that $\lambda_1(T) \neq \lambda_1(H)$) respectively, we get in each case $\gamma^2 \leq \varepsilon^2/2 + \alpha\beta$, $\varepsilon \leq 2\beta$, $\varepsilon \leq 2\alpha$. Thus, we have the following:

PROPOSITION 3.2. *Let G be the simply connected Lie group of* 3×3-*upper triangular real matrices of determinant one with the left invariant metric associated to $(\alpha, \beta, \gamma, \varepsilon)$. Then, $R(\pi) \leq R_0(\pi)$ for all plane $\pi \subset \mathfrak{g}$ if and only if $\gamma^2 \leq \varepsilon^2/2 + \alpha\beta$, $\varepsilon \leq 2\alpha$ and $\varepsilon \leq 2\beta$. Moreover, $R(\pi) = R_0(\pi)$ for all plane $\pi \subset \mathfrak{g}$ if and only if $\gamma^2 = \varepsilon^2/2 + \alpha\beta$, $\varepsilon = 2\alpha$ and $\varepsilon = 2\beta$ (that is, G is symmetric).*

In particular, G is not symmetric if $R(\pi) < R_0(\pi)$ for some plane $\pi \subset \mathfrak{g}$.

3.3. It follows from Theorem 2.7 that:

(i) G admits many different metrics of nonpositive curvature of rank one and only one metric, up to multiplication by a positive constant, of rank two. So the rank in a homogeneous space is not invariant under the change of homogeneous metrics of nonpositive curvature. This situation does not occur for Hadamard manifolds which are compact or have finite volume (see **[4]**).

(ii) G with the left invariant metrics of rank one, gives us examples of homogeneous spaces of rank one having two-flats. In fact, $A = \exp(\mathfrak{a})$ is a flat totally geodesic submanifold isometrically imbedded in G of dimension two.

Acknowledgment. I would like to thank the referee for very detailed suggestions which helped to improve the exposition of this paper, shortening the proof of Theorem 2.1 and strengthening the statement of Theorem 2.7.

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Received April 11, 1989 and in revised form January 31, 1990. Supported in part by CONICOR.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 148, No. 1, 1991

COMBINATORIAL TECHNIQUES AND ABSTRACT WITT RINGS III

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We introduce an equivalence relation on maximal elements (i.e., $D\langle 1, -x\rangle$ is maximal). We present a form theoretic proof of Marshall's classification of reduced Witt rings, thus providing a possible outline for proving the full elementary type conjecture. The same relation restricted to elements of index two yields characterizations of Witt rings with a factor either of local type or a group ring extension of a totally degenerate Witt ring.

(R, G, q) will denote a finitely generated (abstract) Witt ring R, its associated group of one-dimensional forms G and the associated quaternionic mapping q. As in [7, **8**] we use the abstract Witt ring as defined by Marshall-Yucas [**13**] rather than Marshall's modification in [**12**]. The technique introduced here is the formation of equivalence classes of maximal elements ($x \in G$ with $D\langle 1, -x\rangle$ maximal). While forming classes is not combinatorial, it does blend well with the techniques of the previous two papers in this series.

We start by discussing a four step approach to the elementary type conjecture, two of which are statements about classes of maximal elements. The four steps can be verified when R is reduced, thus giving a new, form-theoretic, proof of Marshall's classification theorem [**11**]. Each step is valid for Witt rings of elementary type (as opposed to the main steps in Marshall's proof or in the proofs for $|G| \leq 32$). And, when restricted to maximal elements with $[G : D\langle 1, -x\rangle] = 2$, the proposed approach leads to new results clarifying the structure of such Witt rings. The first section concludes with a verification (with some details omitted) of the four steps when R is reduced. Of interest here is the identification, when R is reduced, of the quotient structure defined in [**8**] with a Pfister quotient as defined by Marshall in [**12**].

For non-reduced R, maximal elements and their classes are difficult to handle. In the second section we consider only elements of index two ($x \in G$ with $i_G D\langle 1, -x\rangle = 2$). The restricted equivalence classes behave well and occur in two types. Using classes of type 1, we slightly improve the characterization of local type factors in **[6]**. Working with classes of type 2 yields an analogous result for factors which are group rings over totally degenerate Witt rings (called S-rings here).

The last section takes up two extreme cases. First we consider the case where there are two classes of elements of index 2, one of each type, which generate $B \equiv q(G, G)$. We show R is then a product of two Witt rings, one of local type and the other an S-ring. Then, since many of the previous results involve conditions of the form $Q(x) \cap Q(y) = 1$, we consider the case where some $Q(x)$ is contained in all $Q(y)$. Under quite general conditions (satisfied if x has index 2, for example) we show R is of local type.

The notation is the same as in **[7, 8]**. Thus for any group H, $H^{\cdot}$ denotes $H - \{1\}$. For $a \in G$, $Q(a) = \{q(a, x) | x \in G\}$ and $Y_R = \{Q(a) | a \in G^{\cdot}\}$. B denotes the image of q, $q(G, G)$. The value set of $\langle 1, -x\rangle$ is $D\langle 1, -x\rangle = \{y \in G | q(x, y) = 1\}$. The radical of G is $\operatorname{rad}(G) = \{x \in G | D\langle 1, -x\rangle = G\}$. We say R is degenerate if $\operatorname{rad} G \neq \{1\}$ and totally degenerate if $\operatorname{rad} G = G$. We will assume throughout that R is non-degenerate.

R is of local type if $|B| = 2$. We let Δ_n denote the group of exponent two and order 2^n. The group ring $R[\Delta_n]$ is again a Witt ring. The direct product in the category of Witt rings is the fiber product over $\mathbb{Z}_2$, which we will denote by the usual product symbol. Thus:

$$R_1 \times R_2 = \{(r_1, r_2) | r_i \in R_i \text{ and } \dim r_1 \equiv \dim r_2 (2)\}.$$

R is of elementary type if it can be built from $\mathbb{Z}_2$, $\mathbb{Z}_4$ and Witt rings of local type by a succession of group ring extensions and products. We will often use orthogonal decompositions as defined in **[3]**. Subgroups $H_1, \dots, H_n$ of G yield an orthogonal decomposition (denoted $H_1 \perp \cdots \perp H_n$) if $G = H_1 \times \cdots \times H_n$ and $x_i \in D\langle 1, -x_j\rangle$ for all $X_i \in H_i$, $x_j \in H_j$, $i \neq j$.

1. Reduced Witt rings. The two notions which form the basis for all three sections are:

DEFINITION. An element $m \in G^{\cdot}$ is *maximal* if $D\langle 1, -m\rangle \subset D\langle 1, -x\rangle$ implies $x = 1$ or $D\langle 1, -m\rangle = D\langle 1, -x\rangle$. The collection of maximal elements of G will be denoted by M.

DEFINITION. For $a, b \in M$ write $a \sim b$ if $a = b$ or $ab \in M$. We say a and b are *equivalent*, and write $a \approx b$, if there exist $c_1, \dots, c_k \in M$ such that: $a \sim c_1 \sim c_2 \sim \cdots \sim c_k \sim b$.

Equivalence is clearly an equivalence relation. Denote by $C(a)$ the equivalence class of $a \in M$. Let $H(a)$ be the subgroup of G generated by $C(a)$.

Recall the quotient structure of [**8**]. For $g \in G^{\cdot}$ let $Q(g) = \{q(g, h) | h \in G\}$ and $H(Q(g)) = \{h \in G | Q(h) \subset Q(g)\}$. Set $G/g = G/H(Q(g))$ and define:

$$q_g : G/g \times G/g \to B/Q(g),$$
$$(aH, bH) \mapsto q(a, b)Q(g).$$

If q_g is linked the resulting Witt ring is denoted R/g. We consider a possible outline for proving the elementary type conjecture:

(1.1) (a) G is generated by M.

(b) If $a \not\approx b$, where $a, b \in M$, and if $x \in H(a)$, $y \in H(b)$ then $x \in D\langle 1, -y\rangle$.

(c) q_a is linked for all $a \in M$.

(d) If $G = H(a)$ for some $a \in M$ and if R/b is of elementary type for all $b \in M$ then R is of local type or a group ring.

Proving these four steps would prove the elementary type conjecture. The first two steps show there is an orthogonal decomposition (cf. [**3**]) $G = H(a_1) \perp \cdots \perp H(a_t)$, where $C(a_1), \ldots, C(a_t)$ are the distinct equivalence classes. Each $H(a)$ generates a Witt ring, if each of these Witt rings is of elementary type then so is R [**3**, 3.8]. We may thus assume $G = H(a)$ for some $a \in M$. Steps (c) and (d) constitute an induction argument on $|G|$ which completes the proof.

There is some evidence for the truth of the elementary type conjecture. It holds if R is reduced (proven for abstract Witt rings by Marshall [**11**], simplified in [**12**]; cf. [**2**], [**4**], [**9**] for the field case) and if $|G| \leq 32$ (proven by a variety of unrelated counting arguments). There is also some evidence that (1.1) will yield a proof of the elementary type conjecture. First, the four steps of (1.1) can be proven if R is reduced, thus given a new proof of Marshall's result. Second, each of the four statements of (1.1) are true for Witt rings of elementary type. This may appear to be an insignificant advantage. However, none of the intermediate results in Marshall's proof of the reduced case are valid for non-reduced Witt rings. Only reduced Witt rings are determined by their space of orderings. Also, very few of the counting arguments used for $|G| \leq 32$ yield information about larger Witt rings. Third, (1.1) can be followed partially for maximal elements with $[G : D\langle 1, -x\rangle] = 2$ yielding significant improvements over previous results (see §2, 3). Unfortunately, we have been unable to prove any new cases of the elementary type conjecture via (1.1).

The remainder of this section is devoted to sketching the proofs of (1.1)(a)–(d) if R is reduced. Thus for this section (R, G, q) will denote a finitely generated, reduced Witt ring. Then $D\langle 1, 1\rangle = \{1\}$ and as a result, if $a \in D\langle 1, b\rangle$ then $D\langle 1, a\rangle \subset D\langle 1, b\rangle$.

LEMMA 1.2. *Let a, $b \in G$.*

(i) *If $D\langle 1, a\rangle = D\langle 1, b\rangle$ then $a = b$.*

(ii) *a is maximal iff a is rigid.*

(iii) *For any $g \in G^{\cdot}$, g is a product of elements of $D\langle 1, g\rangle \cap M$. In particular, G is generated by M.*

Proof. (i) $D\langle 1, a\rangle = D\langle 1, b\rangle \subset D\langle 1, -ab\rangle$. Thus a, b, $-ab$, and hence -1, lie in $D\langle 1, -ab\rangle$. R reduced implies $ab = 1$.

(ii) $\{x \in G | D\langle 1, -a\rangle \subset D\langle 1, -x\rangle\} = \{x \in G | x \in D\langle 1, a\rangle\}$ since R is reduced. Then a is maximal iff this set is $\{1, a\}$ iff a is rigid.

(iii) If $|D\langle 1, g\rangle| = 2$ then $g \in M$ by (ii). If $|D\langle 1, g\rangle| > 2$ write $D\langle 1, q\rangle = \{1, g, x_3, \ldots, x_t\}$. Then $g = x_3 \cdots x_t$. $D\langle 1, x_i\rangle \subsetneq D\langle 1, g\rangle$ by (i), so by induction each x_i is a product of elements in $M \cap D\langle 1, x_i\rangle \subset M \cap D\langle 1, g\rangle$. $\square$

LEMMA 1.3. *Let a, $b \in M$. Then either:*

(i) *$D\langle 1, -ab\rangle = D\langle 1, -a\rangle \cap D\langle 1, -b\rangle$,*

or

(ii) *$a \approx b$.*

In particular, if $a \not\approx b$, $x \in H(a)$ and $y \in H(b)$ then $x \in D\langle 1, -y\rangle$.

Proof. Suppose first that $1 \in D\langle a, b\rangle$. Then a, $b \in D\langle 1, ab\rangle$ and $-ab \in D\langle 1, -a\rangle \cap D\langle 1, -b\rangle$. We obtain (i), since R is reduced. Next suppose that $1 \notin D\langle a, b\rangle$. Then $D\langle a, b\rangle \subset M$ by **[1**, I 1.2] and (1.2)(ii). We may choose $c \in M \cap D\langle 1, ab\rangle$ by (1.2)(iii). Hence $c \in M$ and ac, $bc \in D\langle a, b\rangle \subset M$. So $a \sim c \sim b$. $\square$

The following is of some interest independent of (1.1). We show that when R is reduced, the quotient defined in **[8]** is the same as the Pfister quotient defined by Marshall in **[12]**.

LEMMA 1.4. *Let $a \in M$. Then:*

(i) *$H(Q(a)) = \{1, a\} = D\langle 1, a\rangle$,*

(ii) *R/a is well defined, and*

(iii) *R/a is reduced.*

Proof. (i) Let $h \in H(Q(a))$. Then $q(-1, h) \in Q(h) \subset Q(a)$. By linkage (on G) there exists $z \in G$ with:

$$q(-1, h) = q(-1, z) = q(a, z).$$

Thus $h \in zD\langle 1, 1\rangle = \{z\}$ and $z \in D\langle 1, a\rangle$. So $H(Q(a)) \subset D\langle 1, a\rangle = \{1, a\}$. We have equality since clearly $1, a \in H(Q(a))$.

(ii) Let I be the fundamental ideal of R. Since R is reduced we may assume [**12**, 3.23] that $q : G \times G \to I^2/I^3$ is given by $q(x, y) = \langle\langle -x, -y\rangle\rangle + I^3$. Thus $Q(a) = \langle 1, -a\rangle I + I^3$ and by (i):

$$q_a : G/D\langle 1, a\rangle \times G/D\langle 1, a\rangle \to I^2/\langle 1, -a\rangle I + I^3,$$
$$q_a(\overline{x}, \overline{y}) = \langle\langle -x, y\rangle\rangle + (\langle 1, -a\rangle I + I^3).$$

There is a well-defined Pfister quotient $R/\operatorname{ann}\langle 1, a\rangle$ [**12**, 4.24] which is reduced [**12**, 6.10]. Note that

$$\operatorname{ann}\langle 1, a\rangle = (\{\langle 1, -x\rangle | x \in D\langle 1, a\rangle\}) = (\langle 1, -a\rangle).$$

Set $J = I/(\langle 1, -a\rangle)$. The quaternionic map for $R/\operatorname{ann}\langle 1, a\rangle$ is:

$$q^* : G/D\langle 1, a\rangle \times G/D\langle 1, a\rangle \to J^2/J^3,$$
$$q^*(\overline{x}, \overline{y}) = \langle\langle -x, -y\rangle\rangle + (\langle 1, -a\rangle) + J^3.$$

Map $\alpha : I^2 \to J^2/J^3$ by $\alpha(\varphi) = \varphi + (\langle 1, -a\rangle) + J^3$. This is clearly a surjective homomorphism with $\langle 1, -a\rangle I + I^3 \subset \ker\alpha$. If $\varphi \in \ker\alpha$ then $\varphi - \eta \in (\langle 1, -a\rangle)$, for some $\eta \in I^3$. Thus $\varphi - \eta = \langle 1, -a\rangle\chi$ for some form χ, and indeed $\chi \in I$ as $\varphi - \eta \in I^2$. Hence $\varphi \in \langle 1, -a\rangle I + I^3$.

Thus α is an isomorphism and the linkage of q^* implies q_a is linked.

(iii) Let $\overline{y} = yD\langle 1, a\rangle$ and suppose $q_a(-1, \overline{y}) = 1$. Then $q(-1, y) \in Q(a)$ and as in the proof of (i) we obtain $y \in D\langle 1, a\rangle$. Hence $\overline{y} = \overline{1}$ in $G/D\langle 1, a\rangle$ and so R/a is reduced. □

The proof of (1.1)(d) is long and tedious. We present one part of the proof both to give the flavor of the whole and because a weaker version of this result holds generally for Gorenstein Witt rings (see [**5**]).

PROPOSITION 1.5. *Let $a \in G^{\cdot}$ be maximal. Suppose $G = H(a)$ and R/a is a group ring. Then R is a group ring.*

Proof. We may write $\overline{G} = \overline{G}_0 \times \{1, \overline{t}\}$, where $\{1, a\} \subset G_0 \subset G = G_0 \cdot \{1, t\}$, and $\overline{t}$ is two-sided rigid in R/a. Write $G_0 = \{1, a\} \cdot H_0$ where $a \notin H_0$ and $-1 \in H_0$.

We will assume R is not a group ring and first show that $D\langle 1, -s\rangle$ has index 2 in G. Let $h_0 \in H_0$. Then $\overline{h_0 t}$, $\overline{-ht} \in M^*$, the maximal elements of $\overline{G}$. The value set in $\overline{G}$, $D\langle 1, \overline{x}\rangle$, is

$$D\langle 1, -x\rangle D\langle 1, -ax\rangle/\{1, a\}.$$

If $\overline{x} \in M^*$ then either:

(i) $x, ax \in M$
or
(ii) $x \in M$ and $D\langle 1, ax\rangle = \{1, a, x, ax\}$
or
(iii) $ax \in M$ and $D\langle 1, s\rangle = \{1, a, x, ax\}$.

Now if $\pm h_0 t \in M$ or $\pm a h_0 t \in M$ then R has two-sided rigid elements (1.2) and is thus a group ring. Otherwise, one of two cases occurs:

(i) $h_0 t \in M$, $D\langle 1, a h_0 t\rangle = \{1, h_0 t, a, a h_0 t\}$, $-a h_0 t \in M$ and $D\langle 1, -h_0 t\rangle = \{1, -h_0 t, a, -a h_0 t\}$ or

(ii) $a h_0 t \in M$, $D\langle 1, h_0 t\rangle = \{1, h_0 t, a, a h_0 t\}$, $-h_0 t \in M$ and $D\langle 1, -a h_0 t\rangle = \{1, -h_0 t, a, -a h_0 t\}$.

We see then that for all $h_0 \in H_0$ either $h_0 t$ or $-h_0 t$ lies in $D\langle 1, -a\rangle$. In particular, taking $h_0 = 1$, we have t or $-t$ lies in $D\langle 1, -a\rangle$. Thus

$$|D\langle 1, -a\rangle \cap H_0| = \tfrac{1}{2}|H_0|.$$

If $t \in D\langle 1, -a\rangle$ then $G = \{1, -a, t, -at\}H_0$ and $i_G D\langle 1, -a\rangle = 2$. Similarly, $i_G D\langle 1, -a\rangle = 2$ if $-t \in D\langle 1, -a\rangle$.

We now obtain the desired contradiction by showing that $G = H(a)$ implies $D\langle 1, -a\rangle$ does not have index 2 in G. Note that R/a being a group ring implies $|M^*| \geq 2$ and so $|M| \geq 2$. Since $G = H(a)$, there exists $m \in M - \{a\}$ with $am \in M$. Now $-1 \notin D\langle 1, -a\rangle$ since R is reduced, so either m or $-m$ lies in $D\langle 1, -a\rangle$ (since $i_G D\langle 1, -a\rangle = 2$). But $-m \in D\langle 1, a\rangle$ implies $D\langle 1, -m\rangle \subset D\langle 1, -a\rangle$ and $m \notin M$. And $m \in D\langle 1, -a\rangle$ implies $D\langle 1, -am\rangle \subset D\langle 1, -a\rangle$ and $am \notin M$. Thus we have contradicted the initial assumption that R is not a group ring. □

2. Elements of index 2. We now drop the assumption that R is reduced. (R, G, q) will denote a finitely generated non-degenerate Witt ring. Let $i(x)$ denote the index of $D\langle 1, -x\rangle$ in G (this is a slightly different use of $i(x)$ than in [**6**]). Maximal elements in an arbitrary Witt ring are difficult to work with. If, however, we restrict our attention to those maximal elements with $i(x) = 2$ then the equivalence relation of §1 is a useful tool.

Set $T = \{x \in G | i(x) = 2\}$ and take the same relation of §1 on T, namely, for $x, y \in T$ write $x \sim y$ if $x = y$ or $xy \in T$. Thus for $x, y \in T$, $x \sim y$ iff $i(xy) \leq 2$. In what follows we will frequently use Marshall's result [**12**, 5.2]:

$$|D\langle 1, -xy\rangle / D\langle 1, -x\rangle \cap D\langle 1, -y\rangle| = |Q(x) \cap Q(y)|.$$

LEMMA 2.1. *Let* $x, y \in T$ *and suppose* $x \not\sim y$. *Then*:

(1) $i(xy) = 4$,
(2) $D\langle 1, -xy\rangle = D\langle 1, -x\rangle \cap D\langle 1, -y\rangle$,
(3) $Q(xy) = Q(x)Q(y)$.

Proof. By definition, $i(xy) \geq 4$ while $D\langle 1, -x\rangle \cap D\langle 1, -y\rangle \subset D\langle 1, -xy\rangle$ and $i_G(D\langle 1, -x\rangle \cap D\langle 1, -y\rangle) \leq 4$. This proves (1) and (2). Further, $2 = |D\langle 1, -y\rangle / D\langle 1, -x\rangle \cap D\langle 1, -xy\rangle| = |Q(x) \cap Q(xy)|$. Thus $Q(x) \subset Q(xy)$, as $|Q(x)| = 2$. Similarly, $Q(y) \subset Q(xy)$. Then $Q(x)Q(y) \subset Q(xy) \subset Q(x)Q(y)$ which gives (3). □

LEMMA 2.2. *Let* $x, y \in T$ *and suppose* $x \sim y$. *Then either* $D\langle 1, -x\rangle = D\langle 1, -x\rangle = D\langle 1, -y\rangle$ *or* $Q(x) = Q(y)$. *Further, if both occur then* $x = y$.

Proof. Suppose $D\langle 1, -x\rangle \neq D\langle 1, -y\rangle$. Then

$$|D\langle 1, -xy\rangle / D\langle 1, -x\rangle \cap D\langle 1, -y\rangle| \geq 2.$$

Hence $|Q(x) \cap Q(y)| = 2$ and $Q(x) = Q(y)$. If $D\langle 1, -x\rangle = D\langle 1, -y\rangle$ and $Q(x) = Q(y)$ then

$$2 = |Q(x) \cap Q(y)| = |D\langle 1, -xy\rangle / D\langle 1, -x\rangle|$$

shows $i(xy) = 1$ and $x = y$. □

THEOREM 2.3. $\sim$ *is an equivalence relation on* T.

Proof. We need only check transitivity. Suppose $x, y, z \in T$ with $x \sim y$ and $y \sim z$. We may assume $x \neq y$, $x \neq z$ and $y \neq z$, so that $i(xy) = i(yz) = 2$. We show $i(xz) = 2$.

Suppose not. Then $x \not\sim z$ and $xy \not\sim yz$. By (2.1), $D\langle 1, -xz\rangle$ is contained in $D\langle 1, -x\rangle$, $C\langle 1, -z\rangle$, $D\langle 1, -xy\rangle$, $D\langle 1, -yz\rangle$ and hence $D\langle 1, -y\rangle$. Now $D\langle 1, -x\rangle \neq D\langle 1, -z\rangle$ since otherwise $D\langle 1, -x\rangle = D\langle 1, -xz\rangle$ and $i(xz) = 2$. There can only be three distinct subgroups of index 2 containing $D\langle 1, -xz\rangle$, as $i(xz) = 4$. We must have $D\langle 1, -x\rangle = D\langle 1, -y\rangle$ or $D\langle 1, -y\rangle = D\langle 1, -z\rangle$.

We will assume $D\langle 1, -x\rangle = D\langle 1, -y\rangle = D\langle 1, -xy\rangle$, the other case being similar.

We thus have $D\langle 1, -y\rangle = D\langle 1, -x\rangle \neq D\langle 1, -z\rangle$ and so $Q(y) = Q(z)$ by (2.2). We *claim* $i(xyz) = 4$. Otherwise, $D\langle 1, -xz\rangle \subset D\langle 1, -y\rangle$ implies $D\langle 1, -xz\rangle$ is contained in $D\langle 1, -y\rangle$, $D\langle 1, -z\rangle$, $D\langle 1, -yz\rangle$ and $D\langle 1, -xyz\rangle$, all of index 2. Again there are only three distinct subgroups of index 2 containing $D\langle 1, -xz\rangle$. So $D\langle 1, -xyz\rangle$ equals one of $D\langle 1, -y\rangle$, $D\langle 1, -z\rangle$ or $D\langle 1, -yz\rangle$, which we know are distinct. But $D\langle 1, -y\rangle = D\langle 1, -xyz\rangle$ implies $D\langle 1, -y\rangle = D\langle 1, -xz\rangle$ and $i(xz) = 2$, $D\langle 1, -z\rangle = D\langle 1, -xyz\rangle$ implies $D\langle 1, -z\rangle = D\langle 1, -xy\rangle = D\langle 1, -y\rangle$. And $D\langle 1, -yz\rangle = D\langle 1, xyz\rangle$ implies $D\langle 1, -yz\rangle = D\langle 1, -x\rangle = D\langle 1, -y\rangle = D\langle 1, -z\rangle$. All three possibilities are impossible which proves the *claim*.

We thus have $i(xyz) = 4$ and $D\langle 1, -xz\rangle \subset D\langle 1, -y\rangle$. So $D\langle 1, -xz\rangle = D\langle 1, -xyz\rangle \subset D\langle 1, -y\rangle$, $D\langle 1, -z\rangle$. Hence:

$$1 = \left|\frac{D\langle 1, -xz\rangle}{D\langle 1, -xyz\rangle \cap D\langle 1, -y\rangle}\right| = |Q(xyz) \cap Q(y)|,$$

$$2 = \left|\frac{D\langle 1, -xy\rangle}{D\langle 1, -xyz\rangle \cap D\langle 1, -z\rangle}\right| = |Q(xyz) \cap Q(z)|,$$

which is impossible as $Q(y) = Q(z)$.

NOTATION. For $a \in T$ let $C^{\cdot}(a)$ denote the class of a in T under the relation $\sim$. Let $C(a) = C^{\cdot}(a) \cup \{1\}$.

LEMMA 2.4. *For each* $a \in T$, $C(a)$ *is a subgroup of* G.

Proof. Let x, $y \in C(a)$. If x or y equals 1 then $xy \in C(a)$, so suppose x, $y \in C^{\cdot}(a)$. Then $x \sim y$ and so $i(xy) \leq 2$. If $i(xy) = 1$ then $xy = 1 \in C(a)$. If $i(xy) = 2$ then $xy \in T$, and $xy \sim x \sim a$. Hence $xy \in C(a)$. □

PROPOSITION 2.5. *Let* $a \in T$. *Then either:*

(1) $Q(a) = Q(x)$ *for all* $x \in C^{\cdot}(a)$, *or*

(2) $D\langle 1, -a\rangle = D\langle 1, -x\rangle$ *for all* $x \in C^{\cdot}(a)$.

Proof. Set $C_1(a) = \{x \in C^{\cdot}(a) | Q(x) = Q(a)\} \cup \{1\}$ and $C_2(a) = \{x \in C^{\cdot}(a) | D\langle 1, -x\rangle = D\langle 1, -a\rangle\} \cup \{1\}$. We first claim that $C_1(a)$ is a subgroup of $C(a)$. If x, $y \in C_1(a)$ and $x \neq 1$, $y \neq 1$ then $Q(x) = Q(a) = Q(y)$. So $Q(xy) \subset Q(x)Q(y) = Q(a)$. Hence either

$xy = 1$ or $Q(xy) = Q(a)$. In either case, $xy \in C_1(a)$. Next we claim $C_2(a)$ is a subgroup of $C(a)$. If $x, y \in C_2(a)$ with $x \neq 1$, $y \neq 1$ then $D\langle 1, -x\rangle = D\langle 1, -y\rangle \subset D\langle 1, -xy\rangle$. So either $xy = 1$ or $D\langle 1, -xy\rangle = D\langle 1, -a\rangle$ and so $xy \in C_2(a)$.

Now $C(a) = C_1(a) \cup C_2(a)$ by (2.2). Hence either $C(a) = C_1(a)$, yielding (1), or $C(a) = C_2(a)$, yielding (2). □

Definition. Let $a \in G$ have index 2 (i.e. $a \in T$). We say a has *type* 1 if $Q(a) = Q(x)$ for all $x \in C^{\cdot}(a)$. We say a has *type* 2 if $D\langle 1, -a\rangle = D\langle 1, -x\rangle$ for all $x \in C^{\cdot}(a)$ and $|C(a)| \geq 4$.

Every $a \in T$ thus has type 1 or type 2 (but not both, by the restriction that $|C(a)| \geq 4$ for type 2). We observe that if a has type 2 then $C(a) \subset D\langle 1, -a\rangle$ (namely, if $m \in C^{\cdot}(a)$, $m \neq a$ then $D\langle 1, -m\rangle = D\langle 1, -a\rangle = D\langle 1, -am\rangle$. In particular, $-1 \in D\langle 1, -a\rangle$. Hence $C(a) \subset D\langle 1, -a\rangle$).

Elements of index 2 having type 1 have appeared in the literature before. We reformulate two such results in this language.

Propostion 2.6 (*Marshall* **[12]**). *Suppose G is generated by elements of index* 2. *Then R is a fiber product of Witt rings of local type.*

Proof. We have $G = C(a_1) \cdot \cdots \cdot C(a_k)$, where the $C^{\cdot}(a_i)$ are the distinct classes in T. If $x \in C(a_i)$, $y \in C(a_j)$ with $i \neq j$ then $x \in D\langle 1, -y\rangle$ by (2.1). In particular, no $C(a_i)$ has type 2, else all the $C(a_j)$ are contained in $D\langle 1, -a_i\rangle$ and so $G \subset D\langle 1, -a_i\rangle$. We thus have $G = C(a_1) \times \cdots \times C(a_k)$ and $Q(C(a_i)) \cap Q(C(a_j)) = \{1\}$ if $i \neq j$. Thus R is a fiber product with the ith factor generated by $C(a_i)$. Since $|Q(C(a_i))| = 2$ (a has type 1) each factor is of local type. □

Proposition 2.7 (*Fitzgerald-Yucas* **[6]**). *Let $a \in T$ have type* 1. *Set $H = C(a)$ and $K = \bigcap_{h \in H} D\langle 1, -h\rangle$.*

(1) *If $H \cap K = \{1\}$ then $G = H \perp K$ is an orthogonal decomposition.*

(2) *If, further, $Q(a) \not\subset Q(K)$ then $R = R_1 \times R_2$ is a fiber product with R_1 of local type.*

Proof. We refer to **[6]**. $C^{\cdot}(a) = -M$ and $C(a) = M^2$. The conclusion of (1) is Proposition 2.12(1)–(4) which depends only on

Proposition 2.11 which in turn depends only on the assumption that $H \cap K = \{1\}$. Thus (1) holds; statement (2) is Theorem 1.1. □

We note that, in (2.7), if K generates a Witt ring of elementary type (as in an inductive argument) then condition (1) is sufficient to show R is a fiber product with one factor of local type [**3**, 3.8].

We turn now to elements of index 2 having type 2. Among Witt rings of elementary type these arise from fiber products $R_1 \times R_2$ where $R_1 = S[\Delta]$, S a degenerate Witt ring with radical D_S satisfying $|D_S| \geq 4$ and $|\Delta| = 2$. Here any $a = (g, 1) \in D_S \times 1$ has type 2 and $C(a) = D_S \times 1$. One difficulty is that here the class does not generate a factor of R. In the simplest case where S is totally degenerate (i.e. $D_S = G_S$) then the element $(t, 1)$ (where $\Delta = (1, t)$) is required along with $C(a)$ to generate R_1. Note that $Q(t, 1) = Q(C(a))$.

DEFINITION. A Witt ring R is an *S-ring* if R is a group ring extension $S[\Delta]$ where S is a totally degenerate Witt ring, $|G_S| \geq 4$ and $|\Delta| = 2$.

DEFINITION. Let $a \in T$ have type 2. An element $t \in G$ is a *cap* for a if $Q(t) = Q(C(a))$.

We will concentrate on the easiest case of type 2 elements. We seek conditions on an $a \in T$ having type 2 analogous to (2.7) which will yield an S-ring factor.

In what follows we will often use the observation that $Q(a) \subset Q(b)$ iff $G = D\langle 1, -a\rangle D\langle 1, -ab\rangle$.

LEMMA 2.8. *Let* $a \in T$ *have type* 2.

(1) $Q(C(a)) = \bigcup_{m \in C(a)} Q(m)$.

(2) *For any* $g \in G$ *and* $m, m' \in C^{\cdot}(a)$ *we have* $Q(m) \subset Q(g)$ *iff* $Q(m') \subset Q(mm'g)$.

Proof. (1) We check that the union is a group. Let $p_1 \in Q(m_1)$ and $p_2 \in Q(m_2)$ where $m_1, m_2 \in C(a)$. If either $p_1 = 1$ or $p_2 = 1$ then $p_1 p_2 \in Q(m_1) \cup Q(m_2)$. Suppose $p_1 \neq 1$, $p_2 \neq 1$. Then $p_1 = q(m_1, y)$ for some $y \notin D\langle 1, -m_1\rangle = D\langle 1, -m_2\rangle$, since a has type 2. Thus $p_2 = q(m_2, y)$ and $p_1 p_2 = q(m_1 m_2, y) \in Q(m_1 m_2)$ with $m_1 m_2 \in C(a)$.

(2) $Q(m) \subset Q(g)$ iff $G = D\langle 1, -m\rangle D\langle 1, -mg\rangle$ iff $G = D\langle 1, -m'\rangle D\langle 1, -mg\rangle$ (as $D\langle 1, -m\rangle = D\langle 1, -m'\rangle$) if $Q(m') \subset Q(mm'g)$. □

Proposition 2.9. *Let $a \in T$ have type 2 and set $P = Q(C(a))$. For any $g \in G$ either:*

(1) *$Q(g) \cap P = \{1\}$, or*
(2) *$Q(g) \cap P = Q(m)$, for some $m \in C^{\cdot}(a)$, or*
(3) *$P \subset Q(g)$.*

Proof. Suppose $Q(g) \cap P \neq \{1\}$ or P. Then there exist m_1 and $m_2 \in C^{\cdot}(a)$ with $Q(m_1) \subset Q(g)$ and $Q(m_2) \not\subset Q(g)$ by (2.8). Set $H_1 = H(Q(g))$ and $H_2 = H(Q(m_1 m_2 g))$. We wish to show $H_1 \cap C(a) = \{1, m_1\}$. Now $m_2 \in H_2$ and $m_1 \notin H_2$ by (2.8). Let $m_3 \in H_1 \cap C^{\cdot}(a)$ so that $Q(m_3) \subset Q(g)$. Applying (2.8) with $m = m_3$ and $m' = m_1 m_2 m_3$ yields $Q(m_1 m_2 m_3) \subset Q(m_1 m_2 g)$. Thus $m_1 m_2 m_3$, and so $m_1 m_3$, lies in H_2. This shows $m_1(H_1 \cap C^{\cdot}(a)) \subset H_2$.

If $|H_1 \cap C^{\cdot}(a)| \neq 1$ then there exist distinct x, $y \in H_1 \cap C^{\cdot}(a)$. So $m_1 x$, $m_1 y$, $m_1 xy \in H_2$ and hence $m_1 \in H_2$, a contradiction. Thus $H_1 \cap C^{\cdot}(a) = \{m_1\}$ as desired. □

We can now re-derive a result of Kula **[10]**. We use the counting formula of [7]:

$$(*) \qquad \sum_{x \neq 1, z} \frac{1}{|Q(x) \cap Q(z)|} \cdot \frac{1}{|Q(xz)|} = \frac{-2}{|Q(z)|} + \sum_{y \in D\langle 1, -z\rangle} \frac{1}{|Q(y)|}.$$

Corollary 2.10 (*Kula*). *Let $a \in T$ have type 2 and set $P = Q(C(a))$. If $Q(G) = P$ then R is an S-ring.*

Proof. Choose $b \notin C(a)$ and apply $(*)$ with $z = b$. We split the left-hand sum into sums over $C^{\cdot}(a)$, $bC^{\cdot}(a)$ and $G \backslash \{1, b\} C(a)$. For any $x \notin C(a)$ we have $Q(x) = P$ by (2.9). Set $g = |G|$ and $c = |C(a)| = |P|$. We obtain:

$$\text{LHS} = \frac{c-1}{2c} + \frac{c-1}{2c} + \frac{g-2c}{c^2}.$$

We split the right-hand sum into sums over $\{1\}$, $C^{\cdot}(a) \cap D\langle 1, -b\rangle$ and $D\langle 1, -b\rangle \backslash C(a)$. We obtain:

$$\text{RHS} = \frac{-2}{c} + 1 + \frac{d-1}{2} + \frac{(g/c) - d}{c},$$

where $d = |D\langle 1, -b\rangle \cap C(a)|$. Equating the two sides gives:

$$-1/c = (d-1)/2 - d/c,$$
$$c - 2 = d(c-2)$$

and so $d = 1$. Thus for all $b \notin C(a)$, $D\langle 1, -b\rangle \cap C(a) = \{1\}$. In particular, $C(a) \subset D\langle 1, -a\rangle \subset C(a)$ and so $D\langle 1, -a\rangle = C(a)$.

Fix $b \notin C(a)$. Since $i_G C(a) = i(a) = 2$ and $D\langle 1, -b\rangle \cap C(a) = \{1\}$ we get $D\langle 1, -b\rangle = \{1, -b\}$. Further, $-1 \in D\langle 1, -a\rangle$ (cf. the remarks after (2.5)) and so $D\langle 1, b\rangle = \{1, b\}$ also. Thus b is 2-sided rigid and $E = S[\Delta]$ is a group ring extension. We have $|\Delta| = 2$ and $D\langle 1, -a\rangle = G_S$ since $i(a) = 2$. Moreover we have shown that if $x \in G_S^{\cdot}$ then $x \in D\langle 1, -a\rangle = C(a)$ and so $D\langle 1, -x\rangle = D\langle 1, -a\rangle = G_s$. Thus S is totally degenerate. Finally, by definition of type 2, $|G_S| = |C(a)| \geq 4$. So R is an S-ring. □

We refine (2.9):

Corollary 2.11. *Let $a \in T$ have type 2 and set $P = Q(C(a))$. Let $g \in G$.*

(1) *If $Q(g) \cap P = Q(m)$ for some $m \in C^{\cdot}(a)$ then $Q(mg) \cap P = \{1\}$.*

(2) *If $P \subset Q(g)$ then $Q(g) = Q(mg)$ for all $m \in C(a)$.*

Proof. (1) $Q(m) \subset Q(g)$ implies $Q(mg) \subset Q(g)$. Hence if $Q(mg) \cap P \neq \{1\}$ then $Q(mg) \cap P = Q(m)$ also. Suppose this occurs and choose $n \in C^{\cdot}(a)\backslash\{m\}$. Applying (2.8) with $m = m$ and $m' = n$ to $Q(m) \subset Q(g)$ gives $Q(n) \subset Q(mng)$. Next, using $M = m$ and $m' = mn$ for $Q(m) \subset Q(mg)$ gives $Q(mn) \subset Q(mng)$. Hence $Q(m) \subset Q(mn)Q(n) \subset Q(mng)$. Apply (2.8) to this inclusion with $m = m$ and $m' = n$ to obtain $Q(n) \subset Q(g)$, which is impossible. Thus $Q(mg) \cap P = \{1\}$.

(2) Fix $m_0 \in C^{\cdot}(a)$. Then $Q(mm_0) \subset Q(g)$ for all $m \in C(a)$ and so $Q(m) \subset Q(m_0 g)$ by (2.8). Thus $P \subset Q(m_0 g)$. From $Q(m_0) \subset Q(m_0 g) \subset Q(g)$ we obtain $Q(m_0 g) = Q(g)$. □

We may do better assuming there is a cap for a.

Proposition 2.12. *Let $a \in T$ have type 2 and let t be a cap for a (i.e. $Q(t) = Q(C(a))$). Set $P = Q(C(a))$, $L = \{x \in G | Q(x) \cap P = 1\}$ and $K = D\langle 1, -a\rangle \cap D\langle 1, -t\rangle$. Let $g \in G$. Then:*

(1) *$L \subset K$.*

(2) *If $Q(g) \cap P = Q(m)$ for some $m \in C^{\cdot}(a)$ then $mg \in L$.*

(3) *If $P \subset Q(g)$ then either:*

(i) *There exists a unique $m \in C(a)$ with $mtg \in L$, or*

(ii) *$Q(mg) = Q(m'gt)$ for all m, $m' \in C(a)$.*

Proof. (1) If $Q(x) \cap Q(t) = 1$ then $D\langle 1, -xt\rangle = D\langle 1, -x\rangle \cap D\langle 1, -t\rangle$ and so $x \in D\langle 1, -t\rangle$. Also $Q(x) \cap Q(a) = 1$ so that $x \in D\langle 1, -a\rangle$. Thus $L \subset K$.

(2) is (2.11)(1). For (3) we first show there is at most one $m \in C(a)$ with $mgt \in L$. Suppose not, that is, $Q(m_1gt) \cap P = 1 = Q(m_2gt) \cap P$. then for all m, $Q(m) \cap Q(m_igt) = 1$ $(i = 1, 2)$ and so $D\langle 1, -m_igt\rangle = D\langle 1, -m\rangle \cap D\langle 1, -mm_igt\rangle$. Taking $m = m_1m_2$ and $i = 1, 2$ shows $\langle 1, -m_1gt\rangle = D\langle 1, -m_2gt\rangle$. From $Q(m_igt) \cap Q(t) = 1$ we obtain $D\langle 1, -m_1\rangle = D\langle 1, -t\rangle \cap D\langle 1, -m_1gt\rangle = D\langle 1, -t\rangle \cap D\langle 1, -m_2gt\rangle = D\langle 1, -m_2g\rangle$. Further, $Q(m_1g) = Q(m_2g)$ by (2.11) and hence $m_1 = m_2$.

Suppose now that (i) does not occur. Then $P \subset Q(mgt)$ for all $m \in C(a)$ by (2.11)(1). So $Q(mgt) = Q(m'g)$ for all m, $m' \in C(a)$ by (2.11)(2). □

There are no examples of Case 3(ii) of (2.12) occurring among Witt rings of elementary type. The possibility that it might occur is the major obstacle to showing every a of type 2 with a cap arises from a fiber product where one factor is an S-ring.

We do however have a result analogous to (2.7).

PROPOSITION 2.13. *Let $a \in T$ have type 2 with a cap t. Set $H = \{1, t\}C(a)$ and $K = D\langle 1, -a\rangle \cap D\langle 1, -t\rangle$.*

(1) *If $t \notin D\langle 1, -a\rangle$ then $G = H \perp K$ is an orthogonal decomposition.*

(2) *If further $Q(t) \cap Q(K) = 1$ then $R = R_1 \times R_2$ where R_1 is an S-ring.*

Proof. (1) Let $g \in G$. Either g or $gt \in D\langle 1, -a\rangle$ as $i(a) = 2$. Further, $C(a) \cap D\langle 1, -t\rangle = 1$ and $|C(a)||D\langle 1, -t\rangle| = |G|$ by (2.8). Thus $G = \bigcup_{m \in C(a)} mD\langle 1, -t\rangle$. There exists then an $m \in C(a)$ such that mg or mgt is an $D\langle 1, -a\rangle \cap D\langle 1, -t\rangle = K$ and so $g \in HK$.

If $g \in H \cap K$ then $K \subset D\langle 1, -g\rangle$ since $g \in H$. For all $m \in C(a)$, $g \in D\langle 1, -m\rangle = D\langle 1, -a\rangle$ and $g \in D\langle 1, -t\rangle$, since $g \in K$. Hence $H \subset D\langle 1, -g\rangle$. Then $G = HK \subset D\langle 1, -g\rangle$ and $g = 1$. Thus $H \cap K = 1$ and $G = H \perp K$ is an orthogonal product.

(2) follows from (1) by [**3**, 3.4]. □

We again note that if K in (2.13) generates a Witt ring of elementary type then condition (1) yields $R = R_1 \times R_2$ with R_1 an S-ring by [**3**, 3.8].

We may combine (2.7) and (2.13) with a change in hypotheses.

THEOREM 2.14. *Let $a \in G$ have index 2. Set $P = Q(C(a))$. Suppose:*

(1) *There exists $x \in G$ with $Q(x) = P$, and*

(2) *For no y is $Q(my) = Q(m'xy)$ for all m, $m' \in C(a)$.*

Then $R = R_1 \times R_2$ with R_1 of local type if a has type 1 and R_1 an S-ring if a has type 2.

Proof. Set $H = \{1, x\}C(a)$, $L = \{y \in G | Q(y) \cap P = 1\}$ and $K = \bigcap_{h \in H} D\langle 1, -h\rangle$. We first note that $L \subset K$. If a has type 2 then this is (2.12). If a has type 1 and $y \in L$ then $Q(m) \cap Q(y) = 1$ for all $m \in C(a)$, as $Q(m) = Q(a) = P$. Thus $D\langle 1, -my\rangle = D\langle 1, -my\rangle \cap D\langle 1, -y\rangle$ and $y \in D\langle 1, -m\rangle$ for all $m \in C(a)$. Then $y \in K$ as $Q(x) = Q(a)$ implies $x \in C(a)$ and $H = C(a)$.

We next show $G = HL$. If a has type 2 then this is (2.12) combined with assumption (2) which eliminates Case 3(ii). Suppose a has type 1. If $g \in G$, $g \notin L$ then $Q(a) \subset Q(g)$ since $|Q(a)| = 2$. Assume, by way of contradiction, that $g \notin HL$. Then $Q(a) \subset Q(mg)$ for all $m \in C(a)$. So $Q(m) \subset Q(mg) \subset Q(m)Q(g) = Q(g)$, which implies $Q(mg) = Q(g)$ for all $m \in C(a)$. Again noting that the x of assumption (1) lies in $C(a)$, we see that assumption (2) is contradicted. So $G = HL$.

We thus have $G = HK$ as well and (by [**6**, 1.2]) that $H \cap K = 1$. Then $G = H \times K$, $L \subset K$ and $G = HL$ imply that $L = K$. Thus $G = H \perp K$ and $P \cap Q(k) = 1$ for all $k \in K$. We obtain that $R = R_1 \times R_2$ with R_1 generated by H. If a has type 1 then $|Q(H)| = 2$ shows R_1 is of local type, while if a has type 2 then (2.10) shows R_1 is an S-ring. □

3. Extreme cases. As before, T is the set of $x \in G$ with $i_G D\langle 1, -x\rangle = 2$. We consider the simplest of cases where T has both elements with type 1 and type 2.

THEOREM 3.1. *Suppose $Q(G)$ is generated by the $Q(x)$ with $x \in T$. Suppose further that $T = C^{\cdot}(a) \cup C^{\cdot}(b)$, where a has type 2 and b has type 1. Then $R = R_1 \times R_2$ where R_1 is an S-ring and R_2 is of local type.*

Proof. Set $P = Q(C(a))$ and $B = Q(G)$.

Step 1. For all $g \in G^{\cdot}$, $Q(g) = Q(b)$, $Q(m)$, $Q(m)Q(b)$, P or B, for some $m \in C^{\cdot}(a)$.

There are three possibilities for $Q(g) \cap P$ by (2.9). First suppose $Q(g) \cap P = 1$. Then $[B : P] = 2$ implies $|Q(g)| = 2$. Thus $g \in T = C^\cdot(a) \cup C^\cdot(b)$ and certainly $g \notin C^\cdot(a)$ so that $Q(g) = Q(b)$ (as b has type 1).

Next suppose $Q(g) \cap P = Q(m)$ for some $m \in C^\cdot(a)$. Then $Q(gm) \cap P = 1$ by (2.11), and so $gm = 1$ or $gm \in C^\cdot(b)$. In the first case $Q(g) = Q(m)$ and in the second case $Q(g) = Q(bm) = Q(b)Q(m)$ by (2.1). Lastly, suppose $P \subset Q(g)$. Again $[B : P] = 2$ implies $Q(g) = P$ or B.

Step 2. There exists $t \in G$ with $Q(t) = P$.

Suppose otherwise. Then for any $g \in G$ either $g \in C(a)C(b)$ or $Q(g) = B$, by Step 1. We again use the counting formula from [7]:

$$\sum_{x \neq 1, a} \frac{1}{|Q(x) \cap Q(a)|} \cdot \frac{1}{|Q(ax)|} = \frac{-2}{|Q(a)|} + \sum_{y \in D\langle 1, -a\rangle} \frac{1}{|Q(y)|}.$$

Set $e = |C(b)|$, $g = |G|$ and $c = |C(a)| = |P|$. We note that $H(Q(m)) = \{1, m\}$ and $H(Q(bm)) = \{1, m\}C(b)$ for $m \in C(a)$. Further, $C(a)C(b) \subset D\langle 1, -a\rangle$, by (2.1) and the remark after (2.5). We split the left-hand sum into sums over $C^\cdot(a)\backslash\{a\}$, $C^\cdot(b)$, $aC^\cdot(b)$, $(C^\cdot(a)\backslash\{a\})C^\cdot(b)$ and $G\backslash C(a)C(b)$. We obtain:

$$\text{LHS} = \frac{c-2}{2} + \frac{e-1}{4} + \frac{e-1}{4} + \frac{(c-2)(e-1)}{4} + \frac{g-ce}{4c}.$$

We split the right-hand sum into sums over $\{1\}$, $C^\cdot(a)$, $C^\cdot(b)$, $C^\cdot(a)C^\cdot(b)$ and $D\langle 1, -a\rangle\backslash C(a)C(b)$. We obtain:

$$\text{RHS} = -1 + 1 + \frac{c-1}{2} + \frac{e-1}{2} + \frac{(c-1)(e-1)}{4} + \frac{(g/2) - ce}{2c}.$$

Equating the two sides gives:

$$\frac{c-2}{2} + \frac{e-1}{4} - \frac{3}{4} = \frac{c-2}{2} + \frac{e-1}{4} - \frac{e}{2},$$
$$-5/4 = -3/2,$$

a contradiction.

Step 3. There is a cap t for a with $t \notin D\langle 1, -a\rangle$.

Set $F = H(P)$. We will show $F \not\subset D\langle 1, -a\rangle$, since then if $t \in F\backslash D\langle 1, -a\rangle$ we must have $Q(t) = P$ lest $t \in C(a) \subset D\langle 1, -a\rangle$. Let $f = |F|$ and $i = |F \cap D\langle 1, -a\rangle|$. We use the same formula as Step 2. On the left-hand side we need only replace the sum over $G\backslash C(a)C(b)$

by sums over $F\backslash C(a)$ and $G\backslash(C(a)C(b)\cup F)$. We obtain:

$$\begin{aligned}\text{LHS} = \frac{c-2}{2} + \frac{e-1}{4} + \frac{e-1}{4} + \frac{(c-2)(e-1)}{4} \\ + \frac{f-c}{2c} + \frac{g-ce-f+c}{4c}.\end{aligned}$$

On the right-hand side we need only replace the sum over $D\langle 1, -a\rangle\backslash C(a)C(b)$ by sums over $(F \cap D\langle 1, -a\rangle)\backslash C(a)$ and $D\langle 1, -a\rangle\backslash(C(a)C(b)\cup F)$. We obtain:

$$\text{RHS} = \frac{c-1}{2} + \frac{e-1}{2} + \frac{(c-1)(e-1)}{4} + \frac{i-c}{c} + \frac{(g/2)-ce-i+c}{2c}.$$

Equating the two sides gives:

$$-1 + \frac{e-1}{4} + \frac{f-c}{2c} + \frac{c-ce-f}{4c} = \frac{-1}{2} + \frac{e-1}{2} + \frac{i-c}{c} + \frac{c-ce-i}{2c},$$
$$f/4c = i/2c,$$
$$f = 2i.$$

Thus $F \not\subset D\langle 1, -a\rangle$ as desired

Step 4. Finish.

Let t be the cap of Step 3. Set $H=\{1, t\}C(a)$ and $K = D\langle 1, -a\rangle\cap D\langle 1, -t\rangle$. Then $G = H \perp K$ by (2.13). Arguing as in Step 1, we see that if $k \in K$ then $Q(k) = Q(b)$, P or B. Hence (in the notation of [**8**]) $|Y_K| \leq 3$ and the Witt ring R_2 generated by K is of elementary type [**8**, 3.7]. Indeed, R_2 is of local type since otherwise R_2 is a product of two local factors and $|P| = 2$, which is impossible ($|P| = |C(a)| \geq 4$ since a has type 2). Thus $R = R_1 \times R_2$ by [**3**, 3.4], R_2 is of local type and R_1, generated by H, is an S-ring by (2.10). □

Both (2.7) and (2.13) require a condition of the form $Q(x) \not\subset Q(K)$, where $i(x) = 2$, to deduce that x arises from a fiber product. We consider the case of extreme failure of this condition, namely $Q(x) \subset Q(y)$ for all $y \in G^{\cdot}$. When $i(x) = 2$ we will show that $Q(x) \subset Q(y)$ for all y, implies R is of local type.

LEMMA 3.2. *Suppose $Q(w) \subset Q(g)$ for all $g \in G^{\cdot}$. Let $H = H(Q(w))$. In the quotient R/w set $\overline{z} = zH$ and $Q(\overline{z}) = Q(z)/Q(w)$. Then*:

$$(1) \qquad \sum_{\overline{z}\in D\langle 1, -a\rangle} \frac{(h/q) - \varepsilon(\overline{z})|D\langle 1, -a\rangle \cap H|}{|Q(\overline{z})|} = -\left(1 - \frac{1}{q}\right)\frac{2}{|Q(\overline{a})|},$$

where $A \in G\backslash H$, $h = |H|$, $q = |Q(w)|$ *and*:

$$\varepsilon(\overline{z}) = \begin{cases} 0, & \text{if } zH \cap D\langle 1, -a\rangle = \varnothing, \\ 1, & \text{if } zH \cap D\langle 1, -a\rangle \neq \varnothing. \end{cases}$$

Proof. R/w is well defined by **[8**, 2.4]. We start with the counting formula of [7] for both (R, G) and $(R/w, \overline{G})$:

$$\sum_{x\neq 1, a} \frac{1}{|Q(x)\cap Q(a)|}\cdot\frac{1}{|Q(ax)|} = \frac{-2}{|Q(a)|} + \sum_{y\in D\langle 1, -a\rangle}\frac{1}{|Q(y)|}, \tag{2}$$

$$\sum_{\overline{x}\neq \overline{1}, \overline{a}} \frac{1}{|Q(\overline{x})\cap Q(\overline{a})|}\cdot\frac{1}{|Q(\overline{ax})|} = \frac{-2}{|Q(\overline{a})|} + \sum_{\overline{y}\in D\langle 1, -\overline{a}\rangle}\frac{1}{|Q(\overline{y})|}. \tag{3}$$

Note that for all $g \in G^{\cdot}$ and $h \in H$, we have $Q(g) = Q(gh)$. Namely, $Q(gh) \subset Q(g)Q(h) = Q(g)Q(w) = Q(g)$ and similarly $Q(g) \subset Q(gh)$. Then:

$$\begin{aligned}
\mathrm{LHS}(2) &= \sum_{x\in H^{\cdot}} \frac{1}{|Q(x)\cap Q(a)|}\cdot\frac{1}{|Q(ax)|} \\
&\quad + \sum_{x\in aH^{\cdot}} \frac{1}{|Q(x)\cap Q(a)|}\cdot\frac{1}{|Q(ax)|} \\
&\quad + \sum_{\overline{z}\neq\overline{1},\overline{a}}\ \sum_{x\in zH} \frac{1}{|Q(x)\cap Q(a)|}\cdot\frac{1}{|Q(ax)|} \\
&= \frac{h-1}{q|Q(a)|} + \frac{h-1}{q|Q(a)|} + \sum_{\overline{z}\neq\overline{1},\overline{a}} \frac{1}{|Q(x)\cap Q(a)|}\cdot\frac{1}{|Q(az)|}.
\end{aligned}$$

Now $|Q(g)| = |Q(\overline{g})|q$ and $|Q(x)\cap Q(a)| = |Q(\overline{x})\cap Q(\overline{a})|q$. Thus:

$$\begin{aligned}
\mathrm{LHS}(2) &= \frac{2(h-1)}{q^2|Q(\overline{a})|} + \sum_{\overline{z}\neq\overline{1},\overline{a}} \frac{h}{q^2|Q(\overline{z})\cap Q(\overline{a})||Q(\overline{az})|} \\
&= \frac{h}{q^2}\left[\frac{2}{|Q(\overline{a})|} + \sum_{\overline{z}\neq\overline{1},\overline{a}} \frac{1}{|Q(\overline{z})\cap Q(\overline{a})||Q(\overline{az})|}\right] \\
&\quad - \frac{2}{q^2|Q(\overline{a})|}.
\end{aligned}$$

Equation (3) then implies:

$$\mathrm{LHS}(2) = \frac{h}{q^2}\left(\sum_{\overline{z}\in D\langle 1, -\overline{a}\rangle} \frac{1}{|Q(\overline{z})|}\right) - \frac{2}{q^2|Q(\overline{a})|}.$$

We turn now to the right-hand side of (2):

$$\begin{aligned}\mathrm{RHS}(2) &= \frac{-2}{|Q(a)|} + \sum_{\overline{z}\in\overline{G}} \sum_{y\in D\langle 1, -a\rangle\cap zH} \frac{1}{|Q(y)|}\\ &= \frac{-2}{|Q(a)|} + \sum_{\overline{z}\in\overline{G}} \frac{|D\langle 1, -a\rangle \cap zH|}{|Q(z)|}.\end{aligned}$$

If $\varepsilon(\overline{z}) = 1$ then $D\langle 1, -a\rangle \cap zH = x(D\langle 1, -a\rangle \cap H)$ for some $x \in zH$. Thus $|D\langle 1, -a\rangle \cap zH| = \varepsilon(\overline{z})|D\langle 1, -a\rangle \cap H|$. Further, since $D\langle 1, -\overline{a}\rangle = D\langle 1, -a\rangle D\langle 1, -aw\rangle/H$, if $\overline{z} \notin D\langle 1, -\overline{a}\rangle$ then $D\langle 1, -a\rangle \cap zH = \varnothing$. We obtain:

$$\begin{aligned}\mathrm{RHS}(2) &= \frac{-2}{|Q(a)|} + \sum_{\overline{z}\in D\langle 1, -\overline{a}\rangle} \frac{\varepsilon(\overline{z})|D\langle 1, -a\rangle \cap H|}{|Q(z)|}\\ &= \frac{-2}{q|Q(\overline{a})|} + \sum_{\overline{z}\in D\langle 1, -\overline{a}\rangle} \frac{\varepsilon(\overline{z})|D\langle 1, -a\rangle \cap H|}{q|Q(\overline{z})|}.\end{aligned}$$

Equating the two expressions for LHS(2) and RHS(2) (and multiplying by q) gives:

$$\begin{aligned}&\frac{h}{q}\left(\sum_{\overline{z}\in D\langle 1, -\overline{a}\rangle} \frac{1}{|Q(\overline{z})|}\right) - \frac{2}{q|Q(\overline{a})|}\\ &\quad= \left(\sum_{\overline{z}\in D\langle 1, -\overline{a}\rangle} \frac{\varepsilon(\overline{z})|D\langle 1, -a\rangle \cap H|}{|Q(\overline{z})|}\right) - \frac{2}{|Q(\overline{a})|}.\end{aligned}$$

This is easily seen to be equivalent to (1). □

THEOREM 3.3. *Suppose $Q(w) \subset Q(g)$ for all $g \in G^{\cdot}$. Set $H = H(Q(w))$. If $|Q(w)| \leq |H|$ then R is of local type.*

Proof. Let $q = |Q(w)|$, $h = |H|$, $\overline{z} = zH$ and $Q(\overline{z}) = Q(z)/Q(w)$. We assume, if possible, that $g \neq H$. Set $g = |G|$. If there exists an $a \in G\backslash H$ such that $|D\langle 1, -a\rangle \cap H| \leq h/q$ then $\mathrm{LHS}(1) \geq 0$ while $\mathrm{RHS}(1) < 0$, a contradiction. Hence for all $a \in G\backslash H$ we have $|D\langle 1, -a\rangle \cap H| \geq 2h/q$.

We now count, in two ways, the number N of pairs (a, x) where $a \in G\backslash H$, $x \in H^{\cdot}$ and $a \in D\langle 1, -x\rangle$. If we fix x, the number of pairs with this x is

$$\begin{aligned}|D\langle 1, -x\rangle \cap (G\backslash H)| &= |D\langle 1, x\rangle| = |D\langle 1, -x\rangle \cap H|\\ &= (g/q) - |D\langle 1, -x\rangle \cap H|.\end{aligned}$$

Thus

$$N = \frac{(h-1)g}{q} - \sum_{x \in H^{\cdot}} |D\langle 1, -x\rangle \cap H|.$$

Denote this last sum by β.

Next, if we fix $a \in G \backslash H$ the number of pairs with this a is

$$|D\langle 1, -a\rangle \cap H^{\cdot}| = |D\langle 1, -a\rangle \cap H| - 1.$$

So:

$$N = \sum_{a \in G\backslash H} (|D\langle 1, -a\rangle \cap H| - 1) \geq \frac{(g-h)2h}{q} - (g-h).$$

Comparing the two expressions for N yields:

$$\frac{(g-h)2h}{q} - (g-h) \leq \frac{(h-1)g}{q} - \beta,$$
$$(g-h)2h - (g-h)q \leq (h-1)g - \beta q.$$

Now $\beta = \sum_{x \in H^{\cdot}} |D\langle 1, -x\rangle \cap H| \geq (h-1)$. Thus:

$$\begin{aligned} (g-h)2h - (g-h)q + (h-1)q &\leq (h-1)g, \\ gh - gq + g &\leq 2h^2 - 2hq + q, \\ g(h-q+1) &\leq 2h(h-q) + q. \end{aligned}$$

Note that $h - q + 1 \geq 1$ since $h \geq q$. Thus:

$$g \leq \frac{2h(h-q)+q}{(h-q)+1} < 2h,$$

since $q < 2h$. Thus $g = h$ and so $G = H$. Then $Q(g) = Q(w)$ for all $g \in G^{\cdot}$. R is then of local type [7, Th. 5]. □

The condition $q = |Q(w)| \leq |H| = h$ in (3.3) can be viewed as follows: In the quotient R/w, let $\bar{a} = aH$. Then $|D\langle 1, -\bar{a}\rangle| = |D\langle 1, -a\rangle| q/h$. Thus if $q \leq h$, no value group increases in size upon passing to the quotient.

COROLLARY 3.4. *Suppose $i_G D\langle 1, -w\rangle = 2$ and $Q(w) \subset Q(g)$ for all $g \in G^{\cdot}$ Then R is of local type.*

Proof. Clearly $|Q(w)| = 2 \leq |H(Q(w))|$, so apply (3.3). □

COROLLARY 3.5. *Suppose* $Q(w) \subset Q(g)$ *for all* $g \in G^{\cdot}$. *Suppose further that the counting coefficients of the Hasse diagram for* Y_R (*cf.* [7, *p.* 49]) *are positive. Then* R *is of local type.*

Proof. Here we get $|Q(w)| = 2$ by [7, Th. 13], so apply (3.4). $\square$

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Received September 30, 1988 and in revised form January 18, 1990.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 148, No. 1, 1991

DENTABILITY, TREES, AND DUNFORD-PETTIS OPERATORS ON L_1

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If all bounded linear operators from L_1 into a Banach space $\mathfrak{X}$ are *Dunford-Pettis* (i.e. carry weakly convergent sequences onto norm convergent sequences), then we say that $\mathfrak{X}$ has the *complete continuity property* (CCP). The CCP is a weakening of the Radon-Nikodým property (RNP). Basic results of Bourgain and Talagrand began to suggest the possibility that the CCP, like the RNP, can be realized as an internal geometric property of Banach spaces; the purpose of this paper is to provide such a realization. We begin by showing that $\mathfrak{X}$ has the CCP if and only if every bounded subset of $\mathfrak{X}$ is Bocce dentable, or equivalently, every bounded subset of $\mathfrak{X}$ is weak-norm-one dentable (§2). This internal geometric description leads to another; namely, $\mathfrak{X}$ has the CCP if and only if no bounded separated δ-trees grow in $\mathfrak{X}$, or equivalently, no bounded δ-Rademacher trees grow in $\mathfrak{X}$ (§3).

1. Introduction. Throughout this paper, $\mathfrak{X}$ denotes an arbitrary Banach space, $\mathfrak{X}^*$ the dual space of $\mathfrak{X}$, $B(\mathfrak{X})$ the closed unit ball of $\mathfrak{X}$, and $S(\mathfrak{X})$ the unit sphere of $\mathfrak{X}$. The triple (Ω, Σ, μ) refers to the Lebesgue measure space on $[0, 1]$, Σ^+ to the sets in Σ with positive measure, and L_1 to $L_1(\Omega, \Sigma, \mu)$. All notation and terminology, not otherwise explained, are as in [DU]. For clarity, known results are presented as Facts while new results are presented as Theorems, Lemmas, and Observations.

The following fact provides several equivalent formulations of the CCP.

FACT 1.1. For a bounded linear operator T from L_1 into $\mathfrak{X}$, the following statements are equivalent.

(1) T is Dunford-Pettis.

(2) T maps weak compact sets to norm compact sets.

(3) $T(B(L_\infty))$ is a relatively norm compact subset of $\mathfrak{X}$.

(4) The corresponding vector measure $F\colon \Sigma \to \mathfrak{X}$ given by $F(E) = T(\chi_E)$ has a relatively norm compact range in $\mathfrak{X}$.

(5) The adjoint of the restriction of T to L_∞ from $\mathfrak{X}^*$ into L_∞^* is a compact operator.

(6) As a subset of L_1, $T^*(B(\mathfrak{X}^*))$ is relatively L_1-norm compact.

(7) As a subset of L_1, $T^*(B(\mathfrak{X}^*))$ satisfies the Bocce criterion.

The equivalence of (2) and (3) follows from the fact that the subsets of L_1 that are relatively weakly compact are precisely those subsets that are bounded and uniformly integrable, which in turn, are precisely those subsets that can be uniformly approximated in L_1-norm by uniformly-bounded subsets. As for the equivalence of (6) and (7), **[G]** presents the two definitions below and shows that a relatively weakly compact subset of L_1 is relatively L_1-norm compact if and only if it satisfies the Bocce criterion.

DEFINITION 1.2. For f in L_1 and A in Σ, the *Bocce oscillation of f on A* is given by

$$\text{Bocce-osc}\, f|_A \equiv \frac{\int_A |f - [\int_A f\, d\mu/\mu(A)]|\, d\mu}{\mu(A)},$$

observing the convention that $0/0$ is 0.

DEFINITION 1.3. A subset K of L_1 satisfies the *Bocce criterion* if for each $\varepsilon > 0$ and B in Σ^+ there is a finite collection $\mathcal{F}$ of subsets of B each with positive measure such that for each f in K there is an A in $\mathcal{F}$ satisfying

$$\text{Bocce-osc}\, f|_A < \varepsilon .$$

The other implications in Fact 1.1 are straightforward and easy to verify. Because of (4), the CCP is also referred to as the compact range property (CRP).

Towards a martingale characterization of the CCP, fix an increasing sequence $\{\pi_n\}_{n\geq 0}$ of finite positive interval partitions of Ω such that $\bigvee \sigma(\pi_n) = \Sigma$ and $\pi_0 = \{\Omega\}$. Let $\mathcal{F}_n$ denote the sub-σ-field $\sigma(\pi_n)$ of Σ that is generated by π_n. For f in $L_1(\mathfrak{X})$, let $E_n(f)$ denote the conditional expectation of f given $\mathcal{F}_n$.

DEFINITION 1.4. A sequence $\{f_n\}_{n\geq 0}$ in $L_1(\mathfrak{X})$ is an $\mathfrak{X}$-valued *martingale* with respect to $\{\mathcal{F}_n\}$ if for each n we have that f_n is $\mathcal{F}_n$-measurable and $E_n(f_{n+1}) = f_n$ in L_1. The martingale $\{f_n\}$ is *uniformly bounded* provided that $\sup_n \|f_n\|_{L_\infty}$ is finite. Often the martingale is denoted by $\{f_n, \mathcal{F}_n\}$ in order to display both the functions and the sub-σ-fields involved.

There is a one-to-one correspondence between the bounded linear operators T from L_1 into $\mathfrak{X}$ and the uniformly bounded $\mathfrak{X}$-valued martingales $\{f_n, \mathscr{F}_n\}$. This correspondence is obtained by taking

$$T(g) = \lim_{n\to\infty} \int_\Omega f_n(\omega) g(\omega)\, d\mu(\omega) \quad \text{if } \{f_n\} \text{ is the martingale,}$$

and

$$f_n(\omega) = \sum_{E\in\pi_n} \frac{T(\chi_E)}{\mu(E)} \chi_E(\omega) \quad \text{if } T \text{ is the operator.}$$

Fact 1.1.6 implies that a bounded linear operator T from L_1 into $\mathfrak{X}$ is Dunford-Pettis if and only if

$$\lim_{m,n\to\infty} \sup_{x^*\in B(\mathfrak{X}^*)} \|E_n(T^*x^*) - E_m(T^*x^*)\|_{L_1} = 0.$$

Since $E_n(T^*x^*) = x^*f_n$ in L_1, we have the following martingale characterization of Dunford-Pettis operators, and thus of the CCP.

FACT 1.5. A bounded linear operator from L_1 into $\mathfrak{X}$ is Dunford-Pettis if and only if the corresponding martingale is Cauchy in the Pettis norm. Consequently, a Banach space $\mathfrak{X}$ has the CCP if and only if all uniformly bounded $\mathfrak{X}$-valued martingales are Pettis-Cauchy.

Recall that a bounded linear operator $T: L_1 \to \mathfrak{X}$ is (*Bochner*) *representable* if there is g in $L_\infty(\mu, \mathfrak{X})$ such that for each f in $L_1(\mu)$

$$Tf = \int_\Omega fg\, d\mu.$$

A Banach space $\mathfrak{X}$ has the *Radon-Nikodým property* if all bounded linear operators from L_1 into $\mathfrak{X}$ are Bochner representable. It is clear that a representable operator from L_1 into $\mathfrak{X}$ is Dunford-Pettis. Thus, if $\mathfrak{X}$ has the RNP then $\mathfrak{X}$ has the CCP. Both the Bourgain-Rosenthal space [**BR**] and the dual of the James tree space [**J**] have the CCP yet fail the RNP.

2. Dentability. In this section, we examine in which Banach spaces bounded subsets have certain dentability properties.

Dentability characterizations of the RNP are well-known (cf. [**DU**] and [**GU**]).

FACT 2.1. The following statements are equivalent.

(1) $\mathfrak{X}$ has the RNP.

(2) Every bounded subset D of $\mathfrak{X}$ is dentable.

DEFINITION 2.2. D is *dentable* if for each $\varepsilon > 0$ there is x in D such that $x \notin \overline{\text{co}}(D\backslash B_\varepsilon(x))$ where $B_\varepsilon(x) = \{y \in \mathfrak{X} : \|x - y\| < \varepsilon\}$.

(3) Every bounded subset D of $\mathfrak{X}$ is σ-dentable.
DEFINITION 2.3. D is *σ-dentable* if for each $\varepsilon > 0$ there is an x in D such that if x has the form $x = \sum_{i=1}^{n} \alpha_i z_i$ with $z_i \in D$, $0 \leq \alpha_i$, and $\sum_{i=1}^{n} \alpha_i = 1$, then $\|x - z_i\| < \varepsilon$ for some i.

The natural question to explore next is what dentability condition characterizes the CCP. Towards this, the next definition is a weakening of Definition 2.2.

DEFINITION 2.4. A subset D of $\mathfrak{X}$ is *weak-norm-one dentable* if for each $\varepsilon > 0$ there is a finite subset F of D such that for each x^* in $S(\mathfrak{X}^*)$ there is x in F satisfying

$$x \notin \overline{\text{co}}\{z \in D : |x^*(z - x)| \geq \varepsilon\} \equiv \overline{\text{co}}(D \backslash V_{\varepsilon, x^*}(x)) .$$

Petrakis and Uhl **[PU]** showed that if $\mathfrak{X}$ has the CCP then every bounded subset of $\mathfrak{X}$ is weak-norm-one dentable. For our characterization of the CCP, we introduce the following variations of Definition 2.3 that are useful in showing the converse of the above implication of **[PU]**.

DEFINITION 2.5. A subset D of $\mathfrak{X}$ is *Bocce dentable* if for each $\varepsilon > 0$ there is a finite subset F of D such that for each x^* in $S(\mathfrak{X}^*)$ there is x in F satisfying: if $x = \sum_{i=1}^{n} \alpha_i z_i$ with $z_i \in D$, $0 \leq \alpha_i$, and $\sum_{i=1}^{n} \alpha_i = 1$, then $\sum_{i=1}^{n} \alpha_i |x^*(x - z_i)| < \varepsilon$.

DEFINITION 2.6. A subset D of $\mathfrak{X}$ is *midpoint Bocce dentable* if for each $\varepsilon > 0$ there is a finite subset F of D such that for each x^* in $S(\mathfrak{X}^*)$ there is x in F satisfying: if $x = \frac{1}{2}z_1 + \frac{1}{2}z_2$ with $z_i \in D$ then $|x^*(x - z_1)| \equiv |x^*(x - z_2)| < \varepsilon$.

We obtain equivalent formulations of the above definitions by replacing $S(\mathfrak{X}^*)$ with $B(\mathfrak{X}^*)$.

The next theorem, this section's main result, shows that these dentability conditions provide an internal geometric characterization of the CCP.

THEOREM 2.7. *The following statements are equivalent.*

(1) *$\mathfrak{X}$ has the CCP.*
(2) *Every bounded subset of $\mathfrak{X}$ is weak-norm-one dentable.*
(3) *Every bounded subset of $\mathfrak{X}$ is midpoint Bocce dentable.*
(4) *Every bounded subset of $\mathfrak{X}$ is Bocce dentable.*

The remainder of this section is devoted to the proof of Theorem 2.7. Because of its length and complexity and also for the sake of clarity of the exposition, we present the implications as separate theorems. It is clear from the definitions that (2) implies (3) and that (4) implies (3). [**PU**, Theorem II.7] shows that (1) implies (2) by constructing, in a bounded non-weak-norm-one dentable subset D, a $(\overline{\text{co}}\, D)$-valued martingale that is not Cauchy in the Pettis norm. Using Fact 1.1.7, Theorem 2.10 shows that (3) implies (1). That (1) implies (4) follows from Theorem 2.8 and the martingale characterization of the CCP (Fact 1.5).

THEOREM 2.8. *If a subset D of $\mathfrak{X}$ is not Bocce dentable, then there is an increasing sequence $\{\pi_n\}$ of partitions of $[0, 1)$ and a D-valued martingale $\{f_n, \sigma(\pi_n)\}$ that is not Cauchy in the Pettis norm. Moreover, $\{\pi_n\}$ can be chosen so that $\bigvee \sigma(\pi_n) = \Sigma$, $\pi_0 = \{\Omega\}$, and each π_n partitions $[0, 1)$ into a finite number of half-open intervals.*

Proof. Let D be a subset of $\mathfrak{X}$ that is not Bocce dentable. Accordingly, there is an $\varepsilon > 0$ satisfying:

(*) for each finite subset F of D there is x_F^* in $S(\mathfrak{X}^*)$ such that each x in F has the form $x = \sum_{i=1}^m \alpha_i z_i$ with $\sum_{i=1}^m \alpha_i |x_F^*(x - z_i)| > \varepsilon$ for a suitable choice of $z_i \in D$ and $\alpha_i > 0$ with $\sum_{i=1}^m \alpha_i = 1$.

We shall use property (*) to construct an increasing sequence $\{\pi_n\}_{n \geq 0}$ of finite partitions of $[0, 1)$, a martingale $\{f_n, \sigma(\pi_n)\}_{n \geq 0}$, and a sequence $\{x_n^*\}_{n \geq 1}$ in $S(\mathfrak{X}^*)$ such that for each nonnegative integer n:

(1) f_n has the form $f_n = \sum_{E \in \pi_n} x_E \chi_E$ where x_E is in D,
(2) $\int_\Omega |x_{n+1}^*(f_{n+1} - f_n)| \, d\mu \geq \varepsilon$,
(3) if E is in π_n, then E has the form $[a, b)$ and $\mu(E) < 1/2^n$ and
(4) $\pi_0 = \{\Omega\}$.

Condition (3) guarantees that $\bigvee \sigma(\pi_n) = \Sigma$ while condition (2) guarantees that $\{f_n\}$ is not Cauchy in the Pettis norm.

Towards the construction, pick an arbitrary x in D. Set $\pi_0 = \{\Omega\}$ and $f_0 = x\chi_\Omega$. Fix $n \geq 0$. Suppose that a partition π_n of Ω consisting of intervals of length at most $1/2^n$ and a function $f_n = \sum_{E \in \pi_n} x_E \chi_E$ with $x_E \in D$ have been constructed. We now construct f_{n+1}, π_{n+1} and x_{n+1}^* satisfying conditions (1), (2), and (3).

Apply $(*)$ to $F = \{x_E : E \in \pi_n\}$ and find the associated $x_F^* = x_{n+1}^*$ in $S(\mathfrak{X}^*)$. Fix an element $E = [a, b)$ of π_n. We first define $f_{n+1}\chi_E$. Property $(*)$ gives that x_E has the form

$$x_E = \sum_{i=1}^{m} \alpha_i x_i \quad \text{with } \sum_{i=1}^{m} \alpha_i |x_{n+1}^*(x - x_i)| > \varepsilon$$

for a suitable choice of $x_i \in D$ and positive real numbers $\alpha_1, \dots, \alpha_m$ whose sum is one. Using repetition, we arrange to have $\alpha_i < 1/2^{n+1}$ for each i. It follows that there are real numbers $d_0, d_1, \dots, d_m$ such that

$$a = d_0 < d_1 < \cdots < d_{m-1} < d_m = b$$

and

$$d_i - d_{i-1} = \alpha_i(b - \alpha) \quad \text{for } i = 1, \dots, m.$$

Set

$$f_{n+1}\chi_E = \sum_{i=1}^{m} x_i \chi_{[d_{i-1}, d_i)}.$$

Define f_{n+1} on all of Ω similarly. Let π_{n+1} be the partition consisting of all the intervals $[d_{i-1}, d_i)$ obtained from letting E range over π_n.

Clearly, f_{n+1} and π_{n+1} satisfy conditions (1) and (3). Condition (2) is also satisfied since for each $E = [a, b)$ in π_n we have, using the above notation,

$$\begin{aligned}\int_E |x_{n+1}^*(f_{n+1} - f_n)| \, d\mu &= \sum_{i=1}^{m} \int_{d_{i-1}}^{d_i} |x_{n+1}^*(x_i - x_E)| \, d\mu \\ &= (b - a) \sum_{i=1}^{m} \alpha_i |x_{n+1}^*(x_i - x_E)| \geq \mu(E)\varepsilon.\end{aligned}$$

To insure that $\{f_n\}$ is indeed a martingale, we need to compute $E_n(f_{n+1})$. Fix $E = [a, b)$ in π_n. Using the above notation, we have for almost all t in E,

$$\begin{aligned}E_n(f_{n+1})(t) &= \frac{1}{b-a} \int_a^b f_{n+1} \, d\mu \\ &= \frac{1}{b-a} \sum_{i=1}^{m} \int_{d_{i-1}}^{d_i} f_{n+1} \, d\mu = \sum_{i=1}^{m} \frac{d_i - d_{i-1}}{b-a} x_i \\ &= \sum_{i=1}^{m} \alpha_i x_i = x_E = f_n(t).\end{aligned}$$

Thus $E_n(f_{n+1}) = f_n$ a.e., as needed.

This completes the necessary constructions. □

We need the following lemma which we will prove after the proof of Theorem 2.10.

LEMMA 2.9. *If A is in Σ^+ and f in $L_\infty(\mu)$ is not constant a.e. on A, then there is an increasing sequence $\{\pi_n\}$ of positive finite measurable partitions of A such that $\bigvee \sigma(\pi_n) = \Sigma \cap A$ and for each n*

$$\mu\left(\bigcup\left\{E : E \in \pi_n,\ \frac{\int_E f\,d\mu}{\mu(E)} \geq \frac{\int_A f\,d\mu}{\mu(A)}\right\}\right) = \frac{\mu(A)}{2},$$

and so

$$\mu\left(\bigcup\left\{E : E \in \pi_n,\ \frac{\int_E f\,d\mu}{\mu(E)} < \frac{\int_A f\,d\mu}{\mu(A)}\right\}\right) = \frac{\mu(A)}{2}.$$

THEOREM 2.10. *If all bounded subsets of $\mathfrak{X}$ are midpoint Bocce dentable, then $\mathfrak{X}$ has the complete continuity property.*

Proof. Let all bounded subsets of $\mathfrak{X}$ be midpoint Bocce dentable. Fix a bounded linear operator T from L_1 into $\mathfrak{X}$. We shall show that the subset $T^*(B(\mathfrak{X}^*))$ of L_1 satisfies the Bocce criterion. Then an appeal to Fact 1.1.7 shows that $\mathfrak{X}$ has the complete continuity property.

To this end, fix $\varepsilon > 0$ and B in Σ^+. Let F denote the vector measure from Σ into $\mathfrak{X}$ given by $F(E) = T(\chi_E)$. Since the subset $\{\frac{F(E)}{\mu(E)} : E \subset B \text{ and } E \in \Sigma^+\}$ of $\mathfrak{X}$ is bounded, it is midpoint Bocce dentable. Accordingly, there is a finite collection $\mathscr{F}$ of subsets of B each in Σ^+ such that for each x^* in the unit ball of $\mathfrak{X}^*$ there is a set A in $\mathscr{F}$ such that if

$$\frac{F(A)}{\mu(A)} = \frac{1}{2}\frac{F(E_1)}{\mu(E_1)} + \frac{1}{2}\frac{F(E_2)}{\mu(E_2)}$$

for some subsets E_i of B with $E_i \in \Sigma^+$, then

$$\text{(1)} \qquad \frac{1}{2}\left|\frac{x^*F(E_1)}{\mu(E_1)} - \frac{x^*F(A)}{\mu(A)}\right| + \frac{1}{2}\left|\frac{x^*F(E_2)}{\mu(E_2)} - \frac{x^*F(A)}{\mu(A)}\right| < \varepsilon.$$

Fix x^* in the unit ball of $\mathfrak{X}^*$ and find the associated A in $\mathscr{F}$. By definition, the set $T^*(B(\mathfrak{X}^*))$ will satisfy the Bocce criterion provided that Bocce-osc $(T^*x^*)|_A \leq \varepsilon$.

If $T^*x^* \in L_1$ is constant a.e. on A, then the Bocce-osc $(T^*x^*)|_A$ is zero and we are finished. So assume T^*x^* is not constant a.e. on A.

For a finite positive measurable partition π of A, denote

$$f_\pi = \sum_{E \in \pi} \frac{F(E)}{\mu(E)} \chi_E$$

and

$$E_\pi^+ = \bigcup \left\{ E \in \pi : \frac{x^* F(E)}{\mu(E)} \geq \frac{x^* F(A)}{\mu(A)} \right\}$$

and

$$E_\pi^- = \bigcup \left\{ E \in \pi : \frac{x^* F(E)}{\mu(E)} < \frac{x^* F(A)}{\mu(A)} \right\}.$$

Note that for E in Σ

$$x^* F(E) = \int_E (x^* T^*) \, d\mu.$$

Compute

$$\begin{aligned}
(2) \qquad & \int_A \left| x^* f_\pi - \frac{x^* F(A)}{\mu(A)} \right| d\mu \\
&= \sum_{E \in \pi} \int_E \left| \frac{x^* F(E)}{\mu(E)} - \frac{x^* F(A)}{\mu(A)} \right| d\mu \\
&= \mu(A) \sum_{E \in \pi} \frac{\mu(E)}{\mu(A)} \left| \frac{x^* F(E)}{\mu(E)} - \frac{x^* F(A)}{\mu(A)} \right| \\
&= \mu(A) \left[\frac{\mu(E_\pi^+)}{\mu(A)} \left| \frac{x^* F(E_\pi^+)}{\mu(E_\pi^+)} - \frac{x^* F(A)}{\mu(A)} \right| \right. \\
&\qquad \left. + \frac{\mu(E_\pi^-)}{\mu(A)} \left| \frac{x^* F(E_\pi^-)}{\mu(E_\pi^-)} - \frac{x^* F(A)}{\mu(A)} \right| \right].
\end{aligned}$$

Since the L_1-function $T^* x^*$ is bounded, for now we may view $T^* x^*$ as an element in L_∞. Lemma 2.9 allows us to apply property (1) to equation (2). For applying Lemma 2.9 to A with $f \equiv T^* x^*$ produces an increasing sequence $\{\pi_n\}$ of positive measurable partitions of A satisfying

$$\bigvee \sigma(\pi_n) = \Sigma \cap A \quad \text{and} \quad \mu(E_{\pi_n}^+) = \frac{\mu(A)}{2} = \mu(E_{\pi_n}^-).$$

For $\pi = \pi_n$, condition (2) becomes

$$\begin{aligned}
(3) \qquad & \int_A \left| x^* f_{\pi_n} - \frac{x^* F(A)}{\mu(A)} \right| d\mu \\
&= \mu(A) \left[\frac{1}{2} \left| \frac{x^* F(E_{\pi_n}^+)}{\mu(E_{\pi_n}^+)} - \frac{x^* F(A)}{\mu(A)} \right| \right. \\
&\qquad \left. + \frac{1}{2} \left| \frac{x^* F(E_{\pi_n}^-)}{\mu(E_{\pi_n}^-)} - \frac{x^* F(A)}{\mu(A)} \right| \right].
\end{aligned}$$

Since $F(A)/\mu(A)$ has the form

$$\begin{aligned}\frac{F(A)}{\mu(A)} &= \frac{\mu(E^+_{\pi_n})}{\mu(A)}\frac{F(E^+_{\pi_n})}{\mu(E^+_{\pi_n})} + \frac{\mu(E^-_{\pi_n})}{\mu(A)}\frac{F(E^-_{\pi_n})}{\mu(E^-_{\pi_n})} \\ &= \frac{1}{2}\frac{F(E^+_{\pi_n})}{\mu(E^+_{\pi_n})} + \frac{1}{2}\frac{F(E^-_{\pi_n})}{\mu(E^-_{\pi_n})},\end{aligned}$$

applying property (1) to equation (3) yields that for each π_n

$$\int_A \left| x^* f_{\pi_n} - \frac{x^* F(A)}{\mu(A)} \right| d\mu < \mu(A)\varepsilon .$$

Since $\bigvee \sigma(\pi_n) = \Sigma \cap A$ and

$$\begin{aligned}(x^* f_{\pi_n})|_A &= \sum_{E\in\pi_n} \frac{x^* F(E)}{\mu(E)} \chi_E \\ &= \sum_{E\in\pi_n} \frac{\int_E (T^* x^*)\, d\mu}{\mu(E)} \chi_E = E_{\pi_n}(T^* x^*)|_A ,\end{aligned}$$

we have that $(x^* f_{\pi_n})|_A$ converges to $(T^* x^*)|_A$ in L_1-norm. Hence,

$$\text{Bocce-osc}(T^* x^*)|_A \equiv \frac{\int_A |(T^* x^*) - [\int_A (T^* x^*)\, d\mu/\mu(A)]|\, d\mu}{\mu(A)} \le \varepsilon .$$

Thus $T^*(B(\mathfrak{X}^*))$ satisfies the Bocce criterion, and so as needed, $\mathfrak{X}$ has the complete continuity property. □

We now verify Lemma 2.9.

Proof of Lemma 2.9. Fix A in Σ^+ and f in $L_\infty(\mu)$. Without loss of generality, f is not constant a.e. on A and $\int_A f\, d\mu = 0$. Find P and N in Σ satisfying

$$A = P \cup N , \quad \mu(P) = \frac{\mu(A)}{2} = \mu(N) , \quad P \cap N = \varnothing$$

and

$$\int_P f\, d\mu \equiv 2M > 0 , \qquad \int_N f\, d\mu \equiv -2M < 0 .$$

Approximate f by a simple function $\tilde{f}(\cdot) = \sum \alpha_i \chi_{A_i}(\cdot)$ satisfying

(1) $\|f - \tilde{f}\|_{L_\infty} < M$,
(2) $\bigcup A_i = A$ and the A_i are disjoint,
(3) $A_i \subset P$ if $i \le m$ and $A_i \subset N$ if $i > m$ for some positive integer m.

Note that

$$P = \bigcup_{i \le m} A_i \quad \text{and} \quad N = \bigcup_{i > m} A_i .$$

To find the sequence $\{\pi_n\}$, we shall first find an increasing sequence $\{\pi_n^P\}$ of partitions of P and an increasing sequence $\{\pi_n^N\}$ of partitions of N. Then π_n will be the union of π_n^P and π_n^N. To this end, for each A_i obtain an increasing sequence of partitions of A_i:

$$\begin{tikzcd}[column sep=small]
 & & & A_i \equiv E_1^{i0} \arrow[dll] \arrow[drr] & & & \\
 & E_1^{i1} \arrow[dl] \arrow[dr] & & & & E_2^{i1} \arrow[dl] \arrow[dr] & \\
E_1^{i2} & & E_2^{i2} & & E_3^{i2} & & E_4^{i2}
\end{tikzcd}$$

$$\cdots$$

such that for $n = 0, 1, 2, \ldots$ and $k = 1, \ldots, 2^n$

$$E_{2k-1}^{i\,n+1} \cup E_{2k}^{i\,n+1} = E_k^{in}, \quad E_{2k-1}^{i\,n+1} \cap E_{2k}^{i\,n+1} = \varnothing, \quad \mu(E_k^{in}) = \frac{\mu(A_i)}{2^n}.$$

For each positive integer n, let π_n^P be the partition of P given by

$$\pi_n^P = \{P_k^n : k = 1, \ldots, 2^n\} \quad \text{where } P_k^n = \bigcup_{i \le m} E_k^{in},$$

π_n^N be the partition of N given by

$$\pi_n^N = \{N_k^n : k = 1, \ldots, 2^n\} \quad \text{where } N_k^n = \bigcup_{i > m} E_k^{in},$$

and π_n be the partition of A given by

$$\pi_n = \pi_n^P \cup \pi_n^N .$$

The sequence $\{\pi_n\}$ has the desired properties. Since

$$\mu(P_k^n) = \sum_{i \le m} \frac{\mu(A_i)}{2^n} = \frac{\mu(P)}{2^n} = \frac{\mu(A)}{2^{n+1}}$$

and

$$\mu(N_k^n) = \sum_{i < m} \frac{\mu(A_i)}{2^n} = \frac{\mu(N)}{2^n} = \frac{\mu(A)}{2^{n+1}},$$

any element in π_n has measure $\mu(A)/2^{n+1}$. Thus $\bigvee \sigma(\pi_n) = \Sigma \cap A$.

As for the other properties, since $\tilde{f}$ takes the value α_i on $E_k^{in} \subset A_i$ we have

$$\begin{aligned}\int_{P_k^n} \tilde{f}\, d\mu &= \sum_{i\le m} \int_{E_k^{in}} \tilde{f}\, d\mu = \sum_{i\le m} \alpha_i \mu(E_k^{in}) \\ &= \frac{1}{2^n} \sum_{i\le m} \alpha_i \mu(A_i) = \frac{1}{2^n} \int_P \tilde{f}\, d\mu > 0\end{aligned}$$

and likewise

$$\int_{N_k^n} \tilde{f}\, d\mu = \frac{1}{2^n} \int_N \tilde{f}\, d\mu < 0 .$$

We chose $\tilde{f}$ close enough to f so that the above inequalities still hold when we replace $\tilde{f}$ by f,

$$\begin{aligned}\int_{P_k^n} f\, d\mu &\geq \int_{P_k^n} (\tilde{f} - M)\, d\mu \\ &= \frac{1}{2^n} \int_P \tilde{f}\, d\mu - M\mu(P_k^n) \\ &\geq \frac{1}{2^n} \int_P (f - M)\, d\mu - \frac{M\mu(A)}{2^{n+1}} \\ &= \frac{1}{2^n} \int_P f\, d\mu - \frac{M\mu(A)}{2^{n+1}} - \frac{M\mu(A)}{2^{n+1}} \\ &> \frac{M}{2^n} - \frac{M\mu(A)}{2^n} = \frac{M[1-\mu(A)]}{2^n} \\ &\geq 0\end{aligned}$$

and likewise

$$\int_{N_k^n} f\, d\mu < \frac{M[\mu(A) - 1]}{2^n} \leq 0 .$$

Thus the other properties of the lemma are satisfied since for each n,

$$\begin{aligned}\mu\left(\bigcup\left\{E : E \in \pi_n , \int_E f\, d\mu \geq 0\right\}\right) &= \mu\left(\bigcup\left\{E : E \in \pi_n^P\right\}\right) \\ &= \mu(P) = \frac{\mu(A)}{2}\end{aligned}$$

and so

$$\mu\left(\bigcup\left\{E : E \in \pi_n , \int_E f\, d\mu < 0\right\}\right) = \frac{\mu(A)}{2} .$$

Note that the partitions $\{\pi_n\}$ are nested by construction. □

3. Bushes and trees. In this section, we examine which Banach spaces allow certain types of bushes and trees to grow in them. First let us review some known implications.

A Banach space $\mathfrak{X}$ fails the RNP precisely when a bounded δ-bush grows in $\mathfrak{X}$. Thus if a bounded δ-tree grows in $\mathfrak{X}$ then $\mathfrak{X}$ fails the RNP. The converse is false; the Bourgain-Rosenthal space [**BR**] fails the RNP yet has no bounded δ-trees. However, if $\mathfrak{X}$ is a dual space then the converse does hold.

Bourgain [**B2**] showed that if $\mathfrak{X}$ fails the CCP then a bounded δ-tree grows in $\mathfrak{X}$. The converse is false; the dual of the James Tree space has a bounded δ-tree and the CCP. It is well-known that if a bounded δ-Rademacher tree grows in $\mathfrak{X}$ then $\mathfrak{X}$ fails the CCP. Riddle and Uhl [**RU**] showed that the converse holds in a dual space. This section's main theorem, Theorem 3.1 below, makes precise exactly which types of bushes and trees grow in a Banach space failing the CCP.

THEOREM 3.1. *The following statements are equivalent.*

(1) *$\mathfrak{X}$ fails the CCP.*
(2) *A bounded separated δ-tree grows in $\mathfrak{X}$.*
(3) *A bounded separated δ-bush grows in $\mathfrak{X}$.*
(4) *A bounded δ-Rademacher tree grows in $\mathfrak{X}$.*

The remainder of this section is devoted to proving Theorem 3.1. That (1) implies (2) will follow from Theorem 3.2 below. All the other implications are straightforward and will be verified shortly. As usual, we start with some definitions.

Perhaps it is easiest to define a bush via martingales. If $\{\pi_n\}_{n\geq 0}$ is an increasing sequence of finite positive interval partitions of $[0, 1)$ with $\bigvee \sigma(\pi_n) = \Sigma$ and $\pi_0 = \{\Omega\}$ and if $\{f_n, \sigma(\pi_n)\}_{n\geq 0}$ is an $\mathfrak{X}$-valued martingale, then each f_n has the form

$$f_n = \sum_{E\in\pi_n} x_E^n \chi_E$$

and the system

$$\{x_E^n : n = 0, 1, 2, \ldots \text{ and } E \in \pi_n\}$$

is a bush in $\mathfrak{X}$. Moreover, every bush is realized this way. A bush is a *δ-bush* if the corresponding martingale satisfies for each positive

While typing this paper, I learned that H. P. Rosenthal has also recently obtained the result that if $\mathfrak{X}$ fails the CCP then a bounded δ-Rademacher tree grows in $\mathfrak{X}$.

integer n

(i) $$\|f_n(t) - f_{n-1}(t)\| > \delta\,.$$

A bush is a *separated δ-bush* if there exists a sequence $\{x_n^*\}_{n\geq 1}$ in $S(\mathfrak{X}^*)$ such that the corresponding martingale satisfies for each positive integer n

(ii) $$|x_n^*(f_n(t) - f_{n-1}(t))| > \delta\,.$$

In this case we say that the bush is separated by $\{x_n^*\}$. Clearly a separated δ-bush is also a δ-bush.

Observation that (3) *implies* (1) *in Theorem* 3.1. If a bounded separated δ-bush grows in a subset D of $\mathfrak{X}$, then condition (ii) guarantees that the corresponding D-valued martingale $\{f_n, \sigma(\pi_n)\}$ is not Pettis-Cauchy since

$$\|f_n - f_{n-1}\|_{\text{Pettis}} \geq \int_\Omega |x_n^*(f_n(t) - f_{n-1}(t))|\,d\mu > \delta\,.$$

Thus, if a bounded separated δ-bush grows in $\mathfrak{X}$ then $\mathfrak{X}$ fails the CCP (Fact 1.5). □

If each π_n is the nth dyadic partition then we call the bush a (dyadic) tree. Let us rephrase the above definitions for this case, without the help of martingales. A *tree* in $\mathfrak{X}$ is a system of the form $\{x_k^n : n = 0, 1, \ldots; k = 1, \ldots, 2^n\}$ satisfying for $n = 1, 2, \ldots$ and $k = 1, \ldots, 2^{n-1}$

(iii) $$x_k^{n-1} = \frac{x_{2k-1}^n + x_{2k}^n}{2}\,.$$

Condition (iii) guarantees that $\{f_n\}$ is indeed a martingale. It is often helpful to think of a tree diagrammatically:

$$\begin{tikzcd}
&&&&&&& x_1^0 \arrow[dllll] \arrow[drrrr] &&&&&&& \\
&&& x_1^1 \arrow[dll] \arrow[drr] &&&&&&&& x_2^1 \arrow[dll] \arrow[drr] &&& \\
& x_1^2 \arrow[dl] \arrow[dr] &&&& x_2^2 \arrow[dl] \arrow[dr] &&&& x_3^2 \arrow[dl] \arrow[dr] &&&& x_4^2 \arrow[dl] \arrow[dr] & \\
x_1^3 && x_2^3 && x_3^3 && x_4^3 && x_5^3 && x_6^3 && x_7^3 && x_8^3
\end{tikzcd}$$

$$\cdots$$

It is easy to see that (iii) is equivalent to

$$\text{(iii}'\text{)} \qquad x_{2k-1}^n - x_{2k}^n = 2(x_{2k-1}^n - x_k^{n-1}) = 2(x_k^{n-1} - x_{2k}^n).$$

A tree $\{x_k^n\}$ is a *δ-tree* if for $n = 1, 2, \ldots$ and $k = 1, \ldots, 2^{n-1}$

$$\text{(iv)} \qquad \|x_{2k-1}^n - x_k^{n-1}\| \equiv \|x_{2k}^n - x_k^{n-1}\| > \delta.$$

An appeal to (iii′) shows that (iv) is equivalent to

$$\text{(iv}'\text{)} \qquad \|x_{2k-1}^n - x_{2k}^n\| > 2\delta.$$

A tree $\{x_k^n\}$ is a *separated δ-tree* if there exists a sequence $\{x_n^*\}_{n\geq 1}$ in $S(\mathfrak{X}^*)$ such that for $n = 1, 2, \ldots$ and $k = 1, \ldots, 2^{n-1}$

$$\text{(v)} \qquad |x_n^*(x_{2k-1}^n - x_k^{n-1})| \equiv |x_n^*(x_{2k}^n - x_k^{n-1})| > \delta.$$

Another appeal to (iii′) shows that (v) is equivalent to

$$\text{(v}'\text{)} \qquad |x_n^*(x_{2k-1}^n - x_{2k}^n)| > 2\delta.$$

Furthermore, by switching x_{2k-1}^n and x_{2k}^n when necessary, we may assume that (v′) is equivalent to

$$\text{(v}''\text{)} \qquad x_n^*(x_{2k-1}^n - x_{2k}^n) > 2\delta.$$

Since a separated δ-tree is also a separated δ-bush, (2) implies (3) in Theorem 3.1.

A tree $\{x_k^n : n = 0, 1, \ldots ; k = 1, \ldots, 2^n\}$ is called a *δ-Rademacher tree* **[RU]** if for each positive integer n

$$\left\| \sum_{k=1}^{2^{n-1}} (x_{2k-1}^n - x_{2k}^n) \right\| > 2^n \delta.$$

Perhaps a short word on the connection between Rademacher trees and the Rademacher functions $\{r_n\}$ is in order. In light of our discussion in §1, there is a one-to-one correspondence between all bounded trees in $\mathfrak{X}$ and all bounded linear operators from L_1 into $\mathfrak{X}$. If $\{x_k^n\}$ is a bounded tree in $\mathfrak{X}$ with associated operator T, then it is easy to verify that $\{x_k^n\}$ is a δ-Rademacher tree precisely when $\|T(r_n)\| > \delta$ for all positive integers n.

Fact that (4) *implies* (1) *in Theorem* 3.1 **[RU]**. Let $\{f_n\}$ be the (dyadic) martingale associated with a δ-Rademacher tree $\{x_k^n\}$. If x^*

is in $\mathfrak{X}^*$ and I_k^n is the dyadic interval $[(k-1)/2^n, k/2^n)$ then

$$
\begin{aligned}
\int_\Omega |x^*(f_n - f_{n-1})|\, d\mu &= \sum_{k=1}^{2^{n-1}} \int_{I_k^{n-1}} |x^*(f_n - f_{n-1})|\, d\mu \\
&= \sum_{k=1}^{2^{n-1}} \left[\int_{I_{2k-1}^n} |x^*(x_{2k-1}^n - x_k^{n-1})|\, d\mu \right. \\
&\qquad\qquad \left. + \int_{I_{2k}^n} |x^*(x_{2k}^n - x_k^{n-1})|\, d\mu \right] \\
&= \frac{1}{2^n} \sum_{k=1}^{2^{n-1}} [|x^*(x_{2k-1}^n - x_k^{n-1})| + |x^*(x_{2k}^n - x_k^{n-1})|] \\
&= \frac{1}{2^n} \sum_{k=1}^{2^{n-1}} |x^*(x_{2k-1}^n - x_{2k}^n)| \quad \text{by (iii}') \\
&\geq \frac{1}{2^n} \left| x^* \left(\sum_{k=1}^{2^{n-1}} (x_{2k-1}^n - x_{2k}^n) \right) \right| .
\end{aligned}
$$

From this we see that $\{f_n\}$ is not Cauchy in the Pettis norm since

$$
\begin{aligned}
\|f_n - f_{n-1}\|_{\text{Pettis}} &= \sup_{x^* \in B(\mathfrak{X}^*)} \int_\Omega |x^*(f_n - f_{n-1})|\, d\mu \\
&\geq \sup_{x^* \in B(\mathfrak{X}^*)} \frac{1}{2^n} \left| x^* \left(\sum_{k=1}^{2^{n-1}} (x_{2k-1}^n - x_{2k}^n) \right) \right| \\
&= \frac{1}{2^n} \left\| \sum_{k=1}^{2^{n-1}} (x_{2k-1}^n - x_{2k}^n) \right\| \\
&> \frac{1}{2^n} 2^n \delta = \delta .
\end{aligned}
$$

Thus if a bounded δ-Rademacher tree grows in a subset D of $\mathfrak{X}$, then there is a bounded D-valued martingale that is not Pettis-Cauchy and so $\mathfrak{X}$ fails the CCP (Fact 1.5). □

Observation that (2) *implies* (4) *in Theorem* 3.1. A separated δ-tree can easily be reshuffled so that it is a δ-Rademacher tree. For if $\{x_k^n\}$ is a separated δ-tree then we may assume, by switching x_{2k-1}^n and x_{2k}^n when necessary, that there is a sequence $\{x_n^*\}$ in $S(\mathfrak{X}^*)$ satisfying

$$x_n^*(x_{2k-1}^n - x_{2k}^n) > 2\delta .$$

With this modification $\{x_k^n\}$ is a δ-Rademacher tree since

$$\left\|\sum_{k=1}^{2^{n-1}}(x_{2k-1}^n - x_{2k}^n)\right\| \geq \left|\sum_{k=1}^{2^{n-1}} x_n^*(x_{2k-1}^n - x_{2k}^n)\right|$$
$$= \sum_{k=1}^{2^{n-1}} x_n^*(x_{2k-1}^n - x_{2k}^n) > \sum_{k=1}^{2^{n-1}} 2\delta = 2^n\delta. \qquad \square$$

It should be noted that a bounded $\tilde{\delta}$-Rademacher tree need neither be a δ-tree nor a separated δ-tree. For example, consider the c_0-valued dyadic martingale $\{f_n\}$ given by

$$f_n = (s_0, \ldots, s_n, 0, 0, \ldots),$$

where the function s_n from $[0, 1]$ into $[-1, 1]$ is defined by

$$s_n = \begin{cases} (-1)^k 2^{-n} & \text{if } \omega \in I_k^n \text{ with } k \leq 2; \\ (-1)^k & \text{if } \omega \in I_k^n \text{ with } k > 2. \end{cases}$$

The tree associated with $\{f_n\}$ is a $\frac{1}{4}$-Rademacher tree but is neither a δ-tree nor a separated δ-tree for any positive δ. Thus, since a δ-tree grows in a space failing the CCP, the notion of a separated δ-tree is more desirable than that of a δ-Rademacher tree for characterizing the CCP.

To complete the proof of Theorem 3.1, we need only to show that (1) implies (2). Towards this end, let $\mathfrak{X}$ fail the CCP. An appeal to Theorem 2.7 gives that there is a bounded non-midpoint-Bocce-dentable subset of $\mathfrak{X}$. In such a set, we can construct a separated δ-tree. This construction is made precise in the following theorem.

THEOREM 3.2. *A separated δ-tree grows in a non-midpoint-Bocce-dentable set.*

Proof. Let D be a subset of $\mathfrak{X}$ that is not midpoint Bocce dentable. Accordingly, there is a $\delta > 0$ satisfying:

(∗) for each finite subset F of D there is a norm one linear functional x_F^* such that each x in F has the form $x = (x_1 + x_2)/2$ with $|x_F^*(x_1 - x_2)| > \delta$ for a suitable choice of x_1 and x_2 in D.

We shall use the property $(*)$ to construct a tree $\{x_k^n : n = 0, 1, \ldots; k = 1, \ldots, 2^n\}$ in D that is separated by a sequence $\{x_n^*\}_{n\geq 1}$ of norm one linear functionals.

Towards this construction, pick an arbitrary x_1^0 in D. Apply $(*)$ with $F = \{x_1^0\}$ and find $x_F^* = x_1^*$. Property $(*)$ provides x_1^1 and x_2^1 in D satisfying

$$x_1^0 = \frac{x_1^1 + x_2^1}{2} \quad \text{and} \quad |x_1^*(x_1^1 - x_2^1)| > \delta .$$

Next apply $(*)$ with $F = \{x_1^1, x_2^1\}$ and find $x_F^* = x_2^*$. For $k = 1$ and 2, property $(*)$ provides x_{2k-1}^2 and x_{2k}^2 in D satisfying

$$x_k^1 = \frac{x_{2k-1}^2 + x_{2k}^2}{2} \quad \text{and} \quad |x_2^*(x_{2k-1}^2 + x_{2k}^2)| > \delta .$$

Instead of giving a formal inductive proof we shall be satisfied by finding $x_1^3, x_2^3, \ldots, x_8^3$ in D along with x_3^*. Apply $(*)$ with $F = \{x_1^2, x_2^2, x_3^2, x_4^2\}$ and find $x_F^* = x_3^*$. For $k = 1, 2, 3$ and 4, property $(*)$ provides x_{2k-1}^3 and x_{2k}^3 in D satisfying

$$x_k^2 = \tfrac{1}{2}(x_{2k-1}^3 + x_{2k}^3) \quad \text{and} \quad |x_3^*(x_{2k-1}^3 - x_{2k}^3)| > \delta .$$

It is now clear that a separated δ-tree grows in such a set D. □

REMARK 3.3. Theorem 2.7 presents several dentability characterizations of the CCP. Our proof that (1) implies (2) in Theorem 3.1 uses part of one of these characterizations; namely, if $\mathfrak{X}$ fails the CCP then there is a bounded non-midpoint-Bocce-dentable subset of $\mathfrak{X}$. If $\mathfrak{X}$ fails the CCP, then there is also a bounded non-weak-norm-one-dentable subset of $\mathfrak{X}$ (Theorem 2.7). In the closed convex hull of such a set we can construct a martingale that is not Pettis-Cauchy [**PU**, Theorem II.7]; furthermore, the bush associated with this martingale is a separated δ-bush. However, it is unclear whether this martingale is a dyadic martingale, thus the separated δ-bush may not be a tree. If $\mathfrak{X}$ fails the CCP, then there is also a bounded non-Bocce-dentable subset of $\mathfrak{X}$ (Theorem 2.7). In such a set we can construct a martingale that is not Pettis-Cauchy (Theorem 2.8), but it is unclear whether the bush associated with this martingale is a separated δ-bush.

REMARK 3.4. The δ-tree that Bourgain [**B2**] constructed in a space failing the CCP is neither a separated δ-tree nor a δ-Rademacher tree since the operator associated with his tree is Dunford-Pettis.

4. Localization. We now localize the results thus far. We define the CCP for bounded subsets of $\mathfrak{X}$ by examining the behavior of certain bounded linear operators from L_1 into $\mathfrak{X}$. Before determining precisely which operators let us set some notation and consider an example.

Let $F(L_1)$ denote the positive face of the unit ball of L_1, i.e.

$$F(L_1) = \{f \in L_1 : f \geq 0 \text{ a.e. and } \|f\| = 1\},$$

and let Δ denote the subset of $F(L_1)$ given by

$$\Delta = \left\{ \frac{\chi_E}{\mu(E)} : E \in \Sigma^+ \right\}.$$

Note that the L_1-norm closed convex hull of Δ is $F(L_1)$.

Some care is needed in localizing the CCP. The example below (due to Stegall) illustrates the trouble one can encounter in localizing the RNP.

EXAMPLE 4.1. We would like to define the RNP for sets in such a way that if a subset D has the RNP then the $\overline{\text{co}}\, D$ also has the RNP. For now, let us agree that a subset D of $\mathfrak{X}$ has the RNP if all bounded linear operators from L_1 into $\mathfrak{X}$ with $T(\Delta) \subset D$ are representable. Let $\mathfrak{X}$ be any separable Banach space without the RNP (e.g. L_1). Renorm $\mathfrak{X}$ to be a strictly convex Banach space. Let D be the unit sphere of $\mathfrak{X}$ and $T\colon L_1 \to \mathfrak{X}$ satisfy $T(\Delta) \subset D$. Since $\mathfrak{X}$ is strictly convex, it is easy to verify that $T(\Delta)$ is a singleton in $\mathfrak{X}$. Thus T is representable and so D has the RNP. If this is to imply that $\overline{\text{co}}\, D$ also has the RNP, then the unit ball of $\mathfrak{X}$ would have the RNP. But if the unit ball of $\mathfrak{X}$ has the RNP then $\mathfrak{X}$ has the RNP; but, $\mathfrak{X}$ fails the RNP. The same problem arises if we replace $T(\Delta) \subset D$ by either $T(F(L_1)) \subset D$ or $T(B(L_1)) \subset D$.

Because of such difficulties, we localize propeties to nonconvex sets by considering their closed convex hull. We now make precise the localized definitions.

DEFINITION 4.2. If D is a closed bounded convex subset of $\mathfrak{X}$, then D has the *complete continuity property* if all bounded linear operators T from L_1 into $\mathfrak{X}$ satisfying $T(\Delta) \subset D$ are Dunford-Pettis. If D is an arbitrary bounded subset of $\mathfrak{X}$, then D has the *complete continuity property* if the $\overline{\text{co}}\, D$ has the complete continuity property.

The RNP for subsets is defined similarly. We obtain equivalent formulations of the above definitions by replacing $T(\Delta) \subset D$ with $T(F(L_1)) \subset D$. Because of the definitions we restrict out attention to closed bounded convex subsets of $\mathfrak{X}$.

We can derive a martingale characterization of the CCP for a closed bounded convex subset K of $\mathfrak{X}$. As in §1, fix an increasing sequence $\{\pi_n\}_{n\geq 0}$ of finite positive interval partitions of Ω such that $\bigvee \sigma(\pi_n) = \Sigma$ and $\pi_0 = \{\Omega\}$. Set $\mathscr{F}_n = \sigma(\pi_n)$. It is easy to see that a martingale $\{f_n, \mathscr{F}_n\}$ takes values in K precisely when the corresponding bounded linear operator T satisfies $T(\Delta) \subset K$. In light of Fact 1.5, we now have the following fact.

FACT 4.3. If K is a closed bounded convex subset of $\mathfrak{X}$, then K has the CCP precisely when all K-valued martingales are Cauchy in the Pettis norm.

Theorem 2.7 localizes to provide the following characterization.

THEOREM 4.4. *Let K be a closed bounded convex subset of $\mathfrak{X}$. The following statements are equivalent.*

(1) *K has the CCP.*
(2) *All the subsets of K are weak-norm-one dentable.*
(3) *All the subsets of K are midpoint Bocce dentable.*
(4) *All the subsets of K are Bocce dentable.*

Proof. It is clear from the definitions that (2) implies (3) and that (4) implies (3). Theorem 2.8 and Fact 4.3 show that (1) implies (4) while [**PU**, Theorem II.7] and Fact 4.3 show that (1) implies (2). So we only need to show that (3) implies (1). For this, slight modifications in the proof of Theorem 2.10 suffice.

Let all subsets of K be midpoint Bocce dentable. Fix a bounded linear operator T from L_1 into $\mathfrak{X}$ satisfying $T(\Delta) \subset K$. We shall show that the subset $T^*(B(\mathfrak{X}^*))$ of L_1 satisfies the Bocce criterion. Then an appeal to Fact 1.1.7 gives that K has the complete continuity property. To this end, fix $\varepsilon > 0$ and B in Σ^+. Let F denote the vector measure from Σ into $\mathfrak{X}$ given by $F(E) = T(\chi_E)$. Since $T(\Delta) \subset K$, the set $\{\frac{F(E)}{\mu(E)} : E \subset B \text{ and } E \in \Sigma^+\}$ is a subset of K and thus is midpoint Bocce dentable. The proof now proceeds as the proof of Theorem 2.10. □

Towards a localized tree characterization, let K be a closed bounded convex subset of $\mathfrak{X}$. If K fails the CCP, then there is a subset of K that is not midpoint Bocce dentable (Theorem 4.4) and hence a separated δ-tree grows in K (Theorem 3.2). A separated δ-tree is a separated δ-bush and, with slight modifications, a δ-Rademacher tree. In light of our discussion in §3, if a separated δ-bush or a δ-Rademacher tree grows in K, then the associated K-valued martingale is not Pettis-Cauchy and so K fails the CCP (Fact 4.3). Thus Theorem 3.1 localizes to provide the following characterization.

THEOREM 4.5. *Let K be a closed bounded convex subset of $\mathfrak{X}$. The following statements are equivalent.*

(1) *K fails the CCP.*
(2) *A separated δ-tree grows in K.*
(3) *A separated δ-bush grows in K.*
(4) *A δ-Rademacher tree grows in K.*

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Received June 9, 1989 and in revised form November 14, 1989. This paper is part of the Ph.D. work of the author, who is grateful for the guidance of her advisor Professor J. J. Uhl, Jr.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 148, No. 1, 1991

ULTRAPRODUCTS AND SMALL BOUND PERTURBATIONS

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It is very well-known that two real Banach spaces are isometric if and only if they are linearly-isometric or that two uniform algebras are linearly-isometric if and only if they are isomorphic as algebras. These and similar classical "isometric" results have been extended by E. Behrends, M. Cambern, J. Gevirtz, R. Rochberg, the author and others to "almost isometric" cases. Proofs of the extended results are usually quite technical. In this note we show that using ultraproducts of Banach spaces we can in some cases deduce an "almost isometric" result from the classical one in just a few lines.

0. It is a well-known classical result of Ulam that an isometry T from a real Banach space X onto a real Banach space Y with $T(0) = 0$ is automatically linear. More recently, in 1982, Gevirtz [**5**] proved that this result is stable:

THEOREM. *Let T be a map from a Banach space X onto a Banach space Y with $T(0) = 0$ such that*

$$(1-\varepsilon)\|x-y\| \leq \|Tx-Ty\| \leq (1+\varepsilon)\|x-y\|, \quad \text{for } x, y \in X,$$

then

$$\|T(x+y)-Tx-Ty\| \leq \varepsilon'(\|x\|+\|y\|), \quad \text{for } x, y \in X$$

where $\varepsilon' \to 0$ as $\varepsilon \to 0$.

The proof of the above result repeats, roughly speaking, the basic idea of Ulam's proof but is much longer and much more technical. The intent of this note is to draw attention to the method of ultraproducts of Banach spaces. Using this method we can extend in just a few lines some "isometric" results to "almost isometric" cases. This includes the theorem of Gevirtz.

1. In this section we give a definition of the ultraproduct of Banach spaces and list some basic results. We refer to the paper by Heinrich [**6**] for a more extended exposition.

We denote by $\mathbb{N}$ the set of all positive integers and by $\mathscr{F}$ a non-prime ultrafilter of subsets of $\mathbb{N}$. That is, we assume that $\mathscr{F}$ is a proper subset of $2^{\mathbb{N}}$ which does not contain a one point set and such that

$$\begin{array}{ll} A \cap B \in \mathcal{F} & \text{if } A, B \in \mathcal{F}, \\ B \in \mathcal{F} & \text{if } B \supseteq A \in \mathcal{F}, \\ A \in \mathcal{F} \text{ or } B \in \mathcal{F} & \text{if } A \cup B \in \mathcal{F}. \end{array}$$

Throughout this paper we assume $\mathcal{F}$ is fixed.

Definition. Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence of complex numbers. We write

$$\lim_{\mathcal{F}} a_n = g \quad \text{if } \forall \varepsilon > 0 \; \exists A \in \mathcal{F} \; \forall n \in A \quad |a_n - g| \leq \varepsilon .$$

It is easy to observe that $\lim_{\mathcal{F}} a_n$ exists for any bounded sequence of complex numbers. To get a useful alternative definition let $p \in \beta\mathbb{N} \setminus \mathbb{N}$, where $\beta\mathbb{N}$ is the maximal compactification of $\mathbb{N}$. Since $a = (a_n)_{n=1}^{\infty}$ is a continuous bounded function on $\mathbb{N}$ it can be uniquely extended to a continuous function $\tilde{a}$ on $\beta\mathbb{N}$. We have $\tilde{a}(p) = \lim_{\mathcal{F}} a_n$ where $\mathcal{F}$ is the set of all neighborhoods of p, restricted to $\mathbb{N}$.

Definition. Let $(X_n)_{n=1}^{\infty}$ be a sequence of normed spaces and let $m(X_n)$ be the space of all norm bounded sequences $(x_n)_{n=1}^{\infty}$ with $x_n \in X_n$. We introduce a seminorm $\|\cdot\|_{\mathcal{F}}$ on $m(X_n)$ by $\|(x_n)_{n=1}^{\infty}\|_{\mathcal{F}} = \lim_{\mathcal{F}} \|x_n\|$. The ultraproduct $\prod_{\mathcal{F}} X_n$ of $(X_n)_{n=1}^{\infty}$ is the quotient space of the space $m(X_n)$ mod $\ker \|\cdot\|_{\mathcal{F}}$.

Definition. Let X_n, Y_n, $n \in \mathbb{N}$, be sequences of normed spaces and let $T_n \colon X_n \to Y_n$ be a sequence of maps such that

$$\|T_n(x_n)\| \leq K\|x_n\| \quad \text{for } n \in \mathbb{N}, \; x_n \in X_n . \tag{1}$$

(We do not assume that T_n are linear.) Let $\prod_{\mathcal{F}} T_n$ denote the map from $\prod_{\mathcal{F}} X_n$ into $\prod_{\mathcal{F}} Y_n$ defined by $\prod_{\mathcal{F}} T_n([x_n]_{\mathcal{F}}) = [T_n(x_n)]_{\mathcal{F}}$.

For $(x_n)_{n=1}^{\infty} \in m(X_n)$ we denote by $[x_n]_{\mathcal{F}}$ the corresponding element of $\prod_{\mathcal{F}} X_n$. If X_n are equal to a fixed normed space X then $\prod_{\mathcal{F}} X = \prod_{\mathcal{F}} X_n$ is called an ultrapower of X.

From (1) it follows that $\prod_{\mathcal{F}} T_n$ is well-defined and that

$$\left\| \prod_{\mathcal{F}} T_n([x_n]_{\mathcal{F}}) \right\|_{\mathcal{F}} \leq K \|[x_n]_{\mathcal{F}}\|_{\mathcal{F}}, \qquad [x_n]_{\mathcal{F}} \in \prod_{\mathcal{F}} X_n . \tag{2}$$

Note that if X_n is not only a Banach space but also a Banach algebra then we can carry this multiplicative structure to $\prod_{\mathcal{F}} X_n$ by defining

$$[x_n]_{\mathcal{F}} \cdot [y_n]_{\mathcal{F}} = [x_n \cdot y_n]_{\mathcal{F}}, \quad \text{for } [x_n]_{\mathcal{F}}, [y_n]_{\mathcal{F}} \in \prod_{\mathcal{F}} X_n .$$

Here is a list of some basic properties of ultraproducts:

1°. $\prod_{\mathscr{F}} X_n$ is a Banach space, that is $\prod_{\mathscr{F}} X_n$ is complete even if the X_n are not.

2°. A map from X into $\prod_{\mathscr{F}} X$ defined by $x \mapsto [x]_{\mathscr{F}}$ (mapping x onto the sequence constantly equal to x) is an isometric embedding of X into $\prod_{\mathscr{F}} X$. This map is surjective if and only if X is finite dimensional.

3°. If $T_n \colon X_n \to Y_n$ are all linear then $\prod_{\mathscr{F}} T_n$ is a linear map with $\|\prod_{\mathscr{F}} T_n\| = \lim_{\mathscr{F}} \|T_n\|$.

4°. If $T_n \colon X_n \to Y_n$ is a sequence of invertible maps with

$$\sup\left\{\frac{\|T_n x_n\|}{\|x_n\|}, \frac{\|x_n\|}{\|T_n x_n\|} : x_n \in X_n, \ x_n \neq 0\right\} < \infty$$

then $\prod_{\mathscr{F}} T_n$ is invertible and $(\prod_{\mathscr{F}} T_n)^{-1} = \prod_{\mathscr{F}}(T_n^{-1})$.

5°. If $X_n = C(K_n)$ then $\prod_{\mathscr{F}} X_n = C(K)$, where K is compact.

6°. If X_n are closed subalgebras of $C(K_n)$, then $\prod_{\mathscr{F}} X_n$ is a closed subalgebra of $C(K)$.

7°. With any element $[x_n^*]_{\mathscr{F}}$ of $\prod_{\mathscr{F}} X_n^*$ we can associate a linear functional on $\prod_{\mathscr{F}} X_n$ by putting $[x_n^*]_{\mathscr{F}}([x_n]) = \lim_{\mathscr{F}} x_n^*(x_n)$ for $[x_n]_{\mathscr{F}} \in \prod_{\mathscr{F}} X_n$. This defines a linear isometric embedding of $\prod_{\mathscr{F}} X_n^*$ into $(\prod_{\mathscr{F}} X_n)^*$ which is surjective if the spaces X_n are superreflexive.

Proofs of properties 1°–7° are easy exercises, we show here only 3° and 4° to get some additional information about the structure of the algebra $\prod_{\mathscr{F}} A_n \subseteq \prod_{\mathscr{F}} C(K_n)$. The algebra $m(C(K_n))$ consists of all continuous bounded functions defined on $(\bigcup_{n=1}^{\infty} K_n)$, the disjoint union of K_n. So $m(C(K_n))$ can be identified with the algebra of all continuous funtions on $S = \beta(\bigcup_{n=1}^{\infty} K_n)$. The kernel of the seminorm $\|(f_n)\|_{\mathscr{F}} = \lim_{\mathscr{F}} \|f_n\|$ on $m(C(K_n)) = C(S)$ is a closed ideal. Any closed ideal J in $C(S)$ is of the form $J = J_K = \{f \in C(S) : f_{|K} \equiv 0\}$ where $K = \overline{K} \subseteq S$. We also have $C(S)/J_k \cong C(K)$. Hence, $\prod_{\mathscr{F}} C(K_n)$ can be identified with a subalgebra of $C(K)$ where $K \subset \beta(\bigcup K_n) \setminus \bigcup K_n$. Now, since A_n is a subalgebra of $C(K_n)$, $\prod_{\mathscr{F}} A_n$ is a subalgebra of $C(K)$.

2. In this section we give some applications of the method of ultraproducts. We start with the proof of the theorem of Gevirtz. Assume the result is false. Then there are sequences of Banach spaces X_n and

Y_n, a sequence $T_n\colon X_n \to Y_n$ of surjective maps with $T_n0 = 0$ and

$$(3)\qquad \left(1-\frac{1}{n}\right)\|x-y\| \le \|T_nx - T_ny\| \le \left(1+\frac{1}{n}\right)\|x-y\|, \qquad x, y \in X_n,$$

and sequences $x_n \in X_n$, $y_n \in Y_n$ with

$$(4)\qquad \|T_n(x_n+y_n) - T_nx_n - T_ny_n\| \ge \varepsilon'(\|x_n\| + \|y_n\|), \qquad n \in \mathbb{N},$$

where $\varepsilon' > 0$ is a fixed number.

Without loss of generality, by putting

$$\widetilde{T}_n(x) = \frac{1}{\|x_n\| + \|y_n\|} T_n((\|x_n\| + \|y_n\|)x), \qquad x \in X_n,$$

and

$$\tilde{x}_n = \frac{x_n}{\|x_n\| + \|y_n\|}, \qquad \tilde{y}_n = \frac{y_n}{\|x_n\| + \|y_n\|}$$

in place of T_n, x_n and y_n, respectively, we can assume that $\|x_n\| + \|y_n\| = 1$ for all $n \in \mathbb{N}$.

Put

$$T_\infty = \prod_{\mathscr{F}} T_n : \prod_{\mathscr{F}} X_n \to \prod_{\mathscr{F}} Y_n, \quad x_\infty = [x_n]_{\mathscr{F}}, \quad y_\infty = [y_n]_{\mathscr{F}}.$$

By (3) and the property 4°, T_∞ is a surjective isometry. By the theorem of Ulam T_∞ is linear, but from (4) we get

$$\begin{aligned}&\|T_\infty(x_\infty + y_\infty) - T_\infty(x_\infty) - T_\infty(y_\infty)\|_{\mathscr{F}}\\ &\quad = \lim_{\mathscr{F}} \|T_n(x_n+y_n) - T_n(x_n) - T_n(y_n)\| \ge \varepsilon' > 0\end{aligned}$$

which is a contradiction.

To formulate the next result we need some definitions.

By a uniform algebra we mean a sup-norm closed subalgebra with unit, of the algebra $C(K)$ of all continuous complex functions defined on a compact set K.

A linear map T from a Banach space X onto a Banach space Y is called ε-isometry if $\|T\| \le 1+\varepsilon$ and $\|T^{-1}\| \le 1+\varepsilon$.

A linear map T from a Banach algebra A into a Banach algebra B is called ε-multiplicative if

$$(5)\qquad \|T(fg) - T(f)\cdot T(g)\| \le \varepsilon\|f\|\,\|g\|, \qquad f, g \in A.$$

It is well-known that, in general, a linear and multiplicative map $T\colon A \to B$ need not be continuous **[14]**. It is also well-known that, if

B is commutative and semisimple then a linear, multiplicative map $T: A \to B$ is automatically continuous [**15**]. The same is true for ε-multiplicative maps. In [**8**, p. 37] it is shown that an ε-multiplicative map T from a Banach algebra A into a uniform algebra is automatically continuous, so by (5) we have $\|T\| \le 1+\varepsilon$. The general case of a semisimple commutative algebra B follows easily from this by the same arguments (closed graph theorem) as in the multiplicative case.

THEOREM 2. *Let A and B be uniform algebras. If $T: A \to B$ is ε-multiplicative then T is an ε'-isometry. If $T: A \to B$ is an ε-isometry then $\widetilde{T}: A \to B$ defined by $\widetilde{T}(f) = (Tf)(T1)^{-1}$ is ε''-multiplicative. Here ε, ε', ε'' tend to zero simultaneously.*

This theorem was proved in 1979 by R. Rochberg [**13**] under some additional assumptions about A and B. The general case was proved in 1983 in [7] (see also [**8**, p. 35]). On the other hand, the isometric case of this theorem, that is the case where $\varepsilon = \varepsilon' = \varepsilon'' = 0$, is a classical result proven in 1959 by Nagasawa [**12**]. Using ultraproducts we can simply reduce the general case to the isometric one. We show here, by contradiction, the implication in one direction, the second being equally obvious.

Assume $T_n: A_n \to B_n$ is a $\frac{1}{n}$-isometry between function algebras A_n and B_n.

The map $\prod_{\mathscr{F}} T_n: \prod_{\mathscr{F}} A_n \to \prod_{\mathscr{F}} B_n$ is a linear surjective isometry between function algebras so, by the classical result [**15**] $\prod_{\mathscr{F}} T_n([1]_{\mathscr{F}}) = [T_n(1)]_{\mathscr{F}}$ is an invertible norm one element of $\prod_{\mathscr{F}} B_n$, with the norm of its inverse equal also to one. Let F_n be an element of $\mathfrak{M}(B_n)$, the space of all linear-multiplicative functionals on B_n. Since $\prod_{\mathscr{F}} F_n \in \mathfrak{M}(\prod_{\mathscr{F}} B_n)$, we have

$$1 = \left|\prod_{\mathscr{F}} F_n([T_n(1)]_{\mathscr{F}})\right| = \left|\lim_{\mathscr{F}} F_n(T_n(1))\right|,$$

so for all sufficiently large n, $T_n(1)$ is invertible in B_n with

$$\lim_{\mathscr{F}} \|T_n(1)\| = 1 \quad \text{and} \quad \lim_{\mathscr{F}} \|(T_n(1))^{-1}\| = 1.$$

Hence, we can define a map $\widetilde{T}_n: A_n \to B_n$ by $\widetilde{T}_n f = (T_n f)(T_n(1))^{-1}$ and we have $\lim_{\mathscr{F}} \|\widetilde{T}_n\| = 1 = \lim_{\mathscr{F}} \|\widetilde{T}_n^{-1}\|$.

Assume there are $\varepsilon_0 > 0$ and $f_n, g_n \in A_n$, $\|f_n\| = 1 = \|g_n\|$ such that

$$\|\widetilde{T}_n(f_n \cdot g_n) - \widetilde{T}_n(f_n)\widetilde{T}_n(g_n)\| \ge \varepsilon_0.$$

Then

$$\left\| \prod_{\mathscr{F}} \widetilde{T}_n([f_n]_{\mathscr{F}}[g_n]_{\mathscr{F}}) - \prod_{\mathscr{F}} \widetilde{T}_n([f_n]_{\mathscr{F}}) \prod_{\mathscr{F}} \widetilde{T}_n([g_n]_{\mathscr{F}}) \right\| \geq \varepsilon_0 > 0. \tag{6}$$

On the other hand $\prod_{\mathscr{F}} \widetilde{T}_n$ is a linear isometry from $\prod_{\mathscr{F}} A_n$ onto $\prod_{\mathscr{F}} B_n$ which maps the unit onto the unit, so again by the Nagasawa theorem it is multiplicative, which contradicts (6).

A linear projection $P : X \to X$ is called ε-L^p-projection, $1 \leq p \leq \infty$, if

$$(1 - \varepsilon)\|x\| \leq (\|Px\|^p + \|x - Px\|^p)^{1/p} \leq (1 + \varepsilon)\|x\|, \qquad x \in X,$$

with the obvious modification for $p = \infty$. L^p-projections and ε-L^p-projections play important roles in studying structure, isometries and small-bound isomorphisms of various Banach spaces. The main result here is due to E. Behrends [**2**]. He proved that if $\dim X > 2$ and $p \neq 2$ then X admits a non-trivial L^p-projection for at most one p and any two such projections commute. In [**4**] this result was extended to ε-L^p-projection as follows.

THEOREM 3. *Let X be a Banach space with* $\dim X > 2$. *Let* $1 \leq p, q \leq \infty$, $p \neq 2$, *let* $P, Q : X \to X$ *be ε-L^p and ε-L^q projections, respectively. Then*

$$|p - q| \leq \varepsilon'(p) \quad \textit{and} \quad \|PQ - QP\| \leq \varepsilon'(p), \quad \textit{where } \varepsilon' \to 0 \textit{ as } \varepsilon \to 0.$$

Using the method of ultraproducts we can deduce the above theorem from the result of Behrends in what is now an obvious way. It is enough to notice that $\prod_{\mathscr{F}} P_n$ is an L^p-projection if P_n is an $\frac{1}{n}$-L^{p_n}-projection and $p_n \to p$, as $n \to \infty$.

There are a number of open questions related to the problems discussed here. We conjecture just two of them.

Conjecture 1. Let A be a uniform algebra. Let F be a linear functional on A such that

$$|F(f \cdot g) - F(f)F(g)| \leq \varepsilon\|f\|\,\|g\|, \qquad f, g \in A.$$

Then there is a linear and multiplicative functional G defined on A such that

$$\|G - F\| \leq \varepsilon', \quad \text{where } \varepsilon' \to 0 \text{ as } \varepsilon \to 0.$$

REMARK. The question whether an almost multiplicative functional is close to a multiplicative one was raised in [**8**], in connection with the theory of perturbations of Banach algebras. It was noticed there that any such functional is automatically continuous [8]. B. E. Johnson [10] gave an example of a non-uniform, commutative Banach algebra which does not have the property described in the above conjecture. He proved [11] also that $C(K)$ algebras and the disc algebra $A(D)$ have this property. The problem is open, for uniform algebras in general, e.g. for $H^\infty(D)$—the algebra of all bounded analytic functions defined on the unit disc.

Conjecture 2. Let X, Y be real Banach spaces such that there is a surjective map $T: X \to Y$ with

$$(1-\varepsilon)\|x-y\| \leq \|Tx - Ty\| \leq (1+\varepsilon)\|x-y\|, \qquad \text{for } x, y \in X,$$

where $0 < \varepsilon \leq \varepsilon_0$ and ε_0 is an absolute constant. Then X and Y are linearly isomorphic.

REMARK. The above statement is known to be true for certain special classes of Banach spaces like uniform algebras [9]. It is also known that this is false, even for $C(K)$ spaces, if we do not assume that ε is close to zero. By the theorem of Gevirtz to prove the conjecture it is enough to show that an almost linear map is close to a linear one.

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Received June 15, 1989.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 148, No. 1, 1991

THE LOCAL STRUCTURE OF SOME MEASURE-ALGEBRA HOMOMORPHISMS

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Extending classical theorems, we obtain representations for bounded linear transformations from L-spaces to Banach spaces with a separable predual. In the case of homomorphisms from a convolution measure algebra to a Banach algebra, we obtain a generalization of Šreĭder's representation of the Gelfand spectrum via generalized characters. The homomorphisms from the measure algebra on a LCA group, G, to that on the circle are analyzed in detail. If the torsion subgroup of G is denumerable, one consequence is the following necessary and sufficient condition that a positive finite Borel measure on G be continuous: $\exists\gamma_\alpha \to \infty$ in $\widehat{G}$ such that $\forall n \neq 0$ $\hat{\mu}(\gamma_\alpha^n) \to 0$.

1. Introduction. Given a measurable space X and a (bounded) complex measure μ on X, the Banach space dual of $L^1(\mu)$ is commonly represented as $L^\infty(\mu)$. We shall call M an *L-space* on X if M is a Banach space of complex measures on X (under the measure norm) such that $\nu \ll \mu \in M \Rightarrow \nu \in M$ **[Sc]**. Šreĭder **[Šr]** gave a representation of the dual M^* of M as a space of so-called generalized functions, i.e., families of functions $f_\mu \in L^\infty(\mu)$ satisfying

$$\nu \ll \mu \Rightarrow f_\nu = f_\mu \quad \nu\text{-a.e.}, \tag{1.1}$$

$$\sup_{\mu\in M} \|f_\mu\|_{L^\infty(\mu)} < \infty. \tag{1.2}$$

The representation of M^*, like that of $L^1(\mu)^*$, is by integration:

$$\mu \mapsto \int f_\mu \, d\mu.$$

Now, given two Banach spaces, B_1 and B_2, we denote by $L(B_1, B_2)$ the Banach space of bounded linear transformations from B_1 to B_2. Since $M^* = L(M, \mathbf{C})$, we may ask, in generalizing the above, for a representation of $L(M, B)$, where B is an arbitrary Banach space. Again, the case where $M = L^1(\mu)$ is classical **[DS]**; here, the hypothesis that B has a separable predual is made. In §2, we extend this theorem to general L-spaces M in a manner similar to Šreĭder's representation above. In essence, functions are replaced by

B-valued functions. Our treatment will be entirely self contained, thus giving an apparently new proof of [**DS**, Theorem VI.8.6]. However, another point of view could be adopted. Namely, if we use the Radon-Nikodym theorem to identify $L(\mu) = \{\nu \ll \mu : \nu \text{ bounded}\}$ with $L^1(\mu)$, then we may regard an L-space M as the direct limit $\lim_{\mu \in M} L^1(\mu)$, where M is directed by $\ll$ and for $\nu \ll \mu$, $L^1(\nu)$ is included in $L^1(\mu)$. Now $L(\cdot, B)$ is a functor from the category of Banach spaces to its opposite category and, furthermore, is easily checked to be a left adjoint. Since left adjoints preserve direct limits and inverse limits are dual to direct limits, it follows that $L(M, B)$ is the inverse limit $\lim_{\mu \in M} L(L^1(\mu), B)$, where, for $\nu \ll \mu$, $L(L^1(\mu), B)$ is mapped by restriction to $L(L^1(\nu), B)$. Hence, given a representation of $L(L^1(\mu), B)$ (as in [**DS**]) and a construction of inverse limits, we may obtain a representation of $L(M, B)$. This amounts to the same as our Theorem 2.1.

Now Šreĭder was actually interested in representing ΔM, the multiplicative linear functionals on M, when M was a convolution measure algebra on a locally compact abelian group. He showed that in addition to (1.1) and (1.2), the following property was necessary and sufficient for f_μ to define an element of ΔM:

$$\text{(1.3)} \qquad \forall \mu, \nu \geq 0 \; f_{\mu * \nu}(xy) = f_\mu(x) f_\nu(y) \quad \mu \times \nu\text{-a.e. } [(x, y)].$$

We, too, are mainly interested in the subset of homomorphisms $\operatorname{Hom}(M, B) \subseteq L(M, B)$ when B is a Banach algebra. A similar condition to (1.3) is found in Theorem 3.2. In particular, when $M = M(G)$, the complex Borel measures on a locally compact abelian group, G, and $B = M(\mathbf{T})$, $\mathbf{T}$ the circle, $\operatorname{Hom}(M(G), M(\mathbf{T}))$ contains in a natural way $\operatorname{Hom}(G, \mathbf{T}) = \widehat{G}$. The closure of $\widehat{G}$ in a certain weak topology is related to the behavior of Fourier transforms at infinity and contains much information about a measure μ when regarded locally, i.e., when restricted to $L(\mu)$, or, what is the same, when viewed via the Šreĭder representation. For example, this analysis will lead to the following surprising result: if the torsion subgroup of G is denumerable, then a positive measure $\mu \in M(G)$ is continuous iff there is a net $\{\gamma_\alpha\} \subseteq \widehat{G}$ tending to infinity such that for all $n \neq 0$, $\lim_\alpha \hat{\mu}(\gamma_\alpha^n) = 0$. Characterizations of certain other classes of measures are found in §4; these have proved useful in [**KL**] and [**L4**]. Other analyses of the local structure of the closure of $\widehat{G}$ for certain μ can be found in [**L3**], [**L4**], and [**L5**]. The local structure of $\widehat{G}$ is also related to asymptotic distribution; this relationship, described here, has been used in [**KL**] and [**L4**].

The Šreĭder representation, Theorem 3.2, has been given before in **[IgK]** for the case $\mathrm{Hom}(M, M(\mathbf{T}))$, M being an L-subalgebra of $M(\mathbf{T})$, though in slightly different notation. An alternative representation for $\mathrm{Hom}(M, M(G))$, where M is a semisimple commutative convolution measure algebra in the sense of Taylor and G is a compact abelian group, analogous to Taylor's representation of ΔM via a structure semigroup, has been given in **[InK]**.

2. The Šreĭder representation of linear transformations. Suppose that M is an L-space on a measurable space X and that B is a Banach space with a separable predual, B_*. Let $\mathscr{B}(X, B)$ denote the set of maps $f: X \to B$ which are bounded in B-norm and measurable when B is given the weak* topology from B_*. If $f \in \mathscr{B}(X, B)$ and $\mu \in M$, there is a unique element $\int f\, d\mu \in B$ defined by the relation

$$\forall b_* \in B_* \ \left\langle b_*, \int f\, d\mu \right\rangle = \int_X \langle b_*, f(x)\rangle\, d\mu(x).$$

If D is a countable dense set in the unit ball of B_*, then the equation

$$\|f(x)\|_B = \sup_{b_* \in D} |\langle b_*, f(x)\rangle|$$

shows that $\|f(\cdot)\|_B$ is measurable. It is clear that

$$\left\| \int f\, d\mu \right\|_B \leq \int \|f\|_B\, d|\mu|.$$

The set of equivalence classes of $\mathscr{B}(X, B)$ under equality μ-a.e. will be denoted $\mathscr{B}(X, B)_\mu$, although this distinction will often be ignored.

The following theorem, which we shall term the *Šreĭder representation*, associates to each element of $L(M, B)$ a certain family of maps in $\mathscr{B}(X, B)$. We denote the image of $\mu \in M$ under $\Sigma \in L(M, B)$ by Σ_μ.

THEOREM 2.1. *Let M be an L-space and B a Banach space with a separable predual. There is a bijection between $L(M, B)$ and the set of elements $\{b_{\cdot, \mu}\}_{\mu \in M} \in \prod_{\mu \in M} \mathscr{B}(X, B)_\mu$ which satisfy*

(i) $$\sup_{\mu \in M} \| \|b_{x, \mu}\|_B \|_{L^\infty(\mu)} < \infty$$

and

(ii) $$\forall \nu \ll \mu \in M \ b_{x, \nu} = b_{x, \mu} \quad \nu\text{-a.e. } [x]$$

such that if Σ *corresponds to* $\{b_{\cdot,\mu}\}_{\mu\in M}$ (*written* $\Sigma \sim b_{\cdot,\cdot}$), *then*

$$\text{(iii)} \qquad \forall\mu \in M \quad \Sigma_\mu = \int b_{x,\mu}\, d\mu(x)$$

and

$$\text{(iv)} \qquad \|\Sigma\|_{L(M,B)} = \sup_{\mu\in M} \|\, \|b_{x,\mu}\|_B \|_{L^\infty(\mu)} .$$

Proof. Given $\{b_{\cdot,\mu}\}$ satisfying (i) and (ii), define Σ by (iii). If $\mu, \nu \in M$, then by (ii), we have $b_{x,\mu} = b_{x,|\mu|+|\nu|}$ μ-a.e., whence $\Sigma_\mu = \int b_{x,|\mu|+|\nu|}\, d\mu(x)$. In conjunction with similar equations for Σ_ν and $\Sigma_{\mu+\nu}$, this equation shows that $\Sigma_\mu + \Sigma_\nu = \Sigma_{\mu+\nu}$. Similarly, for $\alpha \in \mathbf{C}$, $\Sigma_{\alpha\mu} = \alpha\Sigma_\mu$, whence Σ is linear. Let K denote the quantity in (i). Then

$$\begin{aligned} \|\Sigma\| = \sup_{\|\mu\|\le 1} \|\Sigma_\mu\| &= \sup_{\|\mu\|\le 1} \left\| \int b_{x,\mu}\, d\mu(x) \right\| \\ &\le \sup_{\|\mu\|\le 1} \int \|b_{x,\mu}\|\, d|\mu|(x) \le K . \end{aligned}$$

To show that $\|\Sigma\| = K$, choose any nonzero $\mu \in M$ and $\varepsilon > 0$. Let $0 \ne \nu \in L(\mu)$ be such that $\left\| \|b_{\cdot,\mu}\|_B - \|\, \|b_{\cdot,\mu}\|_B \|_{L^\infty(\mu)} \right\|_{L^\infty(\nu)} < \varepsilon$. Let S be the unit sphere of B. Since the unit ball of B is weak* compact, there exists a finite number of elements, $b_*^1, \dots, b_*^n$, of the unit ball of B_* such that

$$S = \bigcup_{i=1}^{n} \{b \in S : |\langle b_*^i, b\rangle - 1| < \varepsilon\} .$$

Therefore $\exists 0 < \omega \in L(\nu)$ $\exists i$ $\|\langle b_*^i, b_{x,\mu}/\|b_{x,\mu}\|_B\rangle - 1\|_{L^\infty(\omega)} < \varepsilon$. We have

$$\begin{aligned} \|\Sigma\| \ge \frac{\|\Sigma_\omega\|}{\|\omega\|} \ge \frac{1}{\|\omega\|} |\langle b_*^i, \Sigma_\omega\rangle| &= \frac{1}{\|\omega\|} \left| \int \langle b_*^i, b_{x,\mu}\rangle\, d\omega(x) \right| \\ &\ge \frac{1}{\|\omega\|} \int \|b_{x,\mu}\|_B\, d\omega(x) - \varepsilon K \ge \left\| \|b_{\cdot,\mu}\|_B \right\|_{L^\infty(\mu)} - \varepsilon(K+1) . \end{aligned}$$

Thus $\|\Sigma\| = K$.

Conversely, let $\Sigma \in L(M, B)$. Fix $\mu \in M$. For $b_* \in B_*$, we denote by $b_* \circ \Sigma$ the map $\nu \mapsto \langle b_*, \Sigma_\nu\rangle$. Restricted to $L(\mu)$, this map is a bounded linear functional and hence can be represented by a function $g_{b_*} \in L^\infty(\mu)$. Choose a countable linearly independent set D whose linear span over $\mathbf{Q}$, D', is dense in B_*. If $b_* = \sum_{i=1}^n \alpha_i d_*^i$, $d_*^i \in D$, $\alpha_i \in \mathbf{Q}$, define

$$h_{b_*} = \sum_{i=1}^n \alpha_i g_{d_*^i}.$$

Then $b_* \mapsto h_{b_*}(x)$ is rational-linear on D' for every $x \in X$. Furthermore, $h_{b_*} = g_{b_*}$ μ-a.e., whence by countability of D',

$$\forall b_* \in D' \; |h_{b_*}(x)| \le \|b_* \circ \Sigma\| \le \|b_*\| \cdot \|\Sigma\| \tag{2.1}$$

for μ-a.e. x. Now for every x such that (2.1) holds, $b_* \mapsto h_{b_*}(x)$ extends from D' to all of B_* as a bounded linear functional, hence element of B, call it $f(x)$. This defines $f(x)$ μ-a.e. and shows that, given any $b_* \in B_*$, if $b_* = \lim_{n\to\infty} b_*^n$ $(b_*^n \in D')$, then

$$\langle b_*, f(x)\rangle = \lim_{n\to\infty} \langle b_*^n, f(x)\rangle = \lim_{n\to\infty} h_{b_*^n}(x) \tag{2.2}$$

for every x where f is defined. Write $b_{\cdot,\mu}$ for the equivalence class of f. From Equation (2.1), we see that $\|f(x)\| \le \|\Sigma\|$ for every x where f is defined. Together with (2.2), this shows that $b_{\cdot,\mu} \in \mathscr{B}(X, B)_\mu$ and gives (i). Now for $\nu \in L(\mu)$ and $b_* \in D'$, we have

$$\begin{aligned}\left\langle b_*, \int f\,d\nu \right\rangle &= \int \langle b_*, f(x)\rangle\,d\nu(x) = \int h_{b_*}(x)\,d\nu(x)\\ &= \int g_{b_*}(x)\,d\nu(x) = \langle b_*, \Sigma_\nu\rangle.\end{aligned}$$

Since D' is dense, (iii) follows. We claim that $b_{\cdot,\mu}$ is uniquely determined by the property just established:

$$\forall \nu \in L(\mu)\; \Sigma_\nu = \int b_{x,\mu}\,d\nu(x).$$

Indeed, if we also have that $\forall \nu \in L(\mu)\; \Sigma_\nu = \int b'_{x,\mu}\,d\nu(x)$ for some $b'_{\cdot,\mu} \in \mathscr{B}(X, B)_\mu$, then

$$\forall b_* \in D'\; \forall \nu \in L(\mu)\; \int \langle b_*, b_{x,\mu}\rangle\,d\nu(x) = \int \langle b_*, b'_{x,\mu}\rangle\,d\nu(x),$$

whence for μ-a.e. x $\forall b_* \in D'$ $\langle b_*, b_{x,\mu}\rangle = \langle b_*, b'_{x,\mu}\rangle$, i.e., $b_{x,\mu} = b'_{x,\mu}$ μ-a.e. Thus (ii) follows. The same argument shows that if $\Sigma \sim b_{\cdot,\cdot}$ and $\Sigma \sim b'_{\cdot,\cdot}$, then $b_{\cdot,\cdot} = b'_{\cdot,\cdot}$. □

We define the *weak* operator topology* (W^*OT) on $L(M, B)$ as the weakest topology such that $\forall \mu \in M$ $\forall b_* \in B_*$ $\Sigma \mapsto \langle b_*, \Sigma_\mu\rangle$ is continuous. It is an elementary exercise to show that the unit ball of $L(M, B)$ is W^*OT compact.

For $\mu \in M$, let $L(M, B)_\mu$ denote the set of Šreĭder representations $b._{,\mu}$ of elements of $L(M, B)$. We give $L(M, B)_\mu$ the weak topology generated by the maps $b._{,\mu} \mapsto \int \langle b_*, b_{x,\nu}\rangle \, d\nu(x)$ $(b_* \in B_*$, $\nu \in L(\mu))$. Thus, the W* OT is the inverse limit of these topologies, i.e., it is the weak topology generated by the maps $\Sigma \mapsto b._{,\mu}$ $(\mu \in M)$ from $L(M, B) \to L(M, B)_\mu$, where $\Sigma \sim b._{,}.$.

Every decomposition $M = I \oplus J$ of M as a direct sum of closed subspaces yields an addition on $L(M, B)$ as follows: if $\Pi^1, \Pi^2 \in L(M, B)$, then we may define

$$\Sigma_\mu = \Pi^1_{\mu_I} + \Pi^2_{\mu_J}, \tag{2.3}$$

where $\mu = \mu_I + \mu_J$, $\mu_I \in I$, $\mu_J \in J$. If $\Sigma \sim b._{,}.$, $\Pi^i \sim b^i_{.,.}$, and $I \perp J$, then $b_{x,\mu} = b^1_{x,\mu_I} + b^2_{x,\mu_J}$ μ-a.e.

The case where $B = M(Y)$, the space of complex regular Borel measures on a locally compact metric space, Y, is of interest. A predual of B is the separable space $C_0(Y)$ of continuous functions vanishing at infinity. We shall denote the Šreĭder representation of Σ by $\sigma_{x,\mu}$ in this case; thus, if $f \in C_0(Y)$ and $\mu \in M$,

$$\int_Y f \, d\Sigma_\mu = \int_X \left(\int_Y f \, d\sigma_{x,\mu} \right) d\mu(x). \tag{2.4}$$

(If Y is separable and a countable union of complete subspaces, then (2.4) holds for $f \in \mathscr{B}(Y, \mathbf{C})$ since it is preserved under bounded pointwise limits. In particular, for Borel sets $E \subseteq Y$,

$$\Sigma_\mu(E) = \int_X \sigma_{x,\mu}(E) \, d\mu(x).)$$

Let M^+ denote the nonnegative elements of M and likewise for $M^+(Y)$. We say that $\Sigma \in L(M, M(Y))$ is *positive* if it carries M^+ into $M^+(Y)$. It is easy to see from (2.4) applied to $|\mu|$ that $\Sigma \geq 0$ iff $\forall \mu \in M$ $\forall^e x[\mu]$ $\sigma_{x,\mu} \geq 0$ ("$\forall^e x[\mu]$" means "for μ-a.e. x"–see **[L1]**). It is also easy to show that if $\Sigma \geq 0$, then $\nu \ll \mu \Rightarrow \Sigma_\nu \ll \Sigma_{|\mu|}$ and $|\Sigma_\mu| \leq \Sigma_{|\mu|}$.

3. The Šreĭder representation of homomorphisms. Let G be a locally compact semigroup with separately continuous multiplication. Then $M(G)$ is a Banach algebra under convolution **[W]**. Let M be an L-subalgebra of $M(G)$, i.e., a subalgebra which is also an L-subspace, and let B be a Banach algebra with a separable predual such that multiplication is separately weak* measurable and

(3.1) $\forall f \in \mathscr{B}(G, B)\ \forall b \in B\ \forall \mu \in M$

$$\int f(x) \cdot b\, d\mu(x) = \left(\int f\, d\mu\right) \cdot b$$

$$\& \int b \cdot f(x)\, d\mu(x) = b \cdot \int f\, d\mu .$$

In order to state some sufficient conditions that (3.1) be true, we define the following multiplication on $B^* \times B$. If $b \in B$ and $b^* \in B^*$, then $b' \mapsto \langle b' \cdot b, b^* \rangle$ is a bounded linear functional on B; we denote it by $b^* \cdot b$. Let $\overline{B}_*^{sw^*}$ be the smallest subspace of B^* containing (canonically) B_* which is closed under weak* sequential limits. Let ΔB be the subset of B^* consisting of the multiplicative linear functionals.

PROPOSITION 3.1. *Let B be a Banach algebra with a separable predual. Right multiplication on B is weak* measurable and the first equation of* (3.1) *holds if any of the following conditions is satisfies*:

(i) $B_* \cdot B \subseteq \overline{B}_*^{sw^*}$.
(ii) *Right multiplication is weak* continuous.*
(iii) *Right multiplication is weak* measurable and $\overline{B}_*^{sw^*} \cap \Delta B$ separates points in B.*

Proof. The class of $b^* \in B^*$ such that $b \mapsto \langle b, b^* \rangle$ is weak* measurable contains B_* and is closed under weak* sequential limits. Thus, all elements of $\overline{B}_*^{sw^*}$ are weak* measurable. Now right multiplication is weak* measurable iff $\forall b \in B$ $\forall b_* \in B_*$ $b' \mapsto \langle b_*, b' \cdot b \rangle$ is weak* measurable. But $\langle b_*, b' \cdot b \rangle = \langle b', b_* \cdot b \rangle$, whence this condition is equivalent to all elements of $B_* \cdot B$ being weak* measurable. The sufficiency of (i) for measurability is now obvious. Also, the class of weak* measurable $b^* \in B^*$ such that

$$\left\langle \int f\, d\mu, b^* \right\rangle = \int \langle f, b^* \rangle\, d\mu$$

is closed under weak* sequential limits by the bounded convergence theorem, hence contains $\overline{B}_*^{sw^*}$. Thus, if (i) holds, then $\forall b_* \in B_*$ $\forall b \in B$

$$\left\langle b_*, \int f \cdot b\, d\mu \right\rangle = \int \langle b_*, f \cdot b \rangle\, d\mu = \int \langle f, b_* \cdot b \rangle\, d\mu$$
$$= \left\langle \int f\, d\mu, b_* \cdot b \right\rangle = \left\langle b_*, \left(\int f\, d\mu\right) \cdot b \right\rangle ,$$

whence the first equation of (3.1).

Now (ii) is equivalent to $B_* \cdot B \subseteq B_*$ since B_* is the set of weak* continuous linear functionals on B. Thus, sufficiency follows from that of (i). Finally, if (iii) holds, then for $f \in \mathscr{B}(G, B)$, $b \in B$, $\mu \in M$, and $b^* \in \overline{B_*}^{sw^*} \cap \Delta B$, we have

$$\begin{aligned}\left\langle \int f \cdot b \, d\mu , b^* \right\rangle &= \int \langle f \cdot b , b^* \rangle \, d\mu = \int \langle f , b^* \rangle \langle b , b^* \rangle \, d\mu \\ &= \int \langle f , b^* \rangle \, d\mu \cdot \langle b , b^* \rangle = \left\langle \int f \, d\mu , b^* \right\rangle \cdot \langle b , b^* \rangle \\ &= \left\langle \left(\int f \, d\mu \right) \cdot b , b^* \right\rangle ,\end{aligned}$$

from which the first equation of (3.1) follows. □

Let $\mathscr{B}_0(G, B)$ denote the Baire-measurable functions from G to B, where B is given the weak* topology. For $\mu, \nu \in M(G)$, let $\mu \times \nu$ denote, besides the usual product measure, also its unique extension to a regular Borel measure in $M(G \times G)$. If $f \in \mathscr{B}_0(G, B)$ and $\mu, \nu \in M(G)$, then

$$\begin{aligned}\int f \, d\mu * \nu &= \int f(xy) \, d\mu \times \nu(x, y) \\ &= \iint f(xy) \, d\mu(x) \, d\nu(y) ,\end{aligned}$$

as can be seen by applying any $b_* \in B_*$ [W].

The Šreĭder representation of $\mathrm{Hom}(M, B)$, the continuous homomorphisms from M to B, satisfies one property additional to those in Theorem 2.1.

THEOREM 3.2. *Let G be a locally compact semigroup with separately continuous multiplication and M an L-subalgebra of $M(G)$. Let B be a Banach algebra with a separable predual and separately weak* measurable multiplication satisfying* (3.1). *Let $\Sigma \in L(M, B)$ and choose $b_{\cdot, \mu} \in \mathscr{B}_0(G, B)$ $(\mu \in M)$ so that $\Sigma \sim b_{\cdot,\cdot}$. Then $\Sigma \in \mathrm{Hom}(M, B)$ iff*

$$\text{(3.2)} \qquad \forall \mu, \nu \in M^+ \; b_{xy, \mu * \nu} = b_{x, \mu} \cdot b_{y, \nu} \quad \textit{for } \mu \times \nu\textit{-a.e. } (x, y) .$$

Proof. Suppose first that (3.2) is satisfied. Then for $\mu, \nu \in M$,

$$\begin{aligned}\Sigma_{\mu*\nu} &= \int b_{t,|\mu|*|\nu|}\, d\mu*\nu(t) = \iint b_{xy,|\mu|*|\nu|}\, d\mu(x)\, d\nu(y)\\ &= \iint b_{x,|\mu|}\cdot b_{y,|\nu|}\, d\mu(x)\, d\nu(y)\\ &= \int\left(\int b_{x,|\mu|}\, d\mu(x)\right)\cdot b_{y,|\nu|}\, d\nu(y)\\ &= \int b_{x,|\mu|}\, d\mu(x)\cdot\int b_{y,|\nu|}\, d\nu(y) = \Sigma_\mu\cdot\Sigma_\nu .\end{aligned}$$

Conversely, if $\Sigma \in \operatorname{Hom}(M, B)$, then given $\mu, \nu \in M^+$, we have for all $\mu' \in L(\mu)$ and $\nu' \in L(\nu)$,

$$\begin{aligned}\int b_{xy,\mu*\nu}\, d\mu'\times\nu'(x, y) &= \int b_{t,\mu*\nu}\, d\mu'*\nu'(t) = \Sigma_{\mu'*\nu'}\\ &= \Sigma_{\mu'}\cdot\Sigma_{\nu'} = \int b_{x,\mu}\, d\mu'(x)\cdot\int b_{y,\nu}\, d\nu'(y)\\ &= \iint b_{x,\mu}\cdot b_{y,\nu}\, d\mu'(x)\, d\nu'(y)\\ &= \int b_{x,\mu}\cdot b_{y,\nu}\, d\mu'\times\nu'(x, y).\end{aligned}$$

Since the span of $L(\mu)\times L(\nu)$ is dense in $L(\mu\times\nu)$, (3.2) follows. □

If multiplication in B is jointly weak* continuous (for example, if $B_*\cap\Delta B$ separates points in B), then the unit ball in $\operatorname{Hom}(M, B)$ is easily shown to be $\mathrm{W^*OT}$ compact. An example where compactness fails is $\operatorname{Hom}(M(\mathbf{R}), M(\mathbf{R}))$: define T_n $(n \geq 1)$ in the unit ball by

$$\int_{\mathbf{R}} f(x)\, d(T_n)_\mu(x) = \int_{\mathbf{R}} f(nx)\, d\mu(x) \qquad (f \in C_0(\mathbf{R}))$$

and let $\Sigma\colon \mu \mapsto \mu(\{0\})\delta(0)$, where $\delta(0)$ is the Dirac measure at 0. Then $T_n \to \Sigma$ in $\mathrm{W^*OT}$, but

$$\Sigma \in L(M(\mathbf{R}), M(\mathbf{R}))\setminus\operatorname{Hom}(M(\mathbf{R}), M(\mathbf{R})).$$

We define the following multiplication on $L(M, B)$: if $\Sigma \sim b_{\cdot,\cdot}$ and $\Pi \sim b'_{\cdot,\cdot}$, then $\Sigma\cdot\Pi$ is defined by its Šreĭder representation $b_{x,\mu}\cdot b'_{x,\mu}$. When B is commutative, $\operatorname{Hom}(M, B)$ is closed under multiplication. It is easily verified that if multiplication in B is separately weak* continuous, then multiplication in $L(M, B)$ is separately $\mathrm{W^*OT}$ continuous.

Suppose that $M = I \oplus J$, where I is a closed ideal and J is a closed subalgebra. If $\Pi^1, \Pi^2 \in \mathrm{Hom}(M, B)$ satisfy

$$\forall \mu \in I \; \forall \nu \in J \; \Pi^1_{\mu * \nu} = \Pi^1_\mu \cdot \Pi^2_\nu \quad \& \quad \Pi^1_{\nu * \mu} = \Pi^2_\mu \cdot \Pi^1_\mu, \tag{3.3}$$

then the "sum" Σ of Π^1 and Π^2 defined in (2.3) is a homomorphism.

4. Limit points of group homomorphisms. If H is a locally compact group, then convolution is separately weak* continuous in $M(H)$. Indeed, if $\mu_\alpha, \mu, \nu \in M(H)$ with $\mu_\alpha \xrightarrow{w^*} \mu$, then for $f \in C_0(H)$, the map $x \mapsto \int f(xy)\, d\nu(y)$ lies in $C_0(H)$, whence

$$\begin{aligned}\int f \, d\mu_\alpha * \nu &= \iint f(xy)\, d\nu(y)\, d\mu_\alpha(x) \\ &\to \iint f(xy)\, d\nu(y)\, d\mu(x) = \int f\, d\mu * \nu,\end{aligned}$$

which is to say that $\mu_\alpha * \nu \xrightarrow{w^*} \mu * \nu$. A similar argument applies to $\nu * \mu_\alpha$. Thus, if G is a locally compact semigroup with separately continuous multiplication and H is a locally compact metrizable group, then the preceding section applied to $\mathrm{Hom}(M, M(H))$ for any L-subalgebra M of $M(G)$. Every continuous homomorphism $\varphi\colon G \to H$ yields an element of $\mathrm{Hom}(M, M(H))$, which we also denote by φ, defined by $\langle f, \varphi_\mu \rangle = \langle f \circ \varphi, \mu \rangle$ for $f \in C_0(H)$. The Šreĭder representation of such a φ is particularly simple: $\varphi \sim \delta(\varphi(x))$ (independent of μ), where $\delta(t)$ denotes the Dirac measure at t.

We identify $\mathrm{Hom}(G, H)$ with a subset of $\mathrm{Hom}(M(G), M(H))$ in the above manner. Our aim is to study the set

$$\Lambda = \overline{\mathrm{Hom}(G, H)} \setminus \mathrm{Hom}(G, H)$$

and its local structure

$$\Lambda(\mu) = \{\Sigma_\mu : \Sigma \in \Lambda\}, \qquad \breve{\Lambda}(\mu) = \{\sigma_{\cdot,\mu} : \sigma_{\cdot,\cdot} \in \breve{\Lambda}\},$$

where $\breve{\Lambda}$ consists of the Šreĭder representations of elements of Λ. Since all elements of $\mathrm{Hom}(G, H)$ are positive and lie in the unit ball, the same holds for Λ. (In fact, every positive homomorphism lies in the unit ball: if $0 \le \Sigma \in \mathrm{Hom}(M(G), M(H))$, then for $\mu \in M(G)$ and $n \ge 1$, we have

$$\|\Sigma_\mu\|^n \le \|\Sigma_{|\mu|}\|^n = \|\Sigma^n_{|\mu|}\| = \|\Sigma_{|\mu|^n}\| \le \|\Sigma\| \cdot \|\,\mu|^n\| = \|\Sigma\| \cdot \|\mu\|^n,$$

whence $\|\Sigma\| \le 1$.)

We are particularly interested in the case where G is a locally compact abelian group and H is a circle group, $\mathbf{T}$. In this case,

$\mathrm{Hom}(G, \mathbf{T}) = \widehat{G}$, the dual of G, and the identification of $\widehat{G}$ as a subset of $\mathrm{Hom}(M(G), M(\mathbf{T}))$ preserves the usual topology of $\widehat{G}$ (of uniform convergence on compact subsets). Furthermore, as $\widehat{G}$ lies in the unit ball of $\mathrm{Hom}(M(G), M(\mathbf{T}))$, it follows that $\overline{\widehat{G}} = \widehat{G} \cup \Lambda$ is a compactification of $\widehat{G}$.

Recall that a sequence $\{x_k\}_{k=1}^{\infty} \subseteq G$ is said to have an *asymptotic distribution* σ, written $\{x_k\} \sim \sigma$, if

$$\frac{1}{K}\sum_{k=1}^{K} \delta(x_k) \xrightarrow{w^*} \sigma \quad \text{as} \quad K \to \infty.$$

For $n \in \mathbf{Z}$ and $\Sigma \in \mathrm{Hom}(M(G), M(\mathbf{T}))$, define $\widehat{\Sigma}(n) \in \Delta M(G)$ by $\langle \mu, \widehat{\Sigma}(n)\rangle = \widehat{\Sigma}_\mu(n)$. We write the Šreĭder representation of $\chi \in \Delta M(G)$ as $\chi_\mu(x)$. Thus, if $\Sigma \sim \sigma_{\cdot,\cdot}$ and $\chi = \widehat{\Sigma}(n)$, then

$$\chi_\mu(x) = \hat{\sigma}_{x,\mu}(n).$$

Note that for all n, the map $\Sigma \mapsto \widehat{\Sigma}(n)$ from $(\mathrm{Hom}(M(G), M(\mathbf{T})),$ W* OT) to $\Delta M(G)$ (with its usual Gelfand topology) is continuous. We regard the Fourier transform as a restriction of the Gelfand transform; thus, in accordance with the Šreĭder representation, we have $\hat{\mu}(\gamma) = \int \gamma\, d\mu$ for $\gamma \in \widehat{G}$.

PROPOSITION 4.1. *Let G be a locally compact abelian group and $\Lambda = \overline{\widehat{G}} \backslash \widehat{G}$ in* $\mathrm{Hom}(M(G), M(\mathbf{T}))$. *Then*

(i) *Λ is closed topologically and under multiplication by elements of $\overline{\widehat{G}}$;*

(ii) *if $\sigma_x, \tau_x \in \breve{\Lambda}(\mu)$, then $\sigma_x * \tau_x \in \breve{\Lambda}(\mu)$;*

(iii) $\Lambda(\mu) = \{\nu \in M(\mathbf{T})$: $\exists$ *net* $\{\gamma_\alpha\} \subseteq \widehat{G}$ $(\gamma_\alpha \to \infty \,\&\, \forall n \in \mathbf{Z}$ $\hat{\mu}(\gamma_\alpha^n) \to \hat{\nu}(n))\}$;

(iv) $\breve{\Lambda}(\mu) = \{\sigma. \in \mathscr{B}(G, M(\mathbf{T}))_\mu$: $\exists$ *net* $\{\gamma_\alpha\} \subseteq \widehat{G}$ $(\gamma_\alpha \to \infty \,\&\, \forall n \in \mathbf{Z}$ $\gamma_\alpha^n \to \hat{\sigma}.(n)$ *weak* in* $L^\infty(\mu))\}$;

(v) *if G is metrizable, then the nets in* (iii) *and* (iv) *can be replaced by sequences and* $\breve{\Lambda}(\mu) = \{\sigma. \in \mathscr{B}(G, M(\mathbf{T}))_\mu$: $\exists \gamma_j \in \widehat{G}$ $(\gamma_j \to \infty\,\&$ *for every subsequence* γ_{j_k}, $\forall^e x[\mu]$ $\{\gamma_{j_k}(x)\}_{k=1}^{\infty} \sim \sigma_x)\}$.

Proof. Suppose that $\Sigma \in \Lambda$ is the limit of a net $\{\gamma_\alpha\} \subseteq \widehat{G}$. Then $\widehat{\Sigma}(n) = \lim \gamma_\alpha^n$ in $\Delta M(G)$ for all $n \in \mathbf{Z}$. Now if $\gamma_\alpha \to \gamma \in \widehat{G}$, then $\gamma_\alpha^n \to \gamma^n$, whence $\Sigma = \gamma$. But since $\Lambda \cap \widehat{G} = \varnothing$, this is impossible, and so $\gamma_\alpha \to \infty$ in $\widehat{G}$. In particular, $\widehat{\Sigma}(1)$ is 0 on $L^1(G)$ **[HMP,**

p. 136, Proposition 4] and consequently Λ is closed. It is clear that $\Lambda \cdot \widehat{G} \subseteq \Lambda$, from which (i) now follows. Statement (ii) ensues as well. Now if $\nu \in \Lambda(\mu)$, then let $\widehat{G} \ni \gamma_\alpha \to \Sigma \in \Lambda$ be such that $\nu = \Sigma_\mu$. Then $\gamma_\alpha \to \infty$ and $(\gamma_\alpha)_\mu \xrightarrow{w^*} \Sigma_\mu = \nu$, which gives the inclusion $\subseteq$ of (iii). On the other hand, if $\gamma_\alpha \to \infty$ and $\forall n \ \hat{\mu}(\gamma_\alpha^n) \to \hat{\nu}(n)$, then by compactness of $\overline{\widehat{G}}$, we can choose a subnet $\{\gamma'_\beta\}$ of $\{\gamma_\alpha\}$ converging to some Σ. Since $\gamma'_\beta \to \infty$, it follows that $\Sigma \in \Lambda$ and $\nu = \Sigma_\mu \in \Lambda(\mu)$. This completes the proof of (iii). The proof of (iv) is analogous. Finally, if G is metrizable, then $L^1(\mu)$ is separable for $\mu \in M(G)$ and so $L(M(G), M(\mathbf{T}))_\mu$ is metrizable. Thus, if $\mu \in M(G)$ and $\gamma_\alpha \to \Sigma \sim \sigma_{\cdot,\cdot}$, pick any non-zero $\rho \in L^1(G)$ and a subsequence $\{\delta(\gamma_{\alpha_j}(\cdot))\}$ converging to $\sigma_{\cdot,|\mu|+|\rho|}$ in $L(M(G), M(\mathbf{T}))_{|\mu|+|\rho|}$. Then $\gamma_{\alpha_j} = \delta(\gamma_{\alpha_j}(\cdot))^\frown(1) \xrightarrow{w^*} (\widehat{\Sigma}(1))_\rho = 0$ in $L^\infty(\rho)$, whence $\gamma_{\alpha_j} \to \infty$ in $\widehat{G}$, and $\gamma_{\alpha_j}^n \xrightarrow{w^*} (\widehat{\Sigma}(n))_\mu = \hat{\sigma}_{\cdot,\mu}(n)$ in $L^\infty(\mu)$. This shows the sufficiency of sequences for (iii) and (iv). Furthermore, if $\forall n \ \gamma_j^n \to \hat{\sigma}_\cdot(n)$ weak* in $L^\infty(\mu)$, then by [**L2**, Lemma 5], there is a subsequence $\{\gamma'_j\}$ of $\{\gamma_j\}$ such that every further subsequence $\{\gamma'_{j_k}\}$ satisfies

$$\forall^e x[\mu] \ \{\gamma'_{j_k}(x)\}_{k=1}^\infty \sim \sigma_x . \tag{4.1}$$

Conversely, if $\{\gamma_j\}$ is a sequence, every subsequence of which satisfies (4.1), then we claim $\gamma_j^n \to \hat{\sigma}_\cdot(n)$ weak* for every n. If not, then for some n there would be a subsequence $\{\gamma_{j_k}^n\}$ converging to a different limit χ. Then also

$$\frac{1}{K}\sum_{k=1}^{K} \gamma_{j_k}^n \xrightarrow{w^*} \chi$$

and by (4.1),

$$\frac{1}{K}\sum_{k=1}^{K} \gamma_{j_k}^n \xrightarrow{w^*} \hat{\sigma}_\cdot(n) .$$

Therefore $\chi = \hat{\sigma}_\cdot(n)$, a contradiction. Thus (v) follows from (iv). □

When $\widehat{G}$ is regarded as a subset of $\Delta M(G)$, we shall use the notation Γ rather than $\widehat{G}$ to avoid confusion. Let $T_n \in \mathrm{Hom}(G, G)$ denote the map $x \mapsto x^n$ $(n \in \mathbf{Z})$, as well as the corresponding map induced in $\mathrm{Hom}(M(G), M(G))$. Thus, for $\Sigma \in \mathrm{Hom}(M(G), M(\mathbf{T}))$, we obtain $\Sigma \circ T_n \in \mathrm{Hom}(M(G), M(\mathbf{T}))$; note that if $\Sigma = \gamma \in \widehat{G}$, then $\gamma \circ T_n = \gamma^n$.

PROPOSITION 4.2. *Let G be a LCA group and*

$$\Sigma \in \mathrm{Hom}(M(G), M(\mathbf{T})).$$

Then $\Sigma \in \overline{\widehat{G}}$ iff $\widehat{\Sigma}(1) \in \overline{\Gamma}$ and $\forall n \in \mathbf{Z}$ $\widehat{\Sigma}(n) = \widehat{\Sigma}(1) \circ T_n$. The map $\Sigma \mapsto \widehat{\Sigma}(1)$ is an isomorphism from $\overline{\widehat{G}}$ onto $\overline{\Gamma}$ sending $\widehat{G}$ to Γ.

Proof. If $\Sigma \in \overline{\widehat{G}}$, let $\widehat{G} \ni \gamma_\alpha \xrightarrow{\mathrm{W^*\,OT}} \Sigma$. Since $\hat{\gamma}_\alpha(n) = \gamma_\alpha^n$, we have $\gamma_\alpha^n \to \widehat{\Sigma}(n)$ for all n. In particular, $\widehat{\Sigma}(1) \in \overline{\Gamma}$. Also, $\widehat{\Sigma}(n) = \lim \gamma_\alpha^n = \lim \gamma_\alpha \circ T_n = (\lim \gamma_\alpha) \circ T_n = \widehat{\Sigma}(1) \circ T_n$. Conversely, if $\widehat{\Sigma}(1) \in \overline{\Gamma}$ and $\forall n$ $\widehat{\Sigma}(n) = \widehat{\Sigma}(1) \circ T_n$, then let $\gamma_\alpha \to \widehat{\Sigma}(1)$. Choose a convergent subnet $\gamma'_\beta \to \Pi$ in $\mathrm{Hom}(M(G), M(\mathbf{T}))$. Then from the above, $\widehat{\Pi}(n) = \widehat{\Pi}(1) \circ T_n = \widehat{\Sigma}(1) \circ T_n = \widehat{\Sigma}(n)$ for all n, whence $\Sigma = \Pi \in \overline{\widehat{G}}$.

It follows from this that the map $\Sigma \mapsto \widehat{\Sigma}(1)$ is injective. Surjectivity onto $\overline{\Gamma}$ is proved by a compactness argument similar to the above. □

We write $M(G) = M_c(G) \oplus M_d(G)$ for the decomposition of a measure into its continuous and discrete parts. Then $h_d \colon \mu \mapsto \int_G d\mu_d = \hat{\mu}_d(0)$ is in $\overline{\Gamma} \backslash \Gamma$ [**HMP**, pp. 136–7, (4.1.4)]. We denote the element of Λ corresponding to h_d by Π^d. If G has at most countably many torsion elements, then we claim that

$$\widehat{\Pi}^d(n) = \begin{cases} 0 & \text{if } n = 0, \\ h_d & \text{if } n \neq 0, \end{cases}$$

whence

$$\Pi^d_\mu = \hat{\mu}_c(0)\lambda + \hat{\mu}_d(0)\delta(0),$$

where λ is Lebesgue measure on $\mathbf{T}$. To see this, note first that

$$\widehat{\Pi}^d(0) \colon \mu \mapsto (\mu \circ T_0^{-1})^\frown(0) = \hat{\mu}(0).$$

Second, if $n \neq 0$, then for all $g \in G$, there are, by the supposition, denumerably many $x \in G$ such that $x^n = g$. Therefore

$$(\mu \circ T_n^{-1})(\{g\}) = \sum_{x^n = g} \mu(\{x\}),$$

whence

$$\begin{aligned} \widehat{\Pi}^d(n) \colon \mu \mapsto & \sum_{g \in G} (\mu \circ T_n^{-1})(\{g\}) \\ & = \sum_{g \in G} \sum_{x^n = g} \mu(\{x\}) = \sum_{x \in G} \mu(\{x\}) = \hat{\mu}_d(0). \end{aligned}$$

This proves the claim.

Related elements of Λ are $\Sigma \cdot \Pi^d$ for $\Sigma \in \overline{\overline{\widehat{G}}}$; if, as above, the torsion subgroup of G is denumerable, then

$$(\Sigma \cdot \Pi^d)_\mu = \hat{\mu}_c(0)\lambda + \Sigma_{\mu_d}.$$

Thus, if we set Π: $\mu \mapsto \hat{\mu}(0)\lambda$, then $\Sigma \cdot \Pi^d$ is the sum of Π and Σ defined by (2.3) and (3.3) from the decomposition $M = M_c \oplus M_d$. An interesting example is $G = \mathbf{T}$ and Σ: $\mu \mapsto \mu$; in this case, $(\Sigma \cdot \Pi^d)_\mu = \hat{\mu}_c(0)\lambda + \mu_d$.

Provided still that G has a denumerable torsion subgroup, the Šreĭder representation $\pi^d_{\cdot,\cdot}$ of Π^d is given by

$$\pi^d_{x,\mu} = \begin{cases} \lambda & \text{if } \mu(\{x\}) = 0, \\ \delta(0) & \text{if } \mu(\{x\}) \neq 0. \end{cases} \tag{4.2}$$

Let $\boldsymbol{\lambda} \in \mathscr{B}(G, M(\mathbf{T}))_\mu$ be defined by $\boldsymbol{\lambda}(x) \equiv \lambda$. Then from [**HMP**, p. 70, Corollaire 2] and Proposition 4.2 (or from (4.2) and the following proposition),

$$\mu \in M_c(G) \Leftrightarrow \boldsymbol{\lambda} \in \breve{\Lambda}(\mu). \tag{4.3}$$

This yields other characterizations of $M_c(G)$ when combined with Proposition 4.1 (iv), (v). For example,

$$\begin{aligned} \mu \in M_c(G) &\Leftrightarrow \exists \gamma_\alpha \to \infty \ \forall \nu \in L(\mu) \ \forall n \neq 0 \ \hat{\nu}(\gamma_\alpha^n) \to 0 \\ &\Leftrightarrow \exists \gamma_\alpha \to \infty \ \forall \gamma \in \widehat{G} \ \forall n \neq 0 \ \hat{\mu}(\gamma\gamma_\alpha^n) \to 0. \end{aligned}$$

Our next proposition describes $\breve{\Lambda}(\mu)$ completely when μ is discrete (cf. [**HMP**, pp. 67–68]).

PROPOSITION 4.3. *Let G be a LCA group. Let $\breve{\overline{\overline{\widehat{G}}}}$ denote the Šreĭder representations of $\overline{\overline{\widehat{G}}} \subseteq \mathrm{Hom}(M(G), M(\mathbf{T}))$ and, for $\mu \in M(G)$, $\breve{\overline{\overline{\widehat{G}}}}(\mu) = \{\sigma_{\cdot,\mu} : \sigma_{\cdot,\cdot} \in \breve{\overline{\overline{\widehat{G}}}}\}$. Let G_d denote G with the discrete topology and, for $\mu \in M_d(G)$, let G_d^μ denote the discrete subgroup generated by the mass-points of μ.*

(i) $\forall \Sigma \in \overline{\overline{\widehat{G}}} \ \ \exists \varphi \in \widehat{G}_d \ \ \forall \mu \in M_d(G) \ \ \Sigma_\mu = \sum_{x \in G} \mu(\{x\})\delta(\varphi(x))$ *and* $\sigma_{x,\mu} = \delta(\varphi(x))$, *where* $\Sigma \sim \sigma_{\cdot,\cdot}$.

(ii) $\forall \mu \in M_d(G) \ \ \breve{\overline{\overline{\widehat{G}}}}(\mu) \simeq \widehat{G_d^\mu}$.

(iii) $\mu \in M_d(G) \Leftrightarrow \breve{\overline{\overline{\widehat{G}}}}(\mu)$ *is a group* (*under the multiplication in* $L(M(G), M(\mathbf{T}))$).

Proof. (i) Let $\widehat{G} \ni \gamma_\alpha \xrightarrow{\text{W}^*\,\text{OT}} \Sigma$. Then for $x \in G$,

$$\delta(\gamma_\alpha(\cdot)) \to \sigma_{\cdot\,,\delta(x)} \in L(M(G)\,,\, M(\mathbf{T}))_{\delta(x)}\,,$$

i.e., $\delta(\gamma_\alpha(x)) = \sigma_{x,\delta(x)}$ eventually. Thus, $\gamma_\alpha(x)$ stabilizes at some point $\varphi(x)$ and $\sigma_{x,\delta(x)} = \delta(\varphi(x))$. The assertions now follow from linearity and properties of the Šreĭder representation.

(ii) The fact that $\breve{\overline{\overline{\widehat{G}}}}(\mu)$ can be identified as a compact subgroup of $\widehat{G_d^\mu}$ follows from (i). If it were not the whole group, then there would be a nonzero $x \in G_d^\mu$ such that $\varphi(x) = 1$ for all $\varphi \in \breve{\overline{\overline{\widehat{G}}}}(\mu)$. In particular, $\gamma(x) = 1$ for all $\gamma \in \widehat{G}$, whence $x = 0$, a contradiction.

(iii) This follows from **[HMP**, p. 68, Proposition 10] and (ii).□

We now arrive at the characterization of positive continuous measures mentioned in the introduction.

THEOREM 4.4. *Let G be a LCA group whose torsion subgroup is denumerable and let $\mu \in M^+(G)$ be* positive. *Then $\mu \in M_c^+(G)$ iff there is a net $\widehat{G} \ni \gamma_\alpha \to \infty$ such that for all $n \neq 0$, $\hat{\mu}(\gamma_\alpha^n) \to 0$.*

Proof. By Proposition 4.1 (iii), this is equivalent to $\mu \in M_c^+(G) \Leftrightarrow \hat{\mu}(0)\lambda \in \breve{\Lambda}(\mu)$. For $\mu \in M_c^+(G)$, this follows from $\lambda \in \breve{\Lambda}(\mu)$ (see (4.3)). If $\mu \notin M_c^+(G)$ and $\Sigma \in \Lambda$, then $\Sigma_\mu = \Sigma_{\mu_c} + \Sigma_{\mu_d} \geq \Sigma_{\mu_d}$ since $\mu_c \geq 0$ and $\Sigma \geq 0$. However, by Proposition 4.3(i), Σ_{μ_d} is nonzero and discrete; hence Σ_μ cannot equal $\hat{\mu}(0)\lambda$. □

Because of the interest this theorem may present, we provide the following "elementary" proof and strengthening for the case $G = \mathbf{T}$. If $\mu \in M_c(\mathbf{T})$, then by Wiener's theorem [**K**, p. 42], there is a sequence $\{m_k^{(1)}\}$ of density one in $\mathbf{N}$ such that $\hat{\mu}(m_k^{(1)}) \to 0$. Likewise, there is a sequence $\{m_k^{(n)}\}$ of density one such that $\hat{\mu}(nm_k^{(n)}) = \widehat{(T_n)}_\mu(m_k^{(n)}) \to 0$ since $(T_n)_\mu \in M_c$, for $n \neq 0$. By an elementary intersection argument, we obtain a sequence $\{m_k\}$, still of density one, such that for all $n \neq 0$, $\hat{\mu}(nm_k) \to 0$. (A similar argument produces a sequence $\{m_k\}$ of density one such that for $n \neq 0$ and all r, $\hat{\mu}(r + nm_k) \to 0$, i.e., $\delta(m_k x) \to \lambda$ in $L(M(\mathbf{T})\,,\, M(\mathbf{T}))_\mu$, thereby strengthening (4.3).) For the converse, we use the following proof due to Jean-François Méla. Let $K_l(x)$ be the Fejér kernel of order l. Then if $\mu \geq 0$ and if for all $n \neq 0$, $\hat{\mu}(nm_k) \to 0$, then

$$\mu(\{0\}) \leq \int_{\mathbf{T}} \frac{1}{2l+1} K_l(m_k x)\, d\mu(x) \to \frac{1}{2l+1}\hat{\mu}(0) \quad \text{as } k \to \infty$$

by hypothesis. Since this is true for all l, it follows that $\mu(\{0\}) = 0$. Now apply this result to $\mu * \tilde{\mu}$, where $\tilde{\mu}(E) = \mu(-E)$.

The local structure of Λ can be used to characterize other classes of measures besides M_c and M_d. If $\mathscr{C}$ is a class of subsets of G, let

$$\mathscr{C}^{\perp} = \{\mu \in M(G) \colon \forall E \in \mathscr{C}\ |\mu|(E) = 0\}.$$

Thus, if $\mathscr{D}$ is the class of singletons, $\mathscr{D}^{\perp} = M_c(G)$.

Definition. A set $E \subseteq G$ is called an *H-set* if there is a sequence $\widehat{G} \ni \gamma_k \to \infty$ such that $\{\gamma_k(x) \colon k \geq 1,\ x \in E\}$ is not dense in $\mathbf{T}$. A set $E \subseteq G$ is called a *Dirichlet set* if there is a sequence $\widehat{G} \ni \gamma_k \to \infty$ such that $\lim_{k\to\infty} \sup_{x\in E} |\gamma_k(x) - 1| = 0$. A measure $\mu \in M(G)$ is called a *Dirichlet measure* if $\overline{\lim}_{\gamma\to\infty} |\widehat{|\mu|}(\gamma)| = \|\mu\|$.

For background on H-sets, see [**Z**, Chapters IX, XII]; on Dirichlet sets and measures, see [**HMP**, pp. 34–35, 240–247]. The following proposition is used in [**KL**].

Proposition 4.5. *Let G be a LCA group.*

(i) *If G is metrizable, then*

$$\begin{aligned} H^{\perp} &= \{\mu \colon \forall \sigma_{\cdot} \in \breve{\Lambda}(\mu)\ \forall^e x[\mu]\ \operatorname{supp} \sigma_x = \mathbf{T}\} \\ &= \{\mu \colon \forall \Sigma \in \Lambda\ \forall \nu \in L(\mu)\ \operatorname{supp} \Sigma_\nu = \mathbf{T}\}. \end{aligned}$$

(ii) *μ is a Dirichlet measure iff the constant function $\boldsymbol{\delta}(\mathbf{0}) \in \Lambda(\mu)$.*

(iii) $D^{\perp} = \{\mu \colon \forall \sigma_{\cdot} \in \breve{\Lambda}(\mu)\ \forall^e x[\mu]\ \sigma_x \neq \delta(0)\}$

Proof. Part (i) follows from Proposition 4.1(v) and a straightforward generalization of [**L4**, Theorem 13]. Part (ii) follows from Proposition 4.2 and the fact that μ is a Dirichlet measure iff the constant function $\mathbf{1} \in (\overline{\Gamma}\backslash\Gamma)(\mu)$ [**HMP**, p. 34, Lemma 6]. Part (iii) follows from part (ii) and the fact that $D^{\perp}$ consists of the measures orthogonal to the Dirichlet measures [**HMP**, p. 243, Proposition 9]. □

Our final remarks concern the circle group.

Definition. A positive measure $\mu \in M^+(\mathbf{T})$ is called *C-quasi-symmetric* if for all pairs of adjacent arcs, I and J, on $\mathbf{T}$ of equal length, $\mu I \leq C \cdot \mu J$. We denote the class of C-quasisymmetric measures by $QS(C)$.

Note that quasisymmetric measures are continuous.

PROPOSITION 4.6. *The class $QS(C)$ is weak* closed. If $\mu \in \breve{QS}(C)$, then $\Lambda(\mu) \subseteq QS(C)$, $\breve{\Lambda}(\mu) \subseteq QS(C)$ in the sense that if $\sigma. \in \breve{\Lambda}(\mu)$, then $\forall^e x[\mu]$ $\sigma_x \in QS(C)$, and $\Lambda(\nu) \subseteq QS(C)$ for all $0 \leq \nu \in L(\mu)$.*

Proof. Let $QS(C) \ni \mu_\alpha \xrightarrow{w^*} \nu$. Given adjacent arcs I, J of equal length and $\varepsilon > 0$, pick $f, g \in C(\mathbf{T})$ such that $f \leq \mathbf{1}_I$, $\mathbf{1}_J \leq g$, $\int(\mathbf{1}_I - f)\,d\nu \leq \varepsilon$, and $\int(g - \mathbf{1}_J)\,d\nu \leq \varepsilon$. We have

$$\begin{aligned}\nu I &\leq \int f\,d\nu + \varepsilon = \lim \int f\,d\mu_\alpha + \varepsilon \leq \overline{\lim}\,\mu_\alpha I + \varepsilon \\ &\leq C \cdot \overline{\lim}\,\mu_\alpha J + \varepsilon \leq C \cdot \lim \int g\,d\mu_\alpha + \varepsilon = C \int g\,d\nu + \varepsilon \\ &\leq C \cdot \nu J + (C+1)\varepsilon.\end{aligned}$$

Since ε was arbitrary, we see that $\nu I \leq C \cdot \nu J$, whence $\nu \in QS(C)$.

Choose $\mu \in QS(C)$. Then $\gamma_\mu \in QS(C)$ for any $\gamma \in \widehat{\mathbf{T}}$. Since $\Lambda(\mu)$ is contained in the weak* closure of $\{\gamma_\mu\}_{\gamma \in \widehat{\mathbf{T}}}$, it follows that $\Lambda(\mu) \subseteq QS(C)$. Suppose that $E \subseteq \mathbf{T}$ and $\mu E > 0$. If I and J are adjacent arcs of equal length and $\varepsilon > 0$, then choose U, a finite union of arcs, such that $\mu(U \triangle E) \leq \varepsilon$. By continuity of μ, we have for all large γ,

$$\begin{aligned}\mu(E \cap \gamma^{-1}[I]) &\leq \mu(U \cap \gamma^{-1}[I]) + \varepsilon \leq C \cdot \mu(U \cap \gamma^{-1}[J]) + 2\varepsilon \\ &\leq C \cdot \mu(E \cap \gamma^{-1}[J]) + (C+2)\varepsilon.\end{aligned}$$

Since ε was arbitrary, it follows that $\Lambda(\mu|_E) \subseteq QS(C)$. As $QS(C)$ is a positive cone, we deduce that $\Lambda(\nu) \subseteq QS(C)$ for $0 \leq \nu \in L(\mu)$. Finally, let $\sigma. \in \breve{\Lambda}(\mu)$. Let P be the essential range of $\sigma.$, i.e, the smallest weak* closed set P such that $\sigma_x \in P$ μ-a.e. Then P is contained in the weak* closure of $\{\int \sigma_x\,d\nu(x) : 0 \leq \nu \in L(\mu),\ \|\nu\| = 1\} = \bigcup\{\Lambda(\nu) : 0 \leq \nu \in L(\mu),\ \|\nu\| = 1\}$, which, by the above, is contained in $QS(C)$. □

As an example of the pathology possible for $\Lambda(\mu)$, we present the following observation.

PROPOSITION 4.7. *There is a measure $\mu \in M(\mathbf{T})$ such that for any probability measure $\nu \in M(\mathbf{T})$, there exists $\sigma. \in \breve{\Lambda}(\mu)$ such that $\sigma_x = \nu$ μ-a.e.*

Proof. Let $\{P_k\}_{k\geq 1}$ be a set of trigonometric polynomials such that $\{P_k \cdot \lambda\}$ is weak* dense in the set of probability measures. Let $\{n_k\} \subseteq \mathbf{N}$ satisfy $n_{k+1} \geq 3n_k \cdot \deg P_k$. Form the generalized Riesz product [**HMP**, Chapitre 5] $\mu = \prod_{k\geq 1} P_k(n_k x)$. Then given a probability ν, let $P_{k_l}\lambda \xrightarrow{w^*} \nu$. For any r, $m \in \mathbf{Z}$, it is easy to see that $\hat{\mu}(r+mn_{k_l}) \to \hat{\mu}(r)\hat{\nu}(m)$, i.e., $\delta(n_{k_l}x) \to \nu$ in $L(M(\mathbf{T}), M(\mathbf{T}))_\mu$. □

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Received August 15, 1988 and in revised form December 6, 1989. Partially supported by an AMS Research Fellowship and an NSF Postdoctoral Fellowship.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 148, No. 1, 1991

ASYMPTOTIC BEHAVIOUR OF SUPERCUSPIDAL CHARACTERS OF p-ADIC GL_3 AND GL_4: THE GENERIC UNRAMIFIED CASE

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This paper describes the singular behaviour of the characters of irreducible supercuspidal representations of π of $G = GL_n(F)$ around 1 in terms of the values at 1 of certain weighted orbital integrals. The weighted orbital integrals are computed when $n = 3$ or 4 and π is generic and unramified.

1. Introduction. Let π be an irreducible supercuspidal representation of $G = GL_n(F)$, where F is a p-adic field of characteristic 0. The character Θ_π of π is a locally constant function on the regular set G_{reg} consisting of all $x \in G$ such that the coefficient of λ^n in the polynomial $\det(\lambda + 1 - \operatorname{Ad} x)$ is nonzero. It is well known that, if $d(\pi)$ is the formal degree of π and $x \in G_{\text{reg}}$ is elliptic and close to the identity, $\Theta_\pi(x) = cd(\pi)$ for some constant c depending only on normalizations of Haar measures. For other $x \in G_{\text{reg}}$ near 1, the value of $\Theta_\pi(x)$ is unknown. Kutzko [**K**] has given a formula for Θ_π when n is prime, but it involves a sum over double cosets in G and cannot easily be evaluated.

The two objects of this paper are as follows. The first is to describe the singular behaviour of the character Θ_π of π around 1 in terms of the values at 1 of certain weighted orbital integrals. To do this, we compare results of Howe and Arthur giving asymptotic expansions for Θ_π. The second is to compute the weighted orbital integrals required to give a formula for Θ_π when $n = 3$ or 4 and π is generic and unramified.

Howe showed that

$$\Theta_\pi(\exp X) = \sum_{\mathscr{O} \in (\mathscr{N}_G)} c_{\mathscr{O}}(\pi)\hat{\mu}_{\mathscr{O}}(X),$$

for $X \in \mathscr{G} = \operatorname{Lie}(G)$ close to zero and such that $\exp X \in G_{\text{reg}}$. $(\mathscr{N}_G)$ denotes nilpotent $\operatorname{Ad} G$-orbits in $\mathscr{G}$, $c_{\mathscr{O}}(\pi)$ is a constant, and $\hat{\mu}_{\mathscr{O}}$ is the Fourier transform of the orbital integral over $\mathscr{O}$. In the case of $GL_n(F)$, the functions $\hat{\mu}_{\mathscr{O}}$ are known. The behaviour of $\Theta_\pi(x)$ as

$x \in G_{\mathrm{reg}}$ approaches 1 is determined by the homogeneity properties of those $\hat{\mu}_{\mathscr{O}}$'s for which $c_{\mathscr{O}}(\pi) \neq 0$. These results are outlined in §2.

In §3 we state results of Arthur [A3], [A4] showing that a weighted orbital integral has a germ expansion valid on a neighbourhood of 1, and that Θ_π itself is a multiple of a weighted orbital integral of a sum of matrix coefficients of π.

The equality of Howe's and Arthur's expansions for Θ_π yields one of the main results of this paper—a formula for each constant $c_{\mathscr{O}}(\pi)$ as a multiple of a certain weighted orbital integral evaluated at 1. We derive this formula in §4. It holds for all n and any supercuspidal representation of $\mathrm{GL}_n(F)$.

In §§5 and 7, we consider a generic, unramified, irreducible supercuspidal representation π of $\mathrm{GL}_3(F)$ or $\mathrm{GL}_4(F)$. Such a representation is known to be induced from a representation of some open subgroup of G. The particular sum of matrix coefficients appearing in the weighted orbital integrals is defined in §5 using results of Kutzko which give the character of the inducing representation. §6 contains a description of the normalizations of measures and the evaluation of the weight factor for the weighted orbital integrals. In §7, we obtain explicit expressions for the constants $c_{\mathscr{O}}(\pi)$ as polynomials in the order q of the residue class field of F.

The equality of Arthur's expansion and Harish-Chandra's generalization of Howe's expansion to a reductive p-adic group can be expected to yield information about the character Θ_π of any supercuspidal representation π. However, the functions $\hat{\mu}_{\mathscr{O}}$, which are not known in general, may be difficult to compute, and the germ expansion for weighted orbital integrals is more complicated than that for $\mathrm{GL}_n(F)$.

I would like to thank Paul Sally for helpful discussions and James Arthur for explaining his results about weighted orbital integrals.

2. Fourier transforms and characters of admissible representations. Throughout this section, G will be the F-points of a connected, reductive F-group. Let π be an irreducible admissible representation of G. Θ_π denotes the character of π. We summarize results of Harish-Chandra and Howe relating the values of Θ_π near singular points in G to certain Fourier transforms.

Recall the definition of the Fourier transform on the Lie algebra $\mathscr{G}$ of G. For $f \in C_c^\infty(\mathscr{G})$, the function $\hat{f} \in C_c^\infty(\mathscr{G})$ is given by:

$$\hat{f}(X) = \int_{\mathscr{G}} \psi(B(X, Y)) f(Y)\, dY, \qquad X \in \mathscr{G},$$

where B is a nondegenerate symmetric G-invariant bilinear form on $\mathscr{G}$, ψ is a nontrivial character of F and dY is a Haar measure on the additive group of $\mathscr{G}$. The map $f \mapsto \hat{f}$ is a bijection of $C_c^\infty(\mathscr{G})$. The Fourier transform of a distribution T on $\mathscr{G}$ is defined by $\hat{T}(f) = T(\hat{f})$. Let $\mathscr{G}_{\mathrm{reg}}$ be the set of semisimple elements X in $\mathscr{G}$ such that $\det(\operatorname{ad} X)_{\mathscr{G}/\mathscr{H}} \neq 0$, where $\mathscr{H}$ is a Cartan subalgebra containing X.

THEOREM 2.1 [HC2, *Theorem* 3]. *Let T be a G-invariant distribution on $\mathscr{G}$ which is supported on the closure of* $\operatorname{Ad} G(\omega)$ *for some compact set $\omega \subset \mathscr{G}$. Then there exists a locally integrable function ϕ_T on $\mathscr{G}$ such that*

1. $\hat{T}(f) = \int_{\mathscr{G}} \phi_T(X) f(X)\, dX$, $f \in C_c^\infty(\mathscr{G})$.
2. *ϕ_T is locally constant on $\mathscr{G}_{\mathrm{reg}}$.*

Let $X_0 \in \mathscr{G}$ and $\mathscr{O} = \operatorname{Ad} G(X_0)$. If G_{X_0} is the stabilizer of X_0 in G, let dx^* be a G-invariant measure on $G_{X_0} \backslash G$. Then

$$\mu_{\mathscr{O}}(f) = \int_{G_{X_0} \backslash G} f(\operatorname{Ad} x^{-1}(X_0))\, dx^*$$

converges for $f \in C_c^\infty(\mathscr{G})$ and $f \mapsto \mu_{\mathscr{O}}(f)$ is a G-invariant distribution on $\mathscr{G}$.

COROLLARY 2.2 [HC2]. *There exists a locally integrable function $\hat{\mu}_{\mathscr{O}} : \mathscr{G} \to \mathbf{C}$ which is locally constant on $\mathscr{G}_{\mathrm{reg}}$ and*

$$\hat{\mu}_{\mathscr{O}}(f) = \int_{\mathscr{G}} \hat{\mu}_{\mathscr{O}}(X) f(X)\, dX,$$

for $f \in C_c^\infty(\mathscr{G})$.

Let $(\mathscr{N}_G)$ be the set of nilpotent G-orbits in $\mathscr{G}$. If q is the order of the residue class field of F, $|\cdot|$ denotes the norm on F which satisfies $|\varpi| = q^{-1}$ for any prime element ϖ of F. For $\gamma \in G$, let G_γ be the centralizer of γ in G, and let $\mathscr{G}_\gamma$ be the Lie algebra of G_γ.

PROPOSITION 2.3 [HC2]. *For $\mathscr{O} \in (\mathscr{N}_G)$, $X \in \mathscr{G}$ and $t \in F^*$, $\hat{\mu}_{\mathscr{O}}(t^2 X) = |t|^{-\dim \mathscr{O}} \hat{\mu}_{\mathscr{O}}(X)$.*

Proof. For $f \in C_c^\infty(\mathscr{G})$, define $f_t(X) = f(t^{-1}X)$, $X \in \mathscr{G}$. It is well-known that $\mu_{\mathscr{O}}(f_{t^2}) = |t|^{\dim \mathscr{O}} \mu_{\mathscr{O}}(f)$. This, together with $\widehat{(f_t)} = |t|^{\dim \mathscr{G}} (\hat{f})_{t^{-1}}$, proves the proposition.

THEOREM 2.4 [**HC2**, *Theorem* 5]. *Let γ be a semisimple point in G. For any irreducible admissible representation π of G, there exist unique complex numbers $c_{\mathscr{O}}(\pi)$, one for each nilpotent G_γ-orbit $\mathscr{O}$ in $\mathscr{G}_\gamma$, such that*

$$\Theta_\pi(\gamma \exp X) = \sum_{\mathscr{O}} c_{\mathscr{O}}(\pi)\hat{\nu}_{\mathscr{O}}(X),$$

for $X \in \mathscr{G}_\gamma$ sufficiently near 0. Here $\nu_{\mathscr{O}}$ is the G_γ-invariant measure on $\mathscr{O}$, and $\hat{\nu}_{\mathscr{O}}$ is the Fourier transform of $\nu_{\mathscr{O}}$ on $\mathscr{G}_\gamma$.

REMARK. The case $G = \mathrm{GL}_n(F)$ and $\gamma = 1$ is due to Howe [**H**].

The functions $\{\hat{\mu}_{\mathscr{O}} | \mathscr{O} \in (\mathscr{N}_G)\}$ are linearly independent on $V \cap \mathscr{G}_{\mathrm{reg}}$, for any neighbourhood V of 0 in $\mathscr{G}$ [**HC2**, Theorem 4]. Therefore the functions $\{\hat{\mu}_{\mathscr{O}} | c_{\mathscr{O}}(\pi) \neq 0\}$ determine the singular behaviour of Θ_π near 1. Very little is known about the constants $c_{\mathscr{O}}(\pi)$ in general. If π is supercuspidal with formal degree $d(\pi)$, then, if $\{0\}$ denotes the trivial nilpotent orbit, $c_{\{0\}}(\pi) = cd(\pi)$ where $c \neq 0$ depends on the normalization of measures. Howe [**H**] proved that, if π is a supercuspidal representation of $\mathrm{GL}_n(F)$, then $c_{\mathscr{O}}(\pi) = 1$ for the regular (maximal dimension) nilpotent orbit $\mathscr{O}$. Moeglin and Waldspurger [**MW**] have shown a relation between $c_{\mathscr{O}}(\pi)$, for π admissible and some $\mathscr{O}$, and dimensions of certain Whittaker models. As far as the functions $\hat{\mu}_{\mathscr{O}}$ themselves are concerned, there is some information available in [**MW**] for induced nilpotent classes, and for $G = \mathrm{GL}_n(F)$ the $\hat{\mu}_{\mathscr{O}}$'s are known due to Howe (see Lemma 4.1).

3. Weighted orbital integrals and characters of supercuspidal representations. We state several results due to Arthur which will be used in later sections. Theorem 3.4 relates the character Θ_π of a supercuspidal representation π to a weighted orbital integral of a sum of matrix coefficients of π. Theorem 3.5 gives a germ expansion for weighted orbital integrals. A vanishing property for weighted orbital integrals of cusp forms is stated in Proposition 3.9. In Proposition 3.7, we derive a formula for the weighted germ g_M^G corresponding to the trivial unipotent class in a Levi subgroup M.

Our notation follows that of Arthur [**A2**]–[**A4**] except in one respect: the boldface letter $\mathbf{G}$ will be used to denote an algebraic group defined over F, and $G = \mathbf{G}(F)$ will be the F-rational points of $\mathbf{G}$. By a Levi subgroup M of G, we mean $M = \mathbf{M}(F)$, where $\mathbf{P} = \mathbf{MN}$ is a parabolic subgroup of $\mathbf{G}$. If $A_{\mathbf{M}}$ is the split component of $\mathbf{M}$, then $A_M = A_{\mathbf{M}}(F)$. Let $\mathscr{F}(M)$, resp. $\mathscr{L}(M)$, be the collection of parabolic, resp. Levi, subgroups of G which contain

M. Given a parabolic subgroup $P = \mathbf{P}(F)$, M_P and N_P denote its Levi component and unipotent radical, respectively. Let $\mathscr{P}(M) = \{P \in \mathscr{F}(M) \mid M_P = M\}$. The chambers in the real vector space $\underline{a}_M = \mathrm{Hom}(X(\mathbf{M})_F, \mathbf{R})$ parametrize the set $\mathscr{P}(M)$, where $X(\mathbf{M})_F$ is the group of characters of $\mathbf{M}$ which are defined over F.

We now review the notation required in order to define the weights v_M occurring in the weighted orbital integrals. Given M, choose a special maximal compact subgroup K of G which is in good position relative to M. For $P \in \mathscr{P}(M)$ and $x = n_P(x)m_P(x)k(x)$, with $n_P(x) \in N_P$, $m_P(x) \in M_P$, and $k(x) \in K$, set $H_P(x) = H_M(m_P(x))$. Here $H_M\colon M \to \underline{a}_M$ is given by:

$$e^{\langle H_M(m), \chi\rangle} = |\chi(m)|, \qquad m \in M, \ \chi \in X(\mathbf{M})_F.$$

Let $\underline{a}_M^G$ be the kernel of the canonical map from $\underline{a}_M$ onto $\underline{a}_G$. There is a compatible embedding of $\underline{a}_G$ into $\underline{a}_M$ resulting from the embeddings of $X(\mathbf{M})_F$ and $X(\mathbf{G})_F$ into the character groups $X(A_{\mathbf{M}})$ and $X(A_{\mathbf{G}})$ of $A_{\mathbf{M}}$ and $A_{\mathbf{G}}$, respectively. Therefore, $\underline{a}_M = \underline{a}_M^G \oplus \underline{a}_G$. Fix a Weyl-invariant norm $\|\cdot\|$ on $\underline{a}_{M_0}$, where $M_0 \subset M$ is a minimal Levi subgroup. The restriction of $\|\cdot\|$ to each of the subspaces $\underline{a}_M$, $M \in \mathscr{L}(M_0)$, yields a measure on $\underline{a}_M$. We take the quotient measure on $\underline{a}_M^G \simeq \underline{a}_M/\underline{a}_G$.

Let $P \in \mathscr{P}(M)$. The roots of (P, A_M) will be regarded as characters of A_M or as elements of the dual space $\underline{a}_M^*$ of $\underline{a}_M$. Let Δ_P be the set of simple roots of (P, A_M). If $\alpha \in \Delta_P$, the co-root $\alpha^\vee$ is defined as follows. Choose a minimal Levi subgroup $M_0 \subset M$. If β is a reduced root of (G, A_{M_0}), the co-root $\beta^\vee$ is an element of the lattice $\mathrm{Hom}(X(A_{\mathbf{M}_0}), \mathbf{Z})$ in $\underline{a}_{M_0}$. For $P_0 \in \mathscr{P}(M_0)$, with $P_0 \subset P$, there is exactly one root $\beta \in \Delta_{P_0}$ such that $\beta | A_{M_0} = \alpha$. $\alpha^\vee$ is defined to be the projection of $\beta^\vee$ onto $\underline{a}_M^G$. Set $\Delta_P^\vee = \{\alpha^\vee | \alpha \in \Delta_P\}$. The lattice $\mathbf{Z}(\Delta_P^\vee)$ in $\underline{a}_M^G$ generated by $\Delta_P^\vee$ is independent of the choice of $P \in \mathscr{P}(M)$ [**A4**, p. 12]. For $x \in G$, $v_M(x)$ is equal to the volume of the convex hull of the projection of the points $\{-H_P(x) | P \in \mathscr{P}(M)\}$ onto $\underline{a}_M^G$. Set $\theta_P(\lambda) = \mathrm{vol}(\underline{a}_M^G/\mathbf{Z}(\Delta_{\mathbf{P}}^\vee))^{-1} \prod_{\alpha \in \Delta_{\mathbf{P}}} \lambda(\alpha^\vee)$, $\lambda \in i\underline{a}_M^*$. Then, [**A2**, p. 36]

$$v_M(x) = \lim_{\lambda \to 0} \sum_{P \in \mathscr{P}(M)} e^{-\lambda(H_P(x))} \theta_P(\lambda)^{-1}, \qquad \lambda \in i\underline{a}_M^*,$$

and, [**A2**, p. 46]

$$(3.1) \quad v_M(x) = 1/r! \sum_{P \in \mathscr{P}(M)} (-\lambda(H_P(x)))^r \theta_P(\lambda)^{-1},$$

where $r = \dim(A_M/A_G)$.

For $\gamma \in G$, define $D(\gamma) = D_G(\gamma) = \det(1 - \mathrm{Ad}(\sigma))_{\mathscr{G}/\mathscr{G}_\sigma}$, where σ is the semisimple part of γ. Let $f \in C_c^\infty(G)$. For a Levi subgroup M, set $A_{M,\mathrm{reg}} = \{a \in A_M \mid \mathbf{G}_a \subset \mathbf{M}^0\}$. The weighted orbital integral is defined for $\gamma \in M$. If $G_\gamma \subset M$, then **[A3**, p. 234]

$$J_M(\gamma, f) = |D(\gamma)|^{1/2} \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) v_M(x)\, dx. \tag{3.2}$$

More generally, for any $\gamma \in M$ **[A3**, §5],

$$J_M(\gamma, f) = \lim_{a \to 1} \sum_{L \in \mathscr{L}(M)} r_M^L(\gamma, a) J_L(a\gamma, f), \qquad a \in A_{M,\mathrm{reg}}, \tag{3.3}$$

where $r_M^L(\gamma, a)$, $L \in \mathscr{L}(M)$ is a certain real-valued function. We remark that $f \mapsto J_M(\gamma, f)$ is not an invariant distribution on $C_c^\infty(G)$. If γ_1 and γ_2 are conjugate in M, then $J_M(\gamma_1, f) = J_M(\gamma_2, f)$, so $J_M(\mathscr{O}, f)$ is well-defined for any conjugacy class $\mathscr{O} \subset M$. The restriction of $f \mapsto J_M(\gamma, f)$ to the space of cusp forms is G-invariant.

Let M_{ell} be the set of γ in M which lie in some elliptic Cartan subgroup of M. Recall that an admissible representation π of G is *supercuspidal* if its matrix coefficients are compactly supported modulo A_G.

THEOREM 3.4 **[A4]**. *Let π be a supercuspidal representation of G. Suppose f is a finite sum of matrix coefficients of π. For $\gamma \in M_{\mathrm{ell}} \cap G_{\mathrm{reg}}$, where M is a Levi subgroup,*

$$(-1)^{\dim(A_M/A_G)} \Theta_\pi(f) |D(\gamma)|^{1/2} \Theta_\pi(\gamma) = J_M(\gamma, f).$$

REMARK. 1. Although f is not in $C_c^\infty(G)$, the weighted orbital integrals of f still converge because $\mathrm{supp}\, f$ is compact modulo A_G.

2. The corresponding result for reductive Lie groups appears in **[A1]**.

3. In Theorem 3.4, and, with the exception of the proof of Proposition 3.9, in the remainder of the paper, if $\gamma \in G_{\mathrm{reg}}$, the integral in $J_M(\gamma, f)$ is taken over $A_M \backslash G$ instead of $G_\gamma \backslash G$. The weight factor v_M is invariant under left translation by elements of M, so this is equivalent to multiplying the original definition (3.2) by the measure of $A_M \backslash G_\gamma$.

The measures on $A_G \backslash G$, $A_M \backslash G$, and $\underline{a}_M / \underline{a}_G$ must be normalized correctly in order for Theorem 3.4 to hold. Let $\kappa_M = A_M \cap K$. Given measures on $\underline{a}_M$, $\underline{a}_G$, and $\underline{a}_M / \underline{a}_G$ defined using the restriction of a fixed Weyl-invariant metric on $\underline{a}_{M_0}$, as above, the compatibility requirement for the measures is as follows [A4, p. 5]:

$$\mathrm{vol}_{A_M}(\kappa_M) = \mathrm{vol}(\underline{a}_M/H_M(A_M)),$$
$$\mathrm{vol}_{A_G}(\kappa_G) = \mathrm{vol}(\underline{a}_G/H_G(A_G)).$$

The measures on $A_M\backslash G$ and $A_G\backslash G$ are the quotient measures induced by the measures on G, A_M and A_G.

If $\gamma \in G_{\mathrm{reg}} \cap M$, the weighted orbital integral $J_M(\gamma, f)$ has a germ expansion on neighbourhoods of semisimple points in M. The weighted germs are uniquely determined up to orbital integrals on M. Suppose ϕ_1 and ϕ_2 are functions defined on an open subset Σ of σM_σ which contains an M_σ-invariant neighbourhood of the semisimple element σ. ϕ_1 is (M, σ)-equivalent to ϕ_2, $\phi_1(\gamma) \overset{(M,\sigma)}{\sim} \phi_2(\gamma)$, if $\phi_1(\gamma) - \phi_2(\gamma) = J_M^M(\gamma, h)$ for $\gamma \in \Sigma \cap U$, where U is a neighbourhood of σ in M, and $h \in C_c^\infty(M)$. Let $(\sigma\mathscr{U}_{M_\sigma})$ be the finite set of orbits in $\sigma\mathscr{U}_{M_\sigma}$ under conjugation by $M(\sigma) = \mathbf{M}^0(F)_\sigma$. Let $\gamma \in M$. Generalizing the definition of Lusztig and Spaltenstein [LS], Arthur [A3, p. 255] defines the induced space of orbits $\gamma_M^G = \gamma^G$ in G as the finite union of all $\mathbf{G}^0(F)$-orbits in G which intersect γN_P in an open set for any $P \in \mathscr{P}(M)$.

THEOREM 3.5 [A3, *Prop.* 9.1, *Prop.* 10.2]. 1. *There are uniquely determined (M, σ)-equivalence classes of functions $\gamma \mapsto g_M^G(\gamma, \mathscr{O})$, $\gamma \in \sigma M_\sigma \cap G_{\mathrm{reg}}$ parametrized by the classes $\mathscr{O} \in (\sigma\mathscr{U}_{L_\sigma})$ such that, for any $f \in C_c^\infty(G)$,*

$$J_M(\gamma, f) \overset{(M,\sigma)}{\sim} \sum_{L\in\mathscr{L}(M)} \sum_{\mathscr{O}\in(\sigma\mathscr{U}_{L_\sigma})} g_M^L(\gamma, \mathscr{O}) J_L(\mathscr{O}, f),$$

where $J_L(\mathscr{O}, f) \overset{\mathrm{def}}{=} J_L(\sigma u, f)$ for any $\sigma u \in \mathscr{O}$.

2. *Let $t \in F^*$ and $w \in (\mathscr{U}_G)$. Set $d^G(w) = (1/2)(\dim G_w - \mathrm{rank}\, G)$. If $x = \exp(X)$, let $x^t = \exp(tX)$.*

$$g_M^G(\gamma^t, w^t) \overset{(M,1)}{\sim} |t|^{d^G(w)} \sum_{L\in\mathscr{L}(M)} \sum_{u\in(\mathscr{U}_L)} g_M^L(\gamma, u) c_L(u, t)[u^G : w],$$

where the $c_L(u, t)$ are certain real-valued functions and $[u^G : w]$ is 1 if $w \in u^G$, 0 otherwise.

LEMMA 3.6. *Let π be a supercuspidal representation of G and f a matrix coefficient of π. Then $\Theta_\pi(f) = d(\pi)^{-1} f(1)$, where $d(\pi)$ is the formal degree of π.*

Proof. Let $(\ ,\ )$ denote a G-invariant inner product on the representation space V of π. Let $e_1, e_2, \ldots$ be an orthonormal basis for V. $f(x) = (v, \pi(x)w)$, some $v, w \in V$. We use the orthogonality relations for matrix coefficients of supercuspidal representations [**HC1**, p. 5] to evaluate

$$\begin{aligned}\Theta_\pi(f) = \operatorname{tr}\pi(f) &= \operatorname{tr}\left(\int_{A_G\backslash G} f(x)\pi(x)\,dx^*\right)\\ &= \sum_i \int_{A_G\backslash G}(v, \pi(x)w)(\pi(x)e_i, e_i)\,dx^*\\ &= \sum_i d(\pi)^{-1}\overline{(v, e_i)}(e_i, w)\\ &= d(\pi)^{-1}(v, w) = d(\pi)^{-1}f(1).\end{aligned}$$

PROPOSITION 3.7. *Assume G is connected. Let $\gamma \in M_{\mathrm{ell}} \cap G_{\mathrm{reg}}$. If l is the F-rank of G and $d(\mathrm{St}_G)$ is the formal degree of the Steinberg representation of G, then*

$$g_M^G(\gamma, 1) \overset{(M, 1)}{\sim} (-1)^{(l-\dim A_M)}|D(\gamma)|^{1/2}/d(\mathrm{St}_G).$$

Proof. Let π be a supercuspidal representation of G. Choose a matrix coefficient f of π such that $f(1) \neq 0$. By Lemma 3.6, $\Theta_\pi(f) \neq 0$.

First, let $\gamma \in G_{\mathrm{ell}} \cap G_{\mathrm{reg}}$. From [**R**], the leading term in the Shalika germ expansion of $J_G(\gamma, f)$ is $(-1)^{(l-\dim A_G)}|D(\gamma)|^{1/2}f(1)/d(\mathrm{St}_G)$. We also have, by Theorem 3.4,

$$J_G(\gamma, f) = \Theta_\pi(f)|D(\gamma)|^{1/2}\Theta_\pi(\gamma).$$

The leading term in Harish-Chandra's asymptotic expansion of $|D(\gamma)|^{1/2}\Theta_\pi(\gamma)$ is $c_{\{0\}}(\pi)|D(\gamma)|^{1/2}$, because $\hat{\mu}_{\{0\}} \equiv 1$. By $\{0\}$, we mean the trivial nilpotent orbit in $\mathscr{G}$. Thus the leading term in $J_G(\gamma, g)$ is also equal to $\Theta_\pi(f)|D(\gamma)|^{1/2}c_{\{0\}}(\pi)$, which means

$$c_{\{0\}}(\pi) = (-1)^{(l-\dim A_G)}f(1)/\Theta_\pi(f)d(\mathrm{St}_G),$$

which, by Lemma 3.6, equals $(-1)^{(l-\dim A_G)}d(\pi)/d(\mathrm{St}_G)$.

Now let $\gamma \in M_{\mathrm{ell}} \cap G_{\mathrm{reg}}$. From Theorem 3.4 and Theorem 3.5(1),

$$\begin{aligned}|D(\gamma)|^{1/2}\Theta_\pi(\gamma) &= (-1)^{\dim(A_M/A_G)}\Theta_\pi(f)^{-1}J_G(\gamma, f)\\ &\overset{(M, 1)}{\sim} (-1)^{\dim(A_M/A_G)}\Theta_\pi(f)^{-1}\sum_{L\in\mathscr{L}(M)}\ \sum_{\mathscr{O}\in(\mathscr{U}_L)} g_M^L(\gamma, \mathscr{O})J_L(\mathscr{O}, f).\end{aligned}$$

We will show that $g_M^G(\gamma, 1)$ is the only term occurring in the above expansion having the same homogeneity as $|D(\gamma)|^{1/2}$. Given this, we then have

$$\frac{(-1)^{(l-\dim A_G)}d(\pi)}{d(\mathrm{St}_G)}|D(\gamma)|^{1/2} \overset{(M,1)}{\sim} (-1)^{\dim(A_M/A_G)}\Theta_\pi(f)^{-1}f(1)g_M^G(\gamma, 1),$$

which, using Lemma 3.6, yields the desired expression for $g_M^G(\gamma, 1)$.

Let $L \in \mathscr{L}(M)$ and $u \in (\mathscr{U}_L)$. Since $[u^G : 1] = 1 \Leftrightarrow L = G$ and $u = 1$, and $c_G(1, t) = 1$ (see [A3, §10] for the definition of $c_L(u, t)$), Proposition 3.7(2) reduces to:

$$g_M^G(\gamma^t, 1) \overset{(M,1)}{\sim} |t|^{1/2(\dim G - \mathrm{rank}\, G)} g_M^G(\gamma, 1).$$

Let $w \in (\mathscr{U}_G)$, $w \neq 1$. The power of $|t|$ in $|t|^{d^G(w)}c_L(u, t)$, $u \in (\mathscr{U}_L)$ such that $[u^G : w] = 1$, is less than $d^G(1)$. Therefore, all other terms in the above weighted germ expansion for $|D(\gamma)|^{1/2}\Theta_\pi(\gamma)$ have smaller homogeneity than $g_M^G(\gamma, 1)$.

LEMMA 3.8 [A3, *Cor.* 6.3]. *Let* $L_1 \in \mathscr{L}(M)$. *Then*

$$J_{L_1}(\gamma^{L_1}, f) = \lim_{a\to 1} \sum_{L\in\mathscr{L}(L_1)} r_{L_1}^L(\gamma, a) J_L(a\gamma, f), \qquad a \in A_{M,\mathrm{reg}}.$$

REMARK. $J_{L_1}(\gamma^{L_1}, f) \overset{\mathrm{def}}{=} \sum_i J_L(\mathscr{O}_i, f)$, where $\gamma^{L_1} = \bigcup_i \mathscr{O}_i$.

Recall that a locally constant function ϕ on G is a *cusp form* if, for all $x \in G$ and all proper parabolic subgroups $P = MN$ of G, $\int_N \phi(xn)\,dn = 0$. The following is a generalization of the well-known fact that orbital integrals of cusp forms vanish at nonelliptic semisimple points in G.

PROPOSITION 3.9. *Let* f *be a cusp form on* G *such that* supp f *is compact modulo* A_G. *Suppose* γ *is a semisimple element in a Levi subgroup* M *and* $\gamma \notin M_{\mathrm{ell}}$. *Then* $J_M(\gamma, f) = 0$.

Proof. This is due to Arthur. We give a rough outline of the proof. Using results about products of (G, M)-families from §§6 and 10 of [A2], it is possible to show that, for $M_1 \subset M$,

$$v_M(x) = \sum_{\{Q\in\mathscr{F}(M_1),\, Q\neq G\}} a_Q v_{M_1}^Q(x), \qquad x \in G,$$

where $v_{M_1}^Q(x) = \lim_{\lambda\to 0} \sum_{\{P\in\mathscr{P}(M_1)|P\subset Q\}} e^{-\lambda(H_P(x))}\theta_P^Q(\lambda)^{-1}$ and $a_Q \in \mathbf{R}$. Here, $\theta_P^Q(\lambda)$ is defined in the same way as θ_P, but with respect to the set Δ_P^Q of simple roots of $(P\cap M_Q, A_P)$ and the associated set $\{\alpha^\vee|\alpha\in\Delta_P^Q\}$.

Because $\gamma\notin M_{\mathrm{ell}}$, there is a Levi subgroup M_1 properly contained in M with $\gamma\in M_1$. Assume that $M_\gamma = G_\gamma$. Then

$$\begin{aligned} J_M(\gamma, f) &= |D(\gamma)|^{1/2}\int_{M_\gamma\backslash G} f(x^{-1}\gamma x)v_M(x)\,dx \\ &= |D(\gamma)|^{1/2}\sum_{\{Q\in\mathscr{F}(M_1),\, Q\neq G\}} a_Q\int_{M_\gamma\backslash G} f(x^{-1}\gamma x)v_{M_1}^Q(x)\,dx. \end{aligned}$$

Note that $M_\gamma = M_{1_\gamma}$. By [**A2**, (8.1)], the integral corresponding to Q in the sum above is equal to $J_{M_1}^{M_Q}(\gamma, f_Q)$, where $J_{M_1}^{M_Q}$ is the weighted orbital integral for the Levi subgroup M_1 of M_Q, and $f_Q\colon M_Q\to\mathbf{C}$ is given by $f_Q(m) = \delta_Q(m)^{1/2}\int_{N_Q}\int_K f(k^{-1}mnk)\,dk\,dn$. Since f is a cusp form, $f_Q\equiv 0$ for $Q\neq G$. Therefore, $J_M(\gamma, f) = 0$.

For general γ, and $a\in A_{M,\mathrm{reg}}$ close to 1, the element $a\gamma$ is not elliptic in any $L\in\mathscr{L}(M)$, and $L_{a\gamma} = G_{a\gamma}$. Thus the above argument shows that $J_L(a\gamma, f) = 0$. From (3.3), $J_M(\gamma, f) = 0$.

4. Some results for $G = \mathrm{GL}_n(F)$. Assume π is an irreducible supercuspidal representation of $G = \mathrm{GL}_n(F)$. The main result of this section, Theorem 4.4, expresses the constant $c_{\mathscr{O}}(\pi)$, $\mathscr{O}\in(\mathscr{N}_G)$, as a multiple of a certain weighted orbital integral of a sum of matrix coefficients of π. Because of the one-to-one correspondence between the set $(\mathscr{N}_G)$ of nilpotent G-orbits in $\mathscr{G}$ and the set $(\mathscr{U}_G)$ of unipotent conjugacy classes in G, we can view $c_{\mathscr{O}}(\pi)$ and $\hat{\mu}_{\mathscr{O}}$ as corresponding to $\mathscr{O}\in(\mathscr{U}_G)$. We begin by defining some notation which allows us to state our results in terms of unipotent conjugacy classes. For $\mathscr{O}\in(\mathscr{U}_G)$, let $\mathscr{P}(\mathscr{O}) = \{P = MN \mid \mathscr{O} = 1_M^G\}$. If $P\in\mathscr{P}(\mathscr{O})$, let π_P be the admissible representation of G induced (unitarily) from the character $\delta_P^{-1/2}$ of P, and let Θ_P denote the character of π_P. If $P_1, P_2\in\mathscr{P}(\mathscr{O})$, then P_1 and P_2 are conjugate in G, and π_{P_1} and π_{P_2} are equivalent, so $\Theta_{P_1} = \Theta_{P_2}$. Let $\Theta_{\mathscr{O}}$ denote the common value Θ_P, $P\in\mathscr{P}(\mathscr{O})$.

For a Levi subgroup M of G, set $\mathscr{L}_{\mathscr{O}}(M) = \{L\in\mathscr{L}(M)\mid \mathscr{O} = 1_L^G\}$. If $L_1, L_2\in\mathscr{L}_{\mathscr{O}}(M)$ and K is a special maximal compact subgroup in good position relative to M, then $L_1 = kL_2k^{-1}$ for some $k\in K$ and

[**A3**, p. 235] $J_{L_2}(1, f) = J_{L_1}(1, f^k)$, where $f^k(x) = f(kxk^{-1})$. Assume f is a cusp form. Then $J_{L_1}(1, f^k) = J_{L_1}(1, f)$, so $J_{L_1}(1, f) = J_{L_2}(1, f)$. We denote the common value by $J_{\mathscr{O}}(1, f)$. Similarly, let $d(\mathrm{St}(\mathscr{O}))$ be the formal degree of the Steinberg representation of any $L \in \mathscr{L}_{\mathscr{O}}(M_0)$, where M_0 is a minimal Levi subgroup. We note that $\mathscr{L}_{\mathscr{O}}(M_0) \neq \varnothing$ for any $\mathscr{O} \in (\mathscr{U}_G)$. Finally, we set $w_{\mathscr{O}} = |N_G(A)/Z_G(A)|$, for A equal to the split component of any $P \in \mathscr{P}(\mathscr{O})$, and $N_G(A)$ (resp. $Z_G(A)$) the normalizer (resp. centralizer) of A in G. Let $K = \mathrm{GL}_n(\mathscr{O}_F)$, where $\mathscr{O}_F$ is the ring of integers in F. K is a special maximal compact subgroup of G. For convenience, we consider only those Levi subgroups M which are in $\mathscr{L}(M_0)$, where M_0 is the subgroup of diagonal matrices in G. For all such M, $G = PK = KP$ if $P \in \mathscr{P}(M)$.

LEMMA 4.1 [**H**]. *Measures can be normalized so that* $\hat{\mu}_{\mathscr{O}}(\log \gamma) = \mathscr{Q}_{\mathscr{O}}(\gamma)$, *for* $\gamma \in G_{\mathrm{reg}}$ *in a sufficiently small neighbourhood of* 1.

REMARK. In §6, we normalize measures on G and its Levi subgroups. We will assume that the measure on the Lie algebra $\mathscr{G}$ has been normalized so that Lemma 4.1 holds.

LEMMA 4.2. *Let M be a Levi subgroup of G. If $\gamma \in M_{\mathrm{ell}} \cap K \cap G_{\mathrm{reg}}$ and $\mathscr{O} \in (\mathscr{U}_G)$, then*

$$\hat{\mu}_{\mathscr{O}}(\log \gamma) = |D(\gamma)|^{-1/2} w_{\mathscr{O}} \sum_{L \in \mathscr{L}_{\mathscr{O}}(M)} |D_L(\gamma)|^{1/2}.$$

Proof. Let $P_1 = L_1 N_1 \in \mathscr{P}(\mathscr{O})$ with A_1 the split component of L_1. We have simply rewritten van Dijk's [**D**] formula for the induced character:

$$\Theta_{\mathscr{O}}(\gamma) = \sum_{s \in W(A_1, A_M)} {}^s\delta_{P_1}^{-1/2}(\gamma) \frac{|D_{L_1^s}(\gamma)|^{1/2}}{|D(\gamma)|^{1/2}},$$

where $W(A_1, A_M) = \{s \colon A_1 \to A_M \mid s1 - 1,\, a^s = a^y,\, y \in G\}$, and ${}^s\delta_{P_1}^{-1/2}(\gamma) = \delta_{P_1}^{-1/2}(y^{-1}\gamma y)$. $\delta_{P_1}|K \equiv 1$ and y can be taken in K, so ${}^s\delta_{P_1}^{-1/2}(\gamma) = 1$. $W(A_1, A_M) = \varnothing \Leftrightarrow \mathscr{L}_{\mathscr{O}}(M) = \varnothing$. Assume $\mathscr{L}_{\mathscr{O}}(M) \neq \varnothing$ and $L_1 \in \mathscr{L}_{\mathscr{O}}(M)$. Define a map $s \mapsto L$ from $W(A_1, A_M)$ to $\mathscr{L}_{\mathscr{O}}(M)$ by: $L = L_1^s = yL_1y^{-1}$. If $L \in \mathscr{L}_{\mathscr{O}}(M)$, then $L = L_1^y$ for some $y \in K$ and $a \mapsto a^y$ maps A_1 bijectively onto A_L. Since $M \subset L$, $A_L \subset A$. Thus $a \mapsto a^y$ defines an $s \in W(A_1, A_M)$ which maps to L. Suppose $L = L_2^{s_2}$ for some $s_2 \in W(A_1, A_M)$. Then

$A_L = yA_1y^{-1} = y_2A_1y_2^{-1}$, so $y_2^{-1}y \in N_G(A_1)$. Clearly $s = s_2 \Leftrightarrow y_2^{-1}y \in Z_G(A_1)$. Thus $s \mapsto L$ is onto and $w_{\mathscr{O}}$-to-one, which proves the lemma.

LEMMA 4.3. *Let f be a cusp form on G which is compactly supported modulo A_G.*

1. *If $u \in (\mathscr{U}_M)$, and $u \neq 1$, then $J_M(u, f) = 0$.*
2. *$J_M(1, f) = \lim_{a \to 1} J_M(a, f)$, $a \in A_{M,\mathrm{reg}}$.*

Proof. 1. There exists a Levi subgroup $M_1 \subset M$ such that $u = 1_{M_1}^M$. By [**A3**, Corollary 6.3],

$$J_M^G(u, f) = \lim_{a \to 1} \sum_{L \in \mathscr{L}(M)} r_M^L(1, a) J_L(a, f), \qquad a \in A_{M_1,\mathrm{reg}}.$$

Because $a \in A_{M_1,\mathrm{reg}}$ and $M_1 \neq L$ for each $L \in \mathscr{L}(M)$, a is not elliptic in L. Therefore, by Proposition 3.9, $J_L(a, f) = 0$.

2. For $L \in \mathscr{L}(M)$, $L \neq M$, we have $J_L(a, f) = 0$, since $a \in A_{M,\mathrm{reg}}$ is not elliptic in L. By definition, [**A3**] $r_M^M(1, a) = 1$. Thus

$$\begin{aligned} J_M(1, f) &= \lim_{a \to 1} \sum_{L \in \mathscr{L}(M)} r_M^L(1, a) J_L(a, f) \\ &= \lim_{a \to 1} J_M(a, f). \end{aligned}$$

Let π be a supercuspidal representation of G. We now express the coefficients $c_{\mathscr{O}}(\pi)$ in the asymptotic expansion about 1 of the character Θ_π in terms of the weighted orbital integrals at 1 of the matrix coefficients of π.

THEOREM 4.4. *Let f be a finite sum of matrix coefficients of the supercuspidal representation π. Assume $f(1) \neq 0$. For $\mathscr{O} \in (\mathscr{U}_G)$,*

$$c_{\mathscr{O}}(\pi) = \frac{(-1)^{n-1} J_{\mathscr{O}}(1, f) d(\pi)}{w_{\mathscr{O}} d(\mathrm{St}(\mathscr{O})) f(1)}.$$

Proof. Let $\gamma \in M_{0,\mathrm{ell}} \cap G_{\mathrm{reg}}$. Recall [**HC1**] that the matrix coefficients of π are cusp forms. Applying Theorem 3.4, Theorem 3.5(1), and Lemma 4.3(1),

$$\Theta_\pi(\gamma) \overset{(M_0, 1)}{\sim} (-1)^{n-1} \Theta_\pi(f)^{-1} |D(\gamma)|^{-1/2} \sum_{L \in \mathscr{L}(M_0)} g_{M_0}^L(\gamma, 1) J_L^G(1, f).$$

Writing the sum over $L \in \mathscr{L}(M_0)$ as a double sum over $\mathscr{O} \in (\mathscr{U}_G)$ and $L \in \mathscr{L}_{\mathscr{O}}(M_0)$ and using Proposition 3.7 to substitute $|D_L(\gamma)|^{1/2}/d(\mathrm{St}_L)$ for $g^L_{M_0}(\gamma, 1)$, we obtain

$$\Theta_\pi(\gamma) \overset{(M_0, 1)}{\sim} (-1)^{n-1}\Theta_\pi(f)^{-1}|D(\gamma)|^{-1/2} \sum_{\mathscr{O}\in\mathscr{U}_G} J_{\mathscr{O}}(1, f)/d(\mathrm{St}(\mathscr{O}))$$

$$\times \left(\sum_{L\in\mathscr{L}_{\mathscr{O}}(M_0)} |D_L(\gamma)|^{1/2} \right).$$

For $\gamma \in M_{0,\mathrm{ell}} \cap G_{\mathrm{reg}}$ close to 1, we also have:

$$\Theta_\pi(\gamma) = \sum_{\mathscr{O}\in(\mathscr{U}_G)} c_{\mathscr{O}}(\pi)\hat{\mu}_{\mathscr{O}}(\log\gamma) \tag{4.6}$$

$$= \sum_{\mathscr{O}\in(\mathscr{U}_G)} c_{\mathscr{O}}(\pi) w_{\mathscr{O}} \left(\sum_{L\in\mathscr{L}_{\mathscr{O}}(M_0)} \frac{|D_L(\gamma)|^{1/2}}{|D(\gamma)|^{1/2}} \right).$$

The two expressions (4.5) and (4.6) differ by an orbital integral on $M_0 = A_{M_0}$, that is, by $c \mid D(\gamma)|^{-1/2}$, for some constant c. Let $\mathscr{O}_{\mathrm{reg}}$ be the regular unipotent class in G. By Lemma 4.3(2), $J_{M_0}(1, f) = J_{\mathscr{O}_{\mathrm{reg}}}(1, f) = \lim_{a\to 1} J_{M_0}(a, f)$, $a \in A_{M_0,\mathrm{reg}}$. Multiplying (4.5) by $(-1)^{n-1}\Theta_\pi(f)|D(a)|^{1/2}$ and letting $a \to 1$, we get

$$J_{\mathscr{O}_{\mathrm{reg}}}(1, f)/d(\mathrm{St}(\mathscr{O}_{\mathrm{reg}})) + c,$$

which must equal $J_{\mathscr{O}_{\mathrm{reg}}}(1, f)$. Since M_0 is abelian, the Steinberg representation of M_0 is just the trivial representation, so $d(\mathrm{St}(\mathscr{O}_{\mathrm{reg}})) = 1$. Therefore $c = 0$.

The functions $\sum_{L\in\mathscr{L}_{\mathscr{O}}(M_0)} |D_L(\gamma)|^{1/2}/|D(\gamma)|^{1/2}$, $\mathscr{O} \in (\mathscr{U}_G)$, are linearly independent on any neighbourhood of 1 intersected with $A_{M_0,\mathrm{reg}}$. Therefore, the equality of (4.5) and (4.6) implies:

$$c_{\mathscr{O}}(\pi) = \frac{(-1)^{n-1}\Theta_\pi(f)^{-1}J_{\mathscr{O}}(1, f)}{w_{\mathscr{O}}d(\mathrm{St}(\mathscr{O}))}.$$

From Lemma 3.6, $\Theta_\pi(f) = f(1)/d(\pi)$.

REMARK. 1. It follows from the definition of the Steinberg character, that is, the character of St_G (see **[Ca]**), that

$$c_{\mathscr{O}}(\mathrm{St}_G) = (-1)^{n-d(\mathscr{O})}\mathrm{card}\,\mathscr{L}_{\mathscr{O}}(M_0),$$

where $d(\mathscr{O}) = \dim A_M$, $M \in \mathscr{L}_{\mathscr{O}}(M_0)$.

2. If $\pi = \operatorname{Ind}_P^G(\tau \otimes \mathrm{id})$, $P = MN$, τ a supercuspidal representation of M, then, using van Dijk's formula in **[D]** which expresses Θ_π in terms of Θ_τ, it is possible to write $c_{\mathscr{O}}(\pi)$, $\mathscr{O} \in (\mathscr{U}_G)$, terms of the constants $c_{\mathscr{O}'}(\tau)$, $\mathscr{O}' \in (\mathscr{U}_M)$.

3. If π is in the discrete series of G and π is not supercuspidal or a twist of St_G, there is no formula for $c_{\mathscr{O}}(\pi)$, $\mathscr{O} \neq \{1\}$.

5. Characters of inducing representations. To find the constant $c_{\mathscr{O}}(\pi)$ for a supercuspidal representation π of $G = \mathrm{GL}_n(F)$, we must evaluate $J_{\mathscr{O}}(1, f)$ for f equal to a sum of matrix coefficients of π such that $f(1) \neq 0$ (Theorem 4.4). Here, we outline how to produce such a function f. It will be shown in Lemma 6.1 that only the values of f on the unipotent set $\mathscr{U}_G$ are required to compute $J_{\mathscr{O}}(1, f)$. Lemma 5.2 gives a formula for the values of f on $\mathscr{U}_G$ for π generic and unramified.

Carayol **[C]** has constructed an infinite family of irreducible unitary representations of KA_G which are called *very cuspidal.* To each such representation σ is attached a positive integer h, the *level* of σ. Given any (unitary) character χ of F^*, the representation $\pi = \operatorname{Ind}_{KA_G}^G \sigma \otimes \chi \circ \det$ is irreducible and supercuspidal. We will say that any such π is *generic* and *unramified.*

The reason for this terminology is as follows. Let p be the residual characteristic of F. If $(p, n) = 1$, the irreducible supercuspidal representations of G are parametrized by conjugacy classes of admissible characters of extensions of degree n over F. For definitions and a general description, see **[CMS]**. Let θ be such a character. In this setting, those supercuspidal representations which correspond to the case where θ is generic over F and the extension of F is unramified are precisely the generic and unramified representations defined above. We remark that Carayol's construction is valid for arbitrary p, and thus we do not place any restriction on p.

LEMMA 5.1 **[C]**. *Let H be an open subgroup of G. Suppose φ is a matrix coefficient of a representation σ of H. For $x \in G$, define $\tilde{\varphi}(x)$ to be $\varphi(x)$, if $x \in H$, and* 0 *otherwise. Then $\tilde{\varphi}$ is a matrix coefficient of* $\operatorname{Ind}_H^G \sigma$.

Let $\pi = \operatorname{Ind}_{KA_G}^G \sigma \otimes \chi \circ \det$ be generic and unramified. By Lemma 5.1, if χ_σ is the character of σ, then $\tilde{\chi}_\sigma$ is a sum of matrix coefficients of $\operatorname{Ind}_{KA_G}^G \sigma$, and we may take $f = \tilde{\chi}_\sigma \chi \circ \det$ as a finite sum of matrix coefficients of π. Note that $f(1) = \dim \sigma \neq 0$. This particular f

is chosen because $\int_K f(k^{-1}uk)\,dk = f(u)$, for $u \in \mathscr{U}_G$, which will simplify the computation of $J_{\mathscr{O}}(1, f)$ (see Lemma 6.1).

Let ϖ be a prime element in F, and let $\mathscr{P}_F = \varpi\mathscr{O}_F$. If j is a positive integer, define $K_j = \{k \in K \mid k \in I + \mathbf{M}_n(\mathscr{P}_F^j)\}$.

LEMMA 5.2. *If σ is a very cuspidal representation of KA_G having level h, then, for $u \in \mathscr{U}_G \cap K$,*

$$\chi_\sigma(u) = \begin{cases} q^{n(n-1)(h-1)/2}(-1)^{n+s_h(u)} \displaystyle\sum_{j=1}^{s_h(u)-1} (q^j - 1), & \text{if } u \in K_{h-1}, \\ 0, & \text{otherwise.} \end{cases}$$

For $u \in K_{h-1}$, $s_h(u)$ is the number of blocks in the Jordan form of $\varpi^{1-h}(u-1)$ viewed as a matrix over $\mathscr{O}_F/\mathscr{P}_F$.

Proof [**K**, *Lemma* 6.6]. The proof given by Kutzko is for n prime, but in fact uses only the very cuspidal property of σ and therefore is valid for arbitrary n.

6. Weights for $\mathrm{GL}_4(F)$**.** To compute the coefficients $c_{\mathscr{O}}(\pi)$, it is necessary to evaluate $J_{\mathscr{O}}(1, f)$ for f equal to a suitable sum of matrix coefficients of π. Proposition 6.5 gives explicit integral formulas for $J_M(1, f)$ for non-minimal Levi subgroups M of $\mathrm{GL}_4(F)$.

On $G = \mathrm{GL}_n(F)$, we take the Haar measure with respect to which $K = \mathrm{GL}_n(\mathscr{O}_F)$ has measure one. The Haar measure on K is the restriction of this measure to K. If $P = MN$ is a parabolic subgroup with $G = KP$, the measures on M and N are normalized so that the measures of $M \cap K$ and $N \cap K$ equal one. Then we have

$$\int_G \varphi(x)\,dx = \int_K \int_M \int_N \varphi(mnk)\,dk\,dm\,dn, \qquad \varphi \in C_c^\infty(G).$$

LEMMA 6.1. *Let f be a cusp form on G which is compactly supported modulo A_G. Then, if $G = KP$ and $P = MN$,*

$$J_M(1, f) = \lim_{a \to 1} \int_N f_K(u) v_M(n)\,du, \qquad a \in A_{M,\mathrm{reg}}.$$

where $n \in N$ is defined by $u = a^{-1}n^{-1}an$ and

$$f_K(x) = \int_K f(k^{-1}xk)\,dk,$$

for $x \in G$.

Proof. From Lemma 4.3(2) and (3.2),

$$\begin{aligned} J_M(1, f) &= \lim_{a\to 1} J_M(a, f) \\ &= \lim_{a\to 1} |D(a)|^{1/2} \int_{M\backslash G} f(x^{-1}ax)v_M(x)\,dx, \qquad a \in A_{M,\mathrm{reg}}. \end{aligned}$$

The quotient measure on $M\backslash G$ is $dx = dn\,dk$, and **[A2]** $v_M(mnk) = v_M(n)$ for $m \in M$, $n \in N$, and $k \in K$. Therefore,

$$J_M(1, f) = \lim_{a\to 1} |D(a)|^{1/2} \int_N f_K(n^{-1}an)v_M(n)\,dn.$$

Since $n \mapsto a^{-1}n^{-1}an$, $n \in N$, $a \in A_{M,\mathrm{reg}}$, is an invertible polynomial mapping from N to N, we can make the change of variables $u = a^{-1}n^{-1}an$. This introduces the factor $|D(a)|^{-1/2}\delta_P(a)^{1/2}$. f_K is locally constant on G, and therefore is invariant under left and right translation by some open compact subgroup of G. Thus $f_K(au) = f_K(u)$ for all $u \in N$ if a is sufficiently close to the identity. Also, $\delta_P|K \cap P \equiv 1$.

We now describe, for $\mathrm{GL}_n(F)$, the normalizations of measures on $\underline{a}_M$, $\underline{a}_G$, $\underline{a}_M^G$, A_M, A_G and A_M/A_G required by the compatibility conditions of §3. Fix the Weyl-invariant inner product $((x_1, \dots, x_n), (y_1, \dots, y_n)) = \log^{-2} q \sum_{1\le i\le n} x_i y_i$ on $\underline{a}_{M_0}$. The corresponding measure is $\log^{-n} q\, dx_1 \cdots dx_n$, where dx_i denotes the standard Haar measure on $\mathbf{R}$. On $\underline{a}_M$ we take the measure coming from the restriction of the above inner product to $\underline{a}_M$. Suppose M is conjugate to $\prod_{i=1}^r \mathrm{GL}_{n_i}(F)$. The embeddings of $X(\mathbf{M})_F$ and $X(\mathbf{G})_F$ into the character groups $X(A_{\mathbf{M}})$ and $X(A_{\mathbf{G}})$ result in the embedding $x \mapsto (xn_1/n, \dots, xn_r/n)$ of $\underline{a}_G$ into $\underline{a}_M$. It is compatible with the canonical projection $(x_1, \dots, x_r) \mapsto \sum_{1\le i\le r} x_i$ from $\underline{a}_M$ onto $\underline{a}_G$, whose kernel is denoted by $\underline{a}_M^G$. This results in the decomposition $\underline{a}_M = \underline{a}_M^G \oplus \underline{a}_G$.

Let $\kappa_M = A_M \cap K$. The function H_M maps A_M/κ_M bijectively onto a lattice in $\underline{a}_M$. As stated in **[A4**, p. 5], the measure of κ_M in A_M must equal the volume of $\underline{a}_M/H_M(A_M)$. The measures on $A_M\backslash G$, $A_G\backslash G$, and $\underline{a}_M/\underline{a}_G \simeq \underline{a}_M^G$ are the ones induced by those on G, A_M, A_G, $\underline{a}_M$, and $\underline{a}_G$.

The next lemma gives the measures of the κ_M's. We will use these to determine the formal degree $d(\mathrm{St}(\mathcal{O}))$ which appears in the formula for $c_{\mathcal{O}}(\pi)$. Note that, in order to be consistent, the measure of $M_0 \cap K = A_{M_0} \cap K$ must equal one. This determined our choice of inner product on $\underline{a}_{M_0}$.

Lemma 6.2. *Let M be conjugate to $\prod_{i=1}^{r} \mathrm{GL}_{n_i}(F)$. With the above normalizations, the measure of κ_M is $\sqrt{n_1 \cdots n_r}$.*

Proof. For $m \in M$, $H_M(m) = (\log|\det m_1|, \ldots, \log|\det m_r|)$. Thus $H_M(A_M) = n_1 \log q\mathbf{Z} \times \cdots \times n_r \log q\mathbf{Z}$. The measure on $\underline{a}_M \simeq \mathbf{R}^r$ is $(\log^{-r} q/\sqrt{n_1 \cdots n_r})\, dx_1 \cdots dx_r$. The volume of $\underline{a}_M/H_M(A_M)$ is therefore $\sqrt{n_1 \cdots n_r}$.

In order to evaluate $v_M(x)$, $x \in G$, we need to compute $\mathrm{vol}(\underline{a}_M^G/\mathbf{Z}(\Delta_P^\vee))$ for $P \in \mathscr{P}(M)$. As noted in [A4, p. 12], $\mathbf{Z}(\Delta_P^\vee)$ is independent of the choice of $P \in \mathscr{P}(M)$. Let $\mu_M = \mathrm{vol}(\underline{a}_M^G/\mathbf{Z}(\Delta_P^\vee))$.

Lemma 6.3. $\mu_M = \sqrt{n/(n_1 \cdots n_r)} \log^{-r+1} q$.

Proof. Let $P = MN \in \mathscr{P}(M)$ be chosen so that N is upper triangular. Then $\Delta_P^\vee = \{\alpha_1, \ldots, \alpha_{r-1}\}$, where α_i has 1 in the ith position and 0 elsewhere. Define variables $y_1, \ldots, y_r$ by

$$y_1\alpha_1 + \cdots + y_r\alpha_{r-1} + y_r(n_1/n, \ldots, n_r/n) = (x_1, \ldots, x_r).$$

Then, since $dy_1 \cdots dy_r = dx_1 \cdots dx_r$, the measure on $\underline{a}_M$ is $(\log^{-r} q/\sqrt{n_1 \cdots n_r})\, dy_1 \cdots dy_r$. The measure on $\underline{a}_G$ is $(\log^{-1} q/\sqrt{n})\, dx$ and $x \in \underline{a}_G$ embeds in $\underline{a}_M$ as $(xn_1/n, \ldots, xn_r/n)$. The quotient measure on $\underline{a}_M^G$ is given by $(\log^{-r+1} q\sqrt{n/(n_1 \cdots n_r)}\, dy_1 \cdots dy_{r-1}$.

Let $u \in \operatorname{supp} f_K$. We want to compute the value of $v_M(n)$, where $u = a^{-1}n^{-1}an$, $a \in A_{M,\mathrm{reg}}$. If $a \in A_M$, then $a = \mathrm{diag}(a_1 I_{n_1}, \ldots, a_r I_{n_r})$, with $a_i \in F^*$, and I_{n_i} the $n_i \times n_i$ identity matrix, $1 \le i \le r$. Let $\mathscr{P}_F$ be the maximal ideal in the ring of integers $\mathscr{O}_F$. For each positive integer d, define $A_{M,d} = \{a \in A_{M,\mathrm{reg}} \mid a_i \in 1 + \mathscr{P}_F^d, |a_i - a_j| = q^{-d}, i \ne j\}$. We will compute $v_M(n)$ for $a \in A_{M,d}$ for large values of d, and to evaluate $J_M(1, f)$, we will let $d \to \infty$. The next lemma gives the values of $v_M(n)$ for certain non-minimal Levi subgroups of $\mathrm{GL}_4(F)$. We take n in the corresponding upper triangular unipotent subgroup. For $x \in F^*$, $\nu(x)$ is defined by $|x| = q^{-\nu(x)}$.

Lemma 6.4. *Let $u \in N \cap K$, $a \in A_{M,d}$, and n be given by $u = a^{-1}n^{-1}an$.*

1. *Let $M = \mathrm{GL}_3(F) \times \mathrm{GL}_1(F)$. If*

$$u = \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is such that $\max\{|x|, |y|, |z|\} \neq 0$, *then*

$$v_M(n) = \tfrac{2}{\sqrt{3}}(d - \min\{\nu(x), \nu(y), \nu(z)\}),$$

for large d.

2. *Let* $M = \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$. *If*

$$u = \begin{pmatrix} 1 & 0 & w & x \\ 0 & 1 & y & z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

such that $wz - xy \neq 0$ *then*

$$v_M(n) = 2d - \nu(wz - xy),$$

for large d.

3. *Let* $M = \mathrm{GL}_2(F) \times \mathrm{GL}_1(F) \times \mathrm{GL}_1(F)$. *Let*

$$u = \begin{pmatrix} 1 & 0 & x_1 & y_1 \\ 0 & 1 & x_2 & y_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Define

$$A = \min\{\nu(x_1), \nu(x_2)\},$$
$$B = \min\{\nu(x_1y_2 - x_2y_1), \nu(z) + A\}.$$

If $A \neq 0$, $B \neq 0$, *and* d *is large, then*

$$v_M(n) = 3\sqrt{2}d^2 - d(2\sqrt{2}A + 2\sqrt{2}\nu(z) + \sqrt{2}B) + \frac{1}{\sqrt{2}}B^2 - \sqrt{2}(B-A)^2 + \sqrt{2}B\nu(z).$$

REMARK. Let $P_0 = A_{M_0}N_0$ be the Borel subgroup of $\mathrm{GL}_n(F)$ such that N_0 is the subgroup of upper triangular unipotent matrices. For $x \in \mathrm{GL}_n(F)$, we use the following fact to find $H_{P_0}(x)$. Suppose $x = nak$, with $n \in N_0$, $a = \mathrm{diag}(a_1, \ldots, a_n) \in A_{M_0}$, and $k \in K$. Then, for $1 \leq i \leq n$, $|a_i \cdots a_n|$ is equal to the maximum of the set of norms of determinants of $(n-i+1) \times (n-i+1)$ matrices which can be formed from the last $n-i+1$ rows of x. For example, $|a_{n-1}a_n| = \max_{1 \leq i \neq j \leq n}\{|x_{n-1,i}x_{n,j} - x_{n,i}x_{n-1,j}|\}$. If $P = MN$, $M \in \mathscr{L}(M)$, $N \subset N_0$, then $H_P(x) = (\log|a_1 \cdots a_{n_1}|, \ldots, \log|a_{n_{r-1}+1} \cdots a_n|)$.

Proof of Lemma 6.4. 1. Let $\overline{P} \in \mathscr{P}(M)$ be the opposite parabolic subgroup. It is not hard to see that $H_{\overline{P}}(n) = -H_P(n^{t^{-1}})$, where t denotes transpose. If $a \in A_{M,d} = \mathrm{diag}(a_1, a_1, a_2, a_2)$ then

$$n = \begin{pmatrix} 1 & 0 & 0 & (1-a_1^{-1}a_2)^{-1}x \\ 0 & 1 & 0 & (1-a_1^{-1}a_2)^{-1}y \\ 0 & 0 & 1 & (1-a_1^{-1}a_2)^{-1}z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using the above remark, we obtain

$$\begin{aligned} H_P(n^{t^{-1}}) &= \log\max\{1, q^d|x|, q^d|y|, q^d|z|\}(-1, 1) \\ &= -\log q(d - \min\{\nu(x), \nu(y), \nu(z)\})(1, -1), \qquad d \text{ large.} \end{aligned}$$

By definition, $v_M(n)$ is the volume in $\underline{a}_M^G$ of the convex hull of $H_P(n) = 0$ and $H_{\overline{P}}(n)$, which is, by Lemma 6.3, equal to $\frac{2}{\sqrt{3}}(d - \min\{\nu(x), \nu(y), \nu(z)\})$.

2. We note that, if $a = \mathrm{diag}(a_1, a_1, a_2, a_2) \in A_{M,d}$,

$$n^{t^{-1}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -(1-a_1^{-1}a_2)^{-1}w & -(1-a_1^{-1}a_2)^{-1}y & 0 & 0 \\ -(1-a_1^{-1}a_2)^{-1}x & -(1-a_1^{-1}a_2)^{-1}z & 0 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} H_{\overline{P}}(n) &= \log\max\{1, q^{2d}|wz - xy|, q^d|w|, q^d|x|, q^d|y|, q^d|z|\}(1, -1) \\ &= \log q(2d - \nu(wz - xy))(1, -1), \qquad d \text{ large.} \end{aligned}$$

To obtain 2, proceed as above for 1.

3. Let $a = \mathrm{diag}(a_1, a_1, a_2, a_3) \in A_{M,d}$. The characters $\alpha = (1, -1, 0)$, $\beta = (1, 0, -1)$ and $\gamma = (0, 1, -1)$ of A_M are viewed as elements of the dual space $\underline{a}_M^*$. Given u as in the statement of the lemma,

$$n = \begin{pmatrix} 1 & 0 & \tilde{x}_1 & \tilde{y}_1 \\ 0 & 1 & \tilde{x}_2 & \tilde{y}_2 \\ 0 & 0 & 1 & \tilde{z} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\begin{aligned} \tilde{x}_i &= (1-a_1^{-1}a_2)^{-1}x_i, \\ \tilde{y}_i &= (1-a_1^{-1}a_3)^{-1}(y_i + a_1a_2^{-1}(1-a_1^{-1}a_2)^{-1}x_iz), \qquad i = 1, 2, \\ \tilde{z} &= (1-a_2^{-1}a_3)^{-1}z. \end{aligned}$$

Define $A = \min\{\nu(x_1, \nu(x_2)\}$ and $B = \min\{\nu(x_1y_2 - x_2y_1), \nu(z) + A\}$. For u in an open dense subset of the unipotent radical, A and B are nonzero. For d sufficiently large, the values $H_P(n)$, $P \in \mathscr{P}(M)$, are given by the table below.

Δ_P	$\log^{-1} q H_P(n)$
$\{\alpha, \gamma\}$	0
$\{-\alpha, -\gamma\}$	$(2d - B)\alpha^\vee + (2d - \nu(z) - A)\gamma^\vee$
$\{\alpha, -\beta\}$	$(-d + \nu(z))\alpha^\vee + (2d - \nu(z) - A)\beta^\vee$
$\{-\alpha, \beta\}$	$(d - A)\alpha^\vee$
$\{-\beta, \gamma\}$	$(2d - B)\beta^\vee + (-d + A)\gamma^\vee$
$\{\beta, -\gamma\}$	$(d - \nu(z))\gamma^\vee$

For the pairs $\{-\alpha, -\gamma\}$, $\{-\alpha, \beta\}$ and $\{\beta, -\gamma\}$, $H_P(n)$ can easily be computed using the remark preceding the lemma. We describe the case $\{\beta, -\gamma\}$. If $P \in \mathscr{P}(M)$ has simple roots $\{\beta, -\gamma\}$, then

$$N_P = \left\{ \begin{pmatrix} 1 & 0 & c_{13} & c_{14} \\ 0 & 1 & c_{23} & c_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & c_{43} & 1 \end{pmatrix} \right\}.$$

Note that

$$n = n_P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \tilde{z} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $n_P \in N_P$. Also, $\left(\begin{smallmatrix} 1 & \tilde{z} \\ 0 & 1 \end{smallmatrix}\right)$ is the product of $\left(\begin{smallmatrix} 1 & 0 \\ w & 1 \end{smallmatrix}\right)$ and $\operatorname{diag}(\delta_1, \delta_2)$ with a matrix in $\mathrm{GL}_2(\mathscr{O}_F)$, where $|\delta_1| = |\delta_2|^{-1} = |\tilde{z}|$, for large d. Therefore, $H_P(n) = \log(q^d|z|)(0, 1, -1)$.

The values $H_P(n)$ for $\{\alpha, -\beta\}$ and $\{-\beta, \gamma\}$ are determined by the values for the other parabolic subgroups by using the following property (see [**A4**, p. 5]): If $P, P' \in \mathscr{P}(M)$ are adjacent, and τ is the simple root of (P, A_M) in $\Delta_P \cap (-\Delta_{P'})$ which determines the wall shared by the chambers of P and P' in $\underline{a}_M$, then for any $x \in G$, $-H_P(x) + H_{P'}(x)$ is a nonnegative multiple of $\tau^\vee$. That is, $\{-H_P(x) \mid P \in \mathscr{P}(M)\}$ forms a positive orthogonal set for M.

To compute $v_M(n)$ we use formula (3.1):

$$v_M(x) = 1/r! \sum_{\{P \in \mathscr{P}(M)\}} (-\lambda(H_P(x)))^r \theta_P(\lambda)^{-1},$$

$$\lambda \in i\underline{a}_M^*, \quad r = \dim(A_M/A_G),$$

where $\theta_P(\lambda) = \mu_M^{-1}\prod_{\alpha\in\Delta_P}\lambda(\alpha^\vee)$. Setting $\lambda = (it_1, it_2, it_3)$ with t_1, t_2, t_3 distinct real numbers, $\mu_M = \sqrt{2}\log^{-2} q$, and computing $1/2\sum_{\{P\in\mathscr{P}(M)\}}(\lambda(H_P(n)))^2\theta_P(\lambda)^{-1}$, after some algebra, we obtain the desired expression for $v_M(n)$.

PROPOSITION 6.5. *Let f be a cusp form on $\mathrm{GL}_4(F)$ with $\operatorname{supp} f \subset KZ$. Given M, define the variable $u \in N\cap K$ as in Lemma 6.4.*

1. *If $M = \mathrm{GL}_3(F)\times\mathrm{GL}_1(F)$,*

$$J_M(1, f) = -2/\sqrt{3}\int_N f_K(u)\min\{\nu(x), \nu(y), \nu(z)\}\,du.$$

2. *If $M = \mathrm{GL}_2(F)\times\mathrm{GL}_2(F)$,*

$$J_M(1, f) = -\int_N f_K(u)\nu(wz-xy)\,du.$$

3. *If $M = \mathrm{GL}_2(F)\times\mathrm{GL}_1(F)\times\mathrm{GL}_1(F)$, and A and B are as in Lemma 6.4,*

$$J_M(1, f) = \sqrt{2}\int_N f_K(u)(B^2/2-(B-A)^2+B\nu(z))\,du.$$

Proof. Let $d \geq 1$ and $a \in A_{M,d}$. For $n \in N$ such that $u = a^{-1}n^{-1}an$, set $\tilde{v}_M(n)$ equal to

$$\begin{aligned}
&(2/\sqrt{3})(d-\min\{\nu(x), \nu(y), \nu(z)\}),\\
&2d-\nu(wz-xy),\\
&3\sqrt{2}d^2 - d(2\sqrt{2}A+2\sqrt{2}\nu(z)+\sqrt{2}B)+B^2/\sqrt{2}\\
&\qquad\qquad -\sqrt{2}(B-A)^2+\sqrt{2}B\nu(z),
\end{aligned}$$

in cases 1, 2 and 3, respectively. By Lemma 6.4, for all $u \in N\cap K$, $\lim_{d\to\infty}(v_M(n)-\tilde{v}_M(n)) = 0$. Results of Arthur **[A3]**, imply that $\lim_{d\to\infty}\int_N f_K(u)(v_M(n)-\tilde{v}_M(n))\,du = 0$. Thus

$$\begin{aligned}
J_M(1, f) &= \lim_{d\to\infty}\left(\int_N f_K(u)\tilde{v}_M(n)\,du + \int_N f_K(u)(v_M(n)-\tilde{v}_M(n))\,du\right)\\
&= \lim_{d\to\infty}\int_N f_K(u)\tilde{v}_M(n)\,du.
\end{aligned}$$

Because f, hence f_K, is a cusp form, we have $\int_N f_K(u)\,du = 0$. In the first two cases, $\tilde{v}_M(n)$ is a constant multiple of d plus a term which is independent of d. Thus the lemma follows immediately in these cases.

To prove 3, we first observe that, for large values of d, $v_{M_1}(n)$ is a multiple of $2d - A - \nu(z)$, where $M_1 = \mathrm{GL}_3(F) \times \mathrm{GL}_1(F)$. In the notation used in the proof of the third part of Lemma 6.4,

$$n = \begin{pmatrix} 1 & 0 & \tilde{x}_1 & 0 \\ 0 & 1 & \tilde{x}_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \tilde{y}_1 - \tilde{x}_1\tilde{z} \\ 0 & 1 & 0 & \tilde{y}_2 - \tilde{x}_2\tilde{z} \\ 0 & 0 & 1 & \tilde{z} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$v_{M_1}(n)$ is therefore a multiple of $\log\max\{1, |\tilde{z}|, |\tilde{y}_1 - \tilde{x}_1\tilde{z}|, |\tilde{y}_2 - \tilde{x}_2\tilde{z}|\}$.

$$\begin{gathered} |\tilde{z}| = q^d|z|, \\ |\tilde{y}_i - \tilde{x}_i\tilde{z}| = q^d|y_i - (1 - a_2^{-1}a_3)^{-1}x_i z|, \qquad i = 1, 2. \end{gathered}$$

We assume that $x_i z \neq 0$, $i = 1, 2$, and d is large. Then $|\tilde{y}_i - \tilde{x}_i\tilde{z}| = q^{2d}|x_i z|$.

$J_{M_1}(a, f) = \delta_P(a)^{1/2}\int_N f_K(au)v_{M_1}(n)\,du$. This is obtained by the same change of variables used in the proof of Lemma 6.1. $a \in A_{M,d}$ is not elliptic in M_1, so, by Proposition 3.9, $J_{M_1}(a, f) = 0$. By an argument similar to the one above for $J_M(1, f)$, we get:

$$\begin{aligned} \lim_{a\to 1} J_{M_1}(a, f) &= \lim_{d\to\infty}\int_N f_K(u)v_{M_1}(n)\,du \\ &= \lim_{d\to\infty}\int_N f_K(u)(2d - A - \nu(z))\,du \\ &= -\int_N f_K(u)(A + \nu(z))\,du. \end{aligned}$$

Thus $\int_N f_K(u)(A + \nu(z))\,du = 0$.

Similarly, if $M_2 = \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$, we can show that $v_{M_2}(n)$ is a multiple of $2d - B$ for large d, so $\int_N f_K(u)B\,du = 0$.

Looking at the formula for $\tilde{v}_M(n)$ given at the beginning of the proof, we see that

$$\int_N f_K(u)\tilde{v}_M(n)\,du = \int_N f_K(u)\sqrt{2}(B^2/2 - (B - A)^2 + B\nu(z))\,du.$$

7. Calculation of $c_{\mathcal{O}}(\pi)$ for $\mathrm{GL}_3(F)$ and $\mathrm{GL}_4(F)$. We now compute the coefficients $c_{\mathcal{O}}(\pi)$ for a generic unramified supercuspidal representation π of $\mathrm{GL}_3(F)$ or $\mathrm{GL}_4(F)$.

Let $M = \prod_{1\leq i\leq r}\mathrm{GL}_{n_i}(F)$. Let St_M be the Steinberg representation of M. If $G = \mathrm{GL}_n(F)$, the formal degree $d(\mathrm{St}_G)$ of St_G is given by

[CMS]:

$$d(\mathrm{St}_G) = 1/n\left(\prod_{k=1}^{n-1}(q^k - 1)\right)\mathrm{vol}_{Z\backslash G}(Z\backslash KZ)^{-1}.$$

Here $Z = A_G$ is the centre of G. We are assuming that $\mathrm{vol}_{Z\backslash G}(Z\backslash KZ) = \mathrm{vol}_G(K)/\mathrm{vol}_Z(K\cap Z)$. With the measures normalized as in §6, we have

$$d(\mathrm{St}_M) = \prod_{i=1}^{r} 1/\sqrt{n_i}\prod_{k=1}^{n_i-1}(q^k - 1). \tag{7.1}$$

If $\pi = \mathrm{Ind}_{KZ}^G\sigma$, then, by **[C**, p. 211], the formal degree $d(\pi) = \mathrm{vol}_{Z\backslash G}(Z\backslash KZ)^{-1}\dim\sigma = \sqrt{n}\dim\sigma$.

THEOREM 7.2. *Assume $G = \mathrm{GL}_4(F)$. Given any character χ of F^*, let $\pi = \mathrm{Ind}_{KZ}^G\sigma\otimes\chi\circ\det$ be a generic unramified supercuspidal representation of G, where σ has level h. If M is a Levi subgroup, let $\mathscr{O} = 1_M^G$.*

1. *If $M = G$, $c_{\mathscr{O}}(\pi) = -4q^{6(h-1)}$.*
2. *If $M = \mathrm{GL}_3(F)\times\mathrm{GL}_1(F)$, $c_{\mathscr{O}}(\pi) = 4q^{3(h-1)}$.*
3. *If $M = \mathrm{GL}_2(F)\times\mathrm{GL}_2(F)$, $c_{\mathscr{O}}(\pi) = 2q^{2(h-1)}$.*
4. *If $M = \mathrm{GL}_2(F)\times\mathrm{GL}_1(F)\times\mathrm{GL}_1(F)$, $c_{\mathscr{O}}(\pi) = -4q^{h-1}$.*
5. *If M is minimal, $c_{\mathscr{O}}(\pi) = 1$.*

Proof. 1 and 5 are due to Howe **[H]**. Let $\tilde{\chi}_\sigma$ be defined as in §5. The function $f = \tilde{\chi}_\sigma\otimes\chi\circ\det$ is a sum of matrix coefficients of π. Note that $f(u) = \tilde{\chi}_\sigma(u)$ for any unipotent element $u\in G$, so $J_{\mathscr{O}}(1, f)$, hence $c_{\mathscr{O}}(\pi)$, is independent of χ. Since $\dim\sigma = f(1)$, and $n = 4$, $d(\pi) = 2f(1)$. Putting this together with Theorem 4.4, we obtain $c_{\mathscr{O}}(\pi) = -2J_{\mathscr{O}}(1, f)/(w_{\mathscr{O}}d(\mathrm{St}(\mathscr{O})))$. In cases 1–4, $w_{\mathscr{O}} = 1, 1, 2$ and 2, respectively. The values of f on the unipotent set are given in Lemma 5.2. Substitution of these values into each formula for $J_{\mathscr{O}}(1, f)$ given in Proposition 6.5 (note that $f_K = f$), and evaluation of the integral results in:

1. $f(1) = q^{6(h-1)}(q^3 - 1)(q^2 - 1)(q - 1)$,
2. $(-2/\sqrt{3})q^{3(h-1)}(q^2 - 1)(q - 1)$,
3. $-q^{2(h-1)}(q - 1)^2$,
4. $2\sqrt{2}q^{h-1}(q - 1)$.

The calculations are fairly short in cases 2 and 3, and lengthy in case 4. We do not include them here. Using (7.1) to evaluate $d(\mathrm{St}(\mathscr{O}))$ completes the proof.

REMARK. For arbitrary n, and π and f as in the theorem, if $M = \mathrm{GL}_{n-1}(F) \times \mathrm{GL}_1(F)$, it is easy to compute

$$J_M(1, f) = -f(1)(\sqrt{n}q^{-(n-1)h})/((\sqrt{n-1})(1 - q^{-(n-1)})),$$

which results in $c_{\mathscr{O}}(\pi) = (-1)^{n-2}nq^{(n-1)(n-2)(h-1)/2}$ for $\mathscr{O} = 1_M^G$.

PROPOSITION 7.3. *Under the same assumptions as Theorem 7.2, except that* $G = \mathrm{GL}_3(F)$, $c_{\mathscr{O}}(\pi) = 3q^{3(h-1)}$, $-3q^{h-1}$, *and* 1 *for* $M = G, \mathrm{GL}_2(F) \times \mathrm{GL}_1(F)$, *and* M_0, *respectively.*

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Received June 2, 1989. Research supported in part by NSERC.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 148, No. 1, 1991

QUASI-ROTATION C^*-ALGEBRAS

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The main result in this paper is to classify the isomorphism classes of certain non-commutative 3-tori obtained by taking the crossed product C^*-algebra of continuous functions on the 2-torus $\mathbf{T}^2$ by the irrational affine quasi-rotations. Each such quasi-rotation is represented by a pair (a, A), where $a \in \mathbf{T}^2$ and $A \in \mathrm{GL}(2, \mathbf{Z})$, and its associated C^*-algebra is shown to be determined (up to isomorphism) by an analogue of the rotation angle, namely its primitive eigenvalue, by its orientation $\det(A) = \pm 1$ and a certain positive integer $m(A)$ which comes from the K_1-group of the algebra and which determines the conjugacy class of A in $\mathrm{GL}(2, \mathbf{Z})$.

Introduction. In this paper we study the C^*-crossed products of the continuous functions on the 2-torus $C(\mathbf{T}^2)$ by certain transformations φ of $\mathbf{T}^2$ which we call quasi-rotations. They are like rotations in that they have an eigenvalue $\lambda = e^{2\pi i\theta}$ and a unitary eigenfunction $f \in C(\mathbf{T}^2)$, and unlike rotations in that their degree matrix $D(\varphi) \in \mathrm{GL}(2, \mathbf{Z})$ does not equal the identity matrix I_2. Clearly they contain the rotation C^*-algebra $\mathscr{A}_\theta$.

Recall that an affine transformation of a group G is a mapping $\sigma\colon G \to G$ of the form $\sigma(z) = aA(z)$, (for $z \in G$), where $a \in G$ and $A \in \mathrm{Aut}(G)$.

Let $\mathscr{A}(\varphi)$ denote the associated crossed product C^*-algebra $C(\mathbf{T}^2) \times_{\alpha_\varphi} \mathbf{Z}$, (cf. [**9**, 7.6]) where α_φ is the automorphism on $C(\mathbf{T}^2)$ associated with φ. We shall construct an integer-valued function m defined on the 2×2 matrices $A \in \mathrm{GL}(2, \mathbf{Z})$ which are of the form $D(\varphi)$, for some quasi-rotation φ, such that

(i) $\mathbf{Z}_{m(D(\varphi))}$ is the torsion subgroup of $K_1(\mathscr{A}(\varphi))$,

(ii) $m(A)$ and $\det(A)$ determine the conjugacy class of A in $\mathrm{GL}(2, \mathbf{Z})$.

When this is combined with the computation of the tracial range on $K_0(\mathscr{A}(\varphi))$ (see §4) a classification of the isomorphism classes of these algebras is obtained (Theorem 5.2) for the affine quasi-rotations of $\mathbf{T}^2$ associated with irrational θ. This is the main result, while for the rational case a partial answer is given. The determination of the strong Morita equivalence classes of these algebras has been studied in [**17**].

The K-groups of the crossed products of $C(\mathbf{T}^2)$ by any transformation have been computed elsewhere ([**6**]; and independently in [**15**]) using the Pimsner-Voiculescu six-term exact sequence. Here we shall merely state the results (§1).

Some results concerning the non-affine quasi-rotation algebras are given in [**16**].

1. K-groups. Every continuous function $f\colon \mathbf{T}^2 \to \mathbf{T}$ has the form $f(x, y) = x^m y^n e^{2\pi i F(x,y)}$ for some integers m, n and some continuous real-valued function F on $\mathbf{T}^2$. Call the 1×2 integral matrix $[m\ n]$ the bidegree of f and denote it by $D(f)$. Let φ be a transformation (i.e., a homeomorphism) of the 2-torus $\mathbf{T}^2$. Write φ as $\varphi = (\varphi_1, \varphi_2)$. Define the degree matrix of φ to be the 2×2 integral matrix

$$D(\varphi) = \begin{pmatrix} D(\varphi_1) \\ D(\varphi_2) \end{pmatrix}.$$

It is easy to verify that $D(\varphi \circ \psi) = D(\varphi)D(\psi)$ for any two transformations φ, ψ of $\mathbf{T}^2$. Replacing ψ by φ^{-1} we see that $D(\varphi) \in \mathrm{GL}(2, \mathbf{Z})$, i.e. $\det D(\varphi) = \pm 1$. This latter determinant determines whether φ is orientation preserving or reversing. Let I_2 denote the identity matrix in $\mathrm{GL}(2, \mathbf{Z})$.

THEOREM 1.1 ([**6**], *Chapter* 3; [**15**], *Chapter* 2). *Let φ be a transformation of $\mathbf{T}^2$.*

(1) *If* $\det D(\varphi) = 1$, *then*

$$K_0(\mathscr{A}(\varphi)) \cong \begin{cases} \mathbf{Z}^4 & \text{if } D(\varphi) = I_2, \\ \mathbf{Z}^3 & \text{if } \det(D(\varphi) - I_2) = 0 \text{ and } D(\varphi) \neq I_2, \\ \mathbf{Z}^2 & \text{if } \det(D(\varphi) - I_2) \neq 0. \end{cases}$$

(2) *If* $\det D(\varphi) = -1$, *then*

$$K_0(\mathscr{A}(\varphi)) \cong \begin{cases} \mathbf{Z}^2 \oplus \mathbf{Z}_2 & \text{if } \det(D(\varphi) - I_2) = 0, \\ \mathbf{Z} \oplus \mathbf{Z}_2 & \text{if } \det(D(\varphi) - I_2) \neq 0. \end{cases}$$

(3) *Write* $D(\varphi)^{-1} = \left(\begin{smallmatrix} m & n \\ p & q \end{smallmatrix}\right)$ *and let J denote the quotient group*

$$J = \frac{\mathbf{Z} \oplus \mathbf{Z}}{(m-1, n)\mathbf{Z} + (p, q-1)\mathbf{Z}} = \frac{\mathbf{Z}^2}{\mathrm{Im}(D(\varphi^{-1})^T - I_2)}.$$

Then

$$K_1(\mathscr{A}(\varphi)) \cong \begin{cases} \mathbf{Z}^2 \oplus J & \text{if } \det D(\varphi) = 1, \\ \mathbf{Z} \oplus J & \text{if } \det D(\varphi) = -1. \end{cases}$$

The proof of this theorem relies on the Pimsner-Voiculescu cyclic six-term exact sequence for K-theory **[11]**. A closer look at the proof yields the following corollary.

COROLLARY 1.2. *Let φ be a transformation of $\mathbf{T}^2$ such that*

$$\det(D(\varphi) - I_2) = 0,$$

and let P denote the Bott projection in $M_2(C(\mathbf{T}^2))$. In this case there is an x such that $\delta(x)$ is a generator of $\ker(\alpha_{\varphi_*} - \mathrm{id}_*)$ in $K_1(C(\mathbf{T}^2))$, where δ is the connecting homomorphism in the Pimsner-Voiculescu sequence $\delta\colon K_0(\mathscr{A}(\varphi)) \to K_1(C(\mathbf{T}^2))$.

(i) *If $\det D(\varphi) = 1$ and $D(\varphi) \neq I_2$, then $K_0(\mathscr{A}(\varphi)) \cong \mathbf{Z}^3$ is generated by $[1]$, $[P]-[1]$, and x.*
(ii) *If $\det D(\varphi) = -1$, then $K_0(\mathscr{A}(\varphi)) \cong \mathbf{Z}^2 \oplus \mathbf{Z}_2$ is generated by $[1]$, $[P]-[1]$ (which has order 2 in this case) and x.*

This corollary focuses only on transformations such that $\det(D(\varphi) - I_2) = 0$ because these include the quasi-rotations.

2. Lemmas. In this section we shall construct the integer-valued function m indicated in the introduction which classifies the conjugacy class of certain integral matrices in $\mathrm{GL}(2, \mathbf{Z})$ which arise as $D(\varphi)$ where φ is a quasi-rotation. As it turns out these are the matrices A which have eigenvalue 1, i.e. $\det(A - I_2) = 0$ (cf. §3).

Two matrices $A, B \in \mathrm{GL}(2, \mathbf{Z})$ are conjugate if there exists $S \in \mathrm{GL}(2, \mathbf{Z})$ such that $SAS^{-1} = B$. Let us express this by $A \sim B$. It will be shown later that for quasi-rotations φ and ψ of $\mathbf{T}^2$, if $\mathscr{A}(\varphi) \cong \mathscr{A}(\psi)$, then $D(\varphi) \sim D(\psi)$ (cf. Proposition 2.8). If, in addition, φ and ψ are affine, it will follow that they are topologically conjugate (i.e., there exists a transformation h of $\mathbf{T}^2$ such that $h \circ \psi = \varphi \circ h$).

The construction of m is divided up into two cases.

LEMMA 2.1. *Let $A \in \mathrm{GL}(2, \mathbf{Z})$ be such that $\det(A - I_2) = 0$ and $\det A = 1$, say*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $e = \gcd(a-1, b)$, when $b \neq 0$, and define

$$m(A) = \begin{cases} \dfrac{e^2}{|b|} & \text{if } b \neq 0, \\ |c| & \text{if } b = 0. \end{cases}$$

Then

$$A \sim \begin{pmatrix} 1 & 0 \\ m(A) & 1 \end{pmatrix}.$$

Hence, $A \sim B$ *if and only if* $m(A) = m(B)$, *for all matrices* A, B *satisfying the hypotheses of this lemma.*

Proof. From $(a-1)(d-1)-bc = 0$ and $ad-bc = 1$ one obtains $a+d = 2$ and $-(a-1)^2 = bc$. If $b = 0$, the lemma is clear. Suppose that $b \neq 0$. Since $e = \gcd(a-1, b)$, there exist integers s, t such that

$$\left(\frac{a-1}{e}\right)t - \left(\frac{b}{e}\right)s = 1,$$

so that

$$S = \begin{pmatrix} \dfrac{a-1}{e} & \dfrac{b}{e} \\ s & t \end{pmatrix} \in \mathrm{GL}(2, \mathbf{Z}).$$

One then checks that

$$SA = \begin{pmatrix} 1 & 0 \\ \dfrac{-e^2}{b} & 1 \end{pmatrix} S:$$

$$\begin{aligned} SA &= \begin{pmatrix} \dfrac{a-1}{e}\dfrac{b}{e} & \\ s & t \end{pmatrix} \begin{pmatrix} a & b \\ c & 2-a \end{pmatrix} \\ &= \begin{pmatrix} \left(\dfrac{a-1}{e}\right)a + \dfrac{bc}{e} & \left(\dfrac{a-1}{e}\right)b + \dfrac{b(2-a)}{e} \\ sa+tc & sb+t(2-a) \end{pmatrix} \end{aligned}$$

and

$$\begin{pmatrix} 1 & 0 \\ \dfrac{-e^2}{b} & 1 \end{pmatrix} \begin{pmatrix} \dfrac{a-1}{e} & \dfrac{b}{e} \\ s & t \end{pmatrix} = \begin{pmatrix} \dfrac{a-1}{e} & \dfrac{b}{e} \\ \dfrac{-e^2}{b}\left(\dfrac{a-1}{e}\right) + s & t-e \end{pmatrix}.$$

These can be seen to be equal using the relations $-(a-1)^2 = bc$ and $(a-1)t - bs = e$. Thus

$$SAS^{-1} = \begin{pmatrix} 1 & 0 \\ \pm m(A) & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ m(A) & 1 \end{pmatrix}. \qquad \sqcup$$

Henceforth we shall write $m(\varphi) = m(D(\varphi))$.

COROLLARY 2.2. *Let φ be a transformation of $\mathbf{T}^2$ with* $\det D(\varphi) = 1$ *and* $\det(D(\varphi) - I_2) = 0$. *Then φ is topologically conjugate to a transformation ψ with*

$$D(\psi) = \begin{pmatrix} 1 & 0 \\ m(\varphi) & 1 \end{pmatrix}.$$

Proof. Since $D(\varphi)$ satisfies the hypotheses of the previous lemma, we have

$$SD(\varphi)S^{-1} = \begin{pmatrix} 1 & 0 \\ m(\varphi) & 1 \end{pmatrix}$$

for some $S \in \mathrm{GL}(2, \mathbf{Z})$. We can choose an automorphism σ of $\mathbf{T}^2$ (as a group) with $D(\sigma) = S$. For example, if

$$S = \begin{pmatrix} m & n \\ p & q \end{pmatrix},$$

let $\sigma(x, y) = (x^m y^n, x^p y^q)$. Letting $\psi = \sigma \circ \varphi \circ \sigma^{-1}$, we obtain

$$D(\psi) = D(\sigma)D(\varphi)D(\sigma)^{-1} = \begin{pmatrix} 1 & 0 \\ m(\varphi) & 1 \end{pmatrix}. \qquad \square$$

COROLLARY 2.3. *Let φ be a transformation of $\mathbf{T}^2$ with* $\det D(\varphi) = 1$ *and* $\det(D(\varphi) - I_2) = 0$. *Then*

$$K_1(\mathscr{A}(\varphi)) \cong \mathbf{Z}^3 \oplus \mathbf{Z}_{m(\varphi)}.$$

Proof. Since by the preceding corollary ψ is topologically conjugate to φ, we can use Theorem 1.1 to obtain

$$\begin{aligned} K_1(\mathscr{A}(\varphi)) &\cong K_1(\mathscr{A}(\psi)) \cong \mathbf{Z}^2 \oplus \left(\frac{\mathbf{Z}^2}{(0,0)\mathbf{Z} + (m(\varphi), 0)\mathbf{Z}}\right) \\ &\cong \mathbf{Z}^3 \oplus \mathbf{Z}_{m(\varphi)}. \end{aligned} \qquad \square$$

Consequently, if φ and ψ are transformations of $\mathbf{T}^2$ satisfying the hypotheses of the above corollary and if $\mathscr{A}(\varphi)$ and $\mathscr{A}(\psi)$ are isomorphic, strongly Morita equivalent, or, more generally, have isomorphic K_1-groups, then $m(\varphi) = m(\psi)$ so that $D(\varphi) \sim D(\psi)$.

LEMMA 2.4. *Let $A \in \mathrm{GL}(2, \mathbf{Z})$ be such that* $\det A = -1$ *and* $\det(A - I_2) = 0$, *so that A has the form*

$$A = \begin{pmatrix} k & x \\ y & -k \end{pmatrix},$$

where $k^2+xy=1$. *Let* $e=\gcd(k-1, x)$, *when* $x\neq 0$, *and consider the integer-valued function*

$$m(A)=\begin{cases}\gcd\left(e, \dfrac{e(k+1)}{x}\right) & \text{if } x\neq 0,\\ \gcd(2, y) & \text{if } x=0.\end{cases}$$

Then $m(A)\in\{1, 2\}$, *and*

(i) $m(A)=1 \Leftrightarrow A\sim\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$,

(ii) $m(A)=2 \Leftrightarrow A\sim\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$.

Consequently, for such matrices A *and* B *one has* $A\sim B\Leftrightarrow m(A)=m(B)$. (*Hence there are two conjugacy classes in this case.*)

Proof. Since $(k-1)/e$ and x/e are relatively prime integers and $xy=(1-k)(1+k)$ or $(x/e)y=((1-k)/e)(1+k)$, it follows that x/e divides $k+1$; hence $e(k+1)/x$ is an integer (when $x\neq 0$), so that $m(A)$ makes sense.

To see that $m(A)\in\{1, 2\}$, note that

$$m(A)|e|(k-1) \quad\text{and}\quad m(A)|(e(k+1)/x)|(k+1).$$

Hence $m(A)|(k+1)-(k-1)$ or $m(A)|2$, as desired.

Now assume that $m(A)=1$ and suppose that $k\neq\pm1$, so that $x\neq 0$. We shall seek an integral matrix

$$S=\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

such that

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}k&x\\y&-k\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix},$$

and $ad-bc=1$. This implies that

$$ka+yb=c,\quad xa-kb=d,\quad kc+yd=a,\quad xc-kd=b,$$

and one easily checks that the last two of these equations follow from the first two. Substituting the first two equations into $ad-bc=1$ we get $a(xa-kb)-b(ka+yb)=1$, or $xa^2-2kab-yb^2=1$, which may be factored as

$$\left[\frac{x}{e}a-\frac{k-1}{e}b\right]\left[ea+\frac{ey}{k-1}b\right]=1,$$

where $ey/(k-1)=-e(k+1)/x$ is an integer (since $k\neq 1$). Therefore, the existence of S is guaranteed provided the equations

$$\frac{x}{e}a-\frac{k-1}{e}b=1,\quad ea+\frac{ey}{k-1}b=1,$$

have integer solutions a, b.

Multiplying the first of these equations by e and the second by x/e and subtracting the two gives $2b = e - (x/e)$. Similarly, if we multiply these equations by $k+1$ and $k-1$, respectively, we obtain $2a = (e(k+1)/x) - ((k-1)/e)$. To show that b exists we must show that e and x/e have the same parity, i.e., either both are even or odd. This may be shown as follows.

Assume that x/e is odd and e is even. Then x is even and $k-1$ is even (since $2|e|(k-1)$). So $k+1$ is even. But then $2|(e(k+1)/x)$ since x/e is odd, and hence, $2|m(A) = 1$, a contradiction. A similar contradiction argument follows if x/e is even and e is odd.

To show that a exists one shows that $e(k+1)/x$ and $(k-1)/e$ have the same parity. If $(k-1)/e$ is even, then x/e is odd. Since $k-1$ is even, $k+1$ is even and so $e(k+1)/x$ is even since x/e is odd. Conversely, if $e(k+1)/x$ is even then since $1 = m(A)$, e must be odd. Now as $k+1$ is even, so is $k-1$, and so $(k-1)/e$ is even since e is odd.

Now we assume that $m(A) = 2$ and $k \neq \pm 1$, so that $x \neq 0$. Then e and $e(k+1)/x$ are even so that the matrix

$$S = \begin{pmatrix} \frac{e(k+1)}{2x} & \frac{e}{2} \\ \frac{k-1}{e} & \frac{x}{e} \end{pmatrix}$$

has integer entries and determinant 1. Using the relation $xy = (1-k)(1+k)$ one can easily check that

$$SA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S.$$

Now the cases when $k = \pm 1$ are easily handled by similar arguments as above. □

The matrices satisfying the hypotheses of Lemma 2.4 are the "orientation reversing" square roots of the identity matrix. Using this lemma we can show that there is a quick way to find the conjugacy class of A when its entries have known parity.

COROLLARY 2.5. *Let A satisfy the hypotheses of Lemma 2.4.*

(1) *k even $\Rightarrow m(A) = 1$.*

(2) *Suppose k is odd. Then*

  (i) *x or y is odd $\Rightarrow m(A) = 1$,*

  (ii) *x and y are even $\Rightarrow m(A) = 2$.*

Proof. If $m(A) \neq 1$, then $m(A) = 2$ so that $2|e|(k-1)$ and hence k is odd. This proves (1). We now prove (2).

(i) Without loss of generality suppose x is odd. Since $m(A)|e|x$, it follows that $m(A) = 1$.

(ii) Suppose that x and y are even. Since $k-1$ is even, e is even. We assert that $e(k+1)/x$ is even, so that $m(A) = 2$. To see this, write $y = (e(k+1)/x)((1-k)/e)$ where we may assume $x \neq 0$ (if $x = 0$ then $k = \pm 1$ so $m(A) = 2$). If x/e is even, then $(1-k)/e$ is odd (being relatively prime), so y is even implies that $e(k+1)/x$ is even. Now if x/e is odd, then $k+1$ being even it follows that $e(k+1)/x$ is even, and hence $m(A) = 2$. □

Setting $m(I_2) = 0$, we may now summarize the contents of Lemmas 2.1 and 2.4 as follows:

COROLLARY 2.6. *Let $A, B \in \mathrm{GL}(2, \mathbf{Z})$ be such that* $\det(A - I_2) = \det(B - I_2) = 0$. *Then $A \sim B$ if and only if* $\det A = \det B$ *and* $m(A) = m(B)$.

COROLLARY 2.7. *Let φ be a transformation of* $\mathbf{T}^2$ *such that* $\det D(\varphi) = -1$ *and* $\det(D(\varphi) - I_2) = 0$. *Then*

$$K_1(\mathscr{A}(\varphi)) \cong \mathbf{Z}^2 \oplus \mathbf{Z}_{m(\varphi)}.$$

Proof. Arguing as in the proof of Corollary 2.2 φ is topologically conjugate to a transformation ψ of $\mathbf{T}^2$ such that

$$D(\psi) = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } m(\varphi) = 1, \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } m(\varphi) = 2. \end{cases}$$

On applying Theorem 1.1(3) to ψ we obtain

$$\begin{aligned} K_1(\mathscr{A}(\varphi)) &\cong K_1(\mathscr{A}(\psi)) \\ &\cong \begin{cases} \mathbf{Z} \oplus \left(\dfrac{\mathbf{Z}^2}{(-1, 1)\mathbf{Z} + (1, -1)\mathbf{Z}} \right) & \text{if } m(\varphi) = 1, \\ \mathbf{Z} \oplus \left(\dfrac{\mathbf{Z}^2}{(0, 0)\mathbf{Z} + (0, -2)\mathbf{Z}} \right) & \text{if } m(\varphi) = 2, \end{cases} \\ &\cong \mathbf{Z}^2 \oplus \mathbf{Z}_{m(\varphi)}. \end{aligned}$$

Combining the results of this section together with those of the previous one we arrive at the following result.

PROPOSITION 2.8. *Let* φ_1 *and* φ_2 *be transformations of* $\mathbf{T}^2$ *such that* $\det(D(\varphi_i) - I_2) = 0$, $i = 1, 2$. *If* $\mathscr{A}(\varphi_1)$ *and* $\mathscr{A}(\varphi_2)$ *have isomorphic* K_i*-groups* $(i = 0, 1)$, *then* $\det D(\varphi_1) = \det D(\varphi_2)$ *and* $m(\varphi_1) = m(\varphi_2)$, *so that* $D(\varphi_1) \sim D(\varphi_2)$.

Proof. Since they have isomorphic K_0-groups, Theorem 1.1 implies that $\det D(\varphi_1) = \det D(\varphi_2)$. Since they have isomorphic K_1-groups, we may combine Corollaries 2.3 and 2.7 to get $m(\varphi_1) = m(\varphi_2)$. By Corollary 2.6 we deduce that $D(\varphi_1) \sim D(\varphi_2)$. □

REMARK. The quantity $\det(D(\varphi) - I_2)$ turns out to be the so-called Lefschetz number of φ, which is defined in algebraic topology as the alternating sum of the traces of the induced maps of φ on the cohomology groups of the underlying space (in our case $\mathbf{T}^2$). The Lefschetz fixed point theorem states that if φ is a diffeomorphism on a smooth manifold which has no fixed points, then its Lefschetz number is zero. In our case, for the 2-torus, the Lefschetz number is

$$\det(D(\varphi) - I_2) = 1 - \operatorname{trace}(D(\varphi)) + \det(D(\varphi)).$$

(see Bott and Tu [**1**, Theorem 11.25].)

3. Quasi-rotations.

DEFINITION. A transformation φ of $\mathbf{T}^2$ is said to be a quasi-rotation if $D(\varphi) \neq I_2$ and if φ has a "non-singular" eigenvalue $\lambda \neq 1$. That is, $\exists f \in C(\mathbf{T}^2)$ invertible such that $f \circ \varphi = \lambda f$.

Taking the supremum on both sides of $f \circ \varphi = \lambda f$ yields $|\lambda| = 1$. Thus $f/|f|$ is a unitary eigenfunction with eigenvalue λ. Hence we will always assume, without loss of generality, that f is unitary. It is easy to show that the *affine* quasi-rotations have eigenvalues which are automatically non-singular.

Crossed products of $C(\mathbf{T}^n)$ by affine rotations of $\mathbf{T}^n$, i.e. $D(\varphi) = I_2$, have been classified by Riedel [**13**, Corollary 3.7].

LEMMA 3.1. *Let* φ *be a quasi-rotation with non-singular eigenvalue* $\lambda \neq 1$ *so that* $f \circ \varphi = \lambda f$, *where* $f \in C(\mathbf{T}^2)$ *is unitary. Then*

(i) $D(f) \neq [0\ 0]$,
(ii) $\det(D(\varphi) - I_2) = 0$.

Proof. Assume that $D(f) = [0\ 0]$ so that one can write $f(x, y) = e^{2\pi i F(x,y)}$, for some continuous real-valued function F on $\mathbf{T}^2$. The relation $f \circ \varphi = \lambda f$ then becomes

$$e^{2\pi i (F(\varphi(x,y)) - F(x,y))} = \lambda.$$

Thus $F(\varphi(x, y)) - F(x, y) = c$, for all $(x, y) \in \mathbf{T}^2$, where c is a real constant. By induction this becomes

$$F(\varphi^{(k)}(x, y)) - F(x, y) = kc,$$

for every positive integer k. But since the left-hand side is bounded, it follows that $c = 0$ and so $\lambda = 1$, a contradiction.

Upon taking degrees on both sides of $f \circ \varphi = \lambda f$ we obtain $D(f)D(\varphi) = D(f)$, or $D(f)(D(\varphi) - I_2) = 0$, where $D(f) \neq [0\ 0]$. Therefore, $\det(D(\varphi) - I_2) = 0$. □

DEFINITION. Let φ be a quasi-rotation of $\mathbf{T}^2$ and λ a non-singular eigenvalue of φ. We call λ a "primitive" eigenvalue if it has an associated unitary eigenfunction $f \in C(\mathbf{T}^2)$ such that $D(f)$ has relatively prime entries.

LEMMA 3.2. *Every quasi-rotation φ of $\mathbf{T}^2$ has a primitive non-singular eigenvalue ($\neq 1$), which is unique up to complex conjugation.*

Proof. Suppose that $f \circ \varphi = \lambda f$, $\lambda \neq 1$, and $f \in C(\mathbf{T}^2)$ is a unitary with $D(f) = [m\ n] \neq [0\ 0]$ (by Lemma 3.1). Let $d = \gcd(m, n)$. Choose a unitary $g \in C(\mathbf{T}^2)$ such that $g^d = f$, where g^d is the d-fold pointwise product of g. Thus $g^d \circ \varphi = \lambda g^d$, or $[(g \circ \varphi)\overline{g}]^d = \lambda$. By continuity, $(g \circ \varphi)\overline{g} = \lambda_0$ for some dth-root λ_0 of λ. Hence $g \circ \varphi = \lambda_0 g$ and $\lambda_0 \neq 1$ is primitive since the entries of $D(g) = [(m/d)\ (n/d)]$ are relatively prime.

To prove the uniqueness part suppose that in addition to $g \circ \varphi = \lambda_0 g$ (λ_0 primitive) we have $h \circ \varphi = \mu h$, where μ is primitive and $D(h)$ has relatively prime entries. Taking degrees on both sides of these two equations we get $D(g)(D(\varphi) - I_2) = 0$, and $D(h)(D(\varphi) - I_2) = 0$. Since $D(\varphi) - I_2 \neq 0$, it follows that $D(g)$ and $D(h)$ are rationally dependent, that is, there are non-zero integers a and b such that

$$aD(g) + bD(h) = [0\ 0].$$

But since $D(g)$ and $D(h)$ have relatively prime entries it follows that $D(g) = \pm D(h)$, and so $D(gh^{\pm 1}) = [0\ 0]$. From the above two eigenvalue equations we have

$$(gh^{\pm 1}) \circ \varphi = (\lambda_0 \mu^{\pm 1})(gh^{\pm 1}).$$

Since $gh^{\pm 1}$ has zero bidegree, Lemma 3.1(i) implies that $\lambda_0 \mu^{\pm 1} = 1$ Hence, $\mu = \lambda_0^{\pm 1}$, as desired. □

EXAMPLES. 1. Let $\lambda = e^{2\pi i \theta}$, $0 < \theta < 1$, and consider the Anzai transformation $\varphi_\theta(x, y) = (\lambda x, xy)$. Since $D(\varphi_\theta) \neq I_2$ and $u \circ \varphi_\theta = \lambda u$ where $u(x, y) = x$ and $\lambda \neq 1$, φ_θ is a quasi-rotation. In fact, it is clear that φ_θ is affine. If θ is irrational, then φ_θ is minimal on the 2-torus (using the minimality criterion in [**8**, p. 84], or [**15**, Prop. 1.1.4]). Hence the associated crossed product C^*-algebra $\mathscr{A}(\varphi_\theta)$ is simple (cf. Power [**12**]) and has a unique faithful tracial state since φ_θ is uniquely ergodic, i.e. has a unique invariant Borel probability measure (cf. [**5**, Prop. 1.12] or [**15**, Lemma 1.3.4]). The isomorphism classes of these algebras (for θ irrational) were studied by Packer [**7**], and also by Ji [**5**] in his more general setting of Furstenberg transformations of n-tori. Here we shall classify these crossed products within the slightly broader family of those associated with affine quasi-rotations.

2. Furstenberg [**4**, p. 597] proved that a minimal transformation φ of $\mathbf{T}^2$ which is not homotopic to the identity, i.e. such that $D(\varphi) \neq I_2$, has an irrational eigenvalue λ, so that any (non-zero) eigenfunction will automatically be invertible. Hence φ is a quasi-rotation.

3. There are only two orientation reversing affine quasi-rotations of $\mathbf{T}^2$ up to topological conjugation (by Lemma 2.4 above). The first one is of the form $(x, y) \mapsto (ay, bx)$, with degree matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

having primitive eigenvalue $\lambda = ab$ (say $\lambda \neq 1$) and eigenfunction $f(x, y) = xy$. The second one has the form $(x, y) \mapsto (\lambda x, \bar{y})$, with degree matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and has primitive eigenvalue λ (say $\lambda \neq 1$) and eigenfunction $u(x, y) = x$.

4. In [**16**] certain techniques of Furstenberg have been used to construct a (non-affine) quasi-rotation ψ which does not have topologically quasi-discrete spectrum. This settled a question of Ji [**5**, pp. 75–76] in the negative; namely, whether in general a transformation of the form $(x, y) \mapsto (e^{2\pi i\theta}x, f(x)y)$, where $f\colon \mathbf{T} \to \mathbf{T}$ is continuous with degree ± 1, is topologically conjugate to the Anzai transformation φ_θ or to its inverse. The latter has topologically quasi-discrete spectrum and so cannot be topologically conjugate to ψ. An interesting question in this regard is whether the associated crossed product C^*-algebras are isomorphic. They have the same K-groups, same tracial range, have unique tracial states, and are both simple.

4. The range of the trace. In this section we wish to compute the range of the trace for the algebras $\mathscr{A}(\varphi)$ for any quasi-rotation φ.

This computation follows closely that of the irrational rotation algebras studied by Rieffel [**14**] and Pimsner and Voiculescu [**11**].

Let us note that almost every C^*-crossed product of a commutative unital C^*-algebra by $\mathbf{Z}$ has a tracial state. If X is a compact metric space and φ is a transformation of X, then a theorem of Krylov and Bogolioubov (cf. [**18**, p. 132]) ensures that there is a Borel probability measure μ on X which is φ-invariant, that is, $\mu(\varphi^{-1}(E)) = \mu(E)$ for every Borel subset E of X. The map

$$\tau(f) = \int_X f\,d\mu$$

is a tracial state on $C(X)$ which is α-invariant, where α is the automorphism of $C(X)$ associated with φ, i.e. $\alpha(f) = f \circ \varphi^{-1}$. This means that τ induces a tracial state $\hat{\tau}$ on $C(X) \times_\alpha \mathbf{Z}$.

THEOREM 4.1. *Let φ be a quasi-rotation of $\mathbf{T}^2$ with primitive eigenvalue $\lambda = e^{2\pi i\theta}$. Then for any tracial state τ on $\mathscr{A}(\varphi)$ we have*

$$\tau_* K_0(\mathscr{A}(\varphi)) = \mathbf{Z} + \theta\mathbf{Z}.$$

Note that we did not assume that θ is irrational, only that it is not an integer.

Proof. Let $f \in C(\mathbf{T}^2)$ be a unitary such that $f \circ \varphi = \lambda f$ and $D(f)$ has relatively prime entries. This f induces a C^*-homomorphism ρ: $C(\mathbf{T}) \to C(\mathbf{T}^2)$ given by $\rho(g) = g \circ f$.

If we let β denote the automorphism on $C(\mathbf{T})$ associated with rotation by λ, namely, $\beta(g)(x) = g(\overline{\lambda}x)$, for $g \in C(\mathbf{T})$ and $x \in \mathbf{T}$, then ρ is an equivariant homomorphism between the C^*-dynamical systems $(C(\mathbf{T}), \beta, \mathbf{Z})$ and $(C(\mathbf{T}^2), \alpha_\varphi, \mathbf{Z})$. To see this we verify that $\rho \circ \beta = \alpha_\varphi \circ \rho$ as follows:

$$\begin{aligned}\alpha_\varphi(\rho(g))(z) = \rho(g)(\varphi^{-1}(z)) &= g \circ f \circ \varphi^{-1}(z) = g(\overline{\lambda} f(z)) \\ &= \beta(g)(f(z)) = \rho(\beta(g))(z),\end{aligned}$$

for all $z \in \mathbf{T}^2$ and $g \in C(\mathbf{T})$.

Using the naturality of the Pimsner-Voiculescu sequence, this ρ induces a morphism between their associated Pimsner-Voiculescu sequences yielding the commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \to & K_0(C(\mathbf{T})) & \xrightarrow{i_*} & K_0(C(\mathbf{T}) \times_\beta \mathbf{Z}) & \xrightarrow{\delta_0} & K_1(C(\mathbf{T})) & \to & \cdots \\
 & & \downarrow \rho_* & & \downarrow \tilde{\rho}_* & & \downarrow \rho_* & & \\
\cdots & \to & K_0(C(\mathbf{T}^2)) & \xrightarrow{i_*} & K_0(\mathscr{A}(\varphi)) & \xrightarrow{\delta_0'} & K_1(C(\mathbf{T}^2)) & \to & \cdots
\end{array}$$

where $\tilde{\rho}$: $C(\mathbf{T}) \times_\beta \mathbf{Z} \to \mathscr{A}(\varphi)$ is the induced homomorphism from ρ.

If θ is irrational one can construct the Rieffel projection e in $C(\mathbf{T}) \times_\beta \mathbf{Z} = \mathscr{A}_\theta$ having trace θ (cf. [**14**, pp. 418f]). If θ is rational one can still construct the Rieffel projection in the same way and it can be shown that $\tau'(e) = \theta$, for any tracial state τ' on $\mathscr{A}_\theta$ (cf. Elliott [**3**, Lemma 2.3, pp. 170-171]). In both cases one has $\delta_0[e] = [f_0]$, which is the generator of $K_1(C(\mathbf{T}))$, where $f_0(z) = z$, $z \in \mathbf{T}$. Since the diagram commutes, one has

$$\delta_0'[\tilde{\rho}(e)] = \delta_0'\tilde{\rho}_*[e] = \rho_*\delta_0[e] = \rho_*[f_0] = [f],$$

and since $D(f)$ has relatively prime entries, $[f]$ is generator of $\ker((\alpha_\varphi)_* - \mathrm{id}_*)$ in $K_1(C(\mathbf{T}))$. Hence the projection $\tilde{\rho}(e)$ yields a generator in $K_0(\mathscr{A}(\varphi))$ which, along with the two generators as in Corollary 1.2 (having traces 0 and 1), gives the range of the trace as

$$\begin{aligned} \tau_* K_0(\mathscr{A}(\varphi)) &= \mathbf{Z} + \tau(\tilde{\rho}(e))\mathbf{Z} \\ &= \mathbf{Z} + \tau'(e)\mathbf{Z} \\ &= \mathbf{Z} + \theta\mathbf{Z}, \end{aligned}$$

where $\tau' = \tau \circ \tilde{\rho}$ is a tracial state on $\mathscr{A}_\theta$. □

REMARK. One could use Pimsner's computation of the tracial range [**10**] to prove the above theorem using the concept of the determinant associated with a trace. But for our purposes the above short proof suffices.

Now let us look at some of the consequences of this theorem and the results of the preceding section.

COROLLARY 4.2. *Let* φ_j *be a quasi-rotation of* $\mathbf{T}^2$ *with primitive eigenvalue* $\lambda_j = e^{2\pi i\theta_j}$, $j = 1, 2$. *If* $\mathscr{A}(\varphi_1) \cong \mathscr{A}(\varphi_2)$, *then*

(1) $\mathbf{Z} + \theta_1\mathbf{Z} = \mathbf{Z} + \theta_2\mathbf{Z}$,

(2) $\det D(\varphi_1) = \det D(\varphi_2)$,

(3) $m(\varphi_1) = m(\varphi_2)$.

Consequently, $D(\varphi_1) \sim D(\varphi_2)$.

Proof. The preceding theorem yields (1), and Proposition 2.8 yields (2) and (3). □

COROLLARY 4.3 (*Packer* [**7**, p. 49]; Ji [**5**, p. 39]). *For each irrational number* $0 < \theta < 1$ *and each non-zero integer* k, *let* $H_{\theta,k}$ *denote* *the crossed product C^*-algebra of $C(\mathbf{T}^2)$ by the Anzai transformation $\varphi(x, y) = (e^{2\pi i\theta}x, x^k y)$. Then*

$$H_{\theta,k} \cong H_{\theta',k'} \Leftrightarrow |k| = |k'| \quad \text{and} \quad \theta' \in \{\theta, 1-\theta\}.$$

Proof. (Note that if $k = 0$, then $H_{\theta,k} \cong \mathscr{A}_\theta \otimes C(\mathbf{T})$ and the conclusion easily holds.)

($\Rightarrow$) In this case the tracial ranges being equal (by the preceding corollary) implies that $\theta' \in \{\theta, 1-\theta\}$, as the latter are irrational. Since these algebras have isomorphic K_1-groups, Corollary 2.3 shows that $|k| = |k'|$. The converse easily follows. □

Let us recall that the natural action of the group $\mathrm{GL}(2, \mathbf{Z})$ on the irrational numbers is given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \theta = \frac{a\theta + b}{c\theta + d}.$$

COROLLARY 4.4. *Let φ_j be an irrational quasi-rotation of $\mathbf{T}^2$ with primitive eigenvalue $\lambda_j = e^{2\pi i\theta_j}$, $j = 1, 2$ (i.e. θ_j is irrational). If $\mathscr{A}(\varphi_1)$ and $\mathscr{A}(\varphi_2)$ are strongly Morita equivalent, then*

(1) $\theta_2 = A\theta_1$, *for some* $A \in \mathrm{GL}(2, \mathbf{Z})$,

(2) $\det D(\varphi_1) = \det D(\varphi_2)$,

(3) $m(\varphi_1) = m(\varphi_2)$.

Consequently, $D(\varphi_1) \sim D(\varphi_2)$.

Proof. Conclusions (2) and (3) follow from Proposition 2.8 since strongly Morita equivalent C^*-algebras have isomorphic K-groups. Theorem 4.1 allows one to apply Rieffel's argument [**14**, Proposition 2.5] to derive (1). □

COROLLARY 4.5. *No $\mathscr{A}_\theta$ is isomorphic to any $C(\mathbf{T}^2) \times_\alpha \mathbf{Z}$. For θ irrational, no $\mathscr{A}_\theta$ is strongly Morita equivalent to any $C(\mathbf{T}^2) \times_\alpha \mathbf{Z}$.*

Proof. Assume that $\mathscr{A}_\theta \cong C(\mathbf{T}^2) \times_\alpha \mathbf{Z}$. Then $K_0(C(\mathbf{T}^2) \times_\alpha \mathbf{Z}) \cong K_0(\mathscr{A}_\theta) \cong \mathbf{Z}^2$, and the proof of Theorem 1.1(1) shows that $K_0(C(\mathbf{T}^2) \times_\alpha \mathbf{Z})$ is generated by the classes $[1]$ and $[P]$, where P is the Bott projection. These, however, have traces equal to 1, and so looking at their tracial ranges yields $\mathbf{Z} = \mathbf{Z} + \theta\mathbf{Z}$. Thus $\theta \in \mathbf{Z}$ and hence $\mathscr{A}_\theta \cong C(\mathbf{T}^2)$ which is isomorphic to $C(\mathbf{T}^2) \times_\alpha \mathbf{Z}$, and being therefore commutative implies that $\alpha = \mathrm{id}$. Thus, $C(\mathbf{T}^2) \cong C(\mathbf{T}^2) \times_{\mathrm{id}} \mathbf{Z} \cong C(\mathbf{T}^2) \otimes C(\mathbf{T}) \cong C(\mathbf{T}^3)$, a contradiction. A similar argument shows the second assertion of the corollary. □

The second assertion of this corollary is still true for θ rational, but it requires a little more work which we defer to a future paper **[17]**.

Let us now extend Theorem 4.1 to matrix algebras over $\mathscr{A}(\varphi)$.

If A is a unital C^*-algebra, then any tracial state τ on $M_n \otimes A$ has the form $(1/n)\mathrm{tr} \otimes \tau'$ for some tracial state τ' on A, where tr is the usual trace on matrices (for instance see **[5**, Lemma 3.3]). Furthermore, if all tracial states on A induce the same map on $K_0(A)$, then all tracial states on $M_n \otimes A$ induce the same map on $K_0(M_n \otimes A)$ (cf. **[5**, Lemma 3.5]). In fact one has in this case

$$\tau_* K_0(M_n \otimes A) = \frac{1}{n} \tau'_* K_0(A),$$

for all tracial states τ, τ' on $M_n \otimes A$ and A, respectively. This yields the following.

COROLLARY 4.6. *Let φ be a quasi-rotation of $\mathbf{T}^2$ with primitive eigenvalue $\lambda = e^{2\pi i \theta}$. Then*

$$\tau_* K_0(M_n \otimes \mathscr{A}(\varphi)) = \frac{1}{n}(\mathbf{Z} + \theta \mathbf{Z}),$$

for any tracial state τ on $M_n \otimes \mathscr{A}(\varphi)$.

COROLLARY 4.7. *Let φ_j be a quasi-rotation of $\mathbf{T}^2$ with primitive eigenvalue $\lambda_j = e^{2\pi i \theta_j}$, $j = 1, 2$. If $M_n \otimes \mathscr{A}(\varphi_1) \cong M_k \otimes \mathscr{A}(\varphi_2)$, then*

(1) $n = k$,
(2) $\mathbf{Z} + \theta_1 \mathbf{Z} = \mathbf{Z} + \theta_2 \mathbf{Z}$,
(3) $\det D(\varphi_1) = \det D(\varphi_2)$,
(4) $m(\varphi_1) = m(\varphi_2)$.

Proof. It will suffice to prove (1) since the other conclusions will then follow from Corollaries 4.2 and 4.6. For brevity denote $B_j = \mathscr{A}(\varphi_j)$, $j = 1, 2$. The proof of (1) is easy if θ_j is irrational, but requires a little more work otherwise. To do so it suffices (by symmetry) to prove that if M_k can be unitally embedded in $M_n \otimes B_1$, then $k | n$.

Recall that $K_0(B_1)$ is generated by a projection $e \in B_1$ of trace θ_1, and two other classes $[1]$ and $x = [P] - [1]$.

Let $\{e_{ij}^{(n)}\}_{i,j=1,\dots,n}$ be the standard matrix units for M_n, so that $K_0(M_n \otimes B_1)$ has independent generators $[e_{11}^{(n)} \otimes e]$, $[e_{11}^{(n)} \otimes 1]$, and $e_{11}^{(n)} \otimes x = [e_{11}^{(n)} \otimes P] - [e_{11}^{(n)} \otimes 1]$.

Suppose that $\sigma: M_k \to M_n \otimes B_1$ is a unital embedding and σ_*: $K_0(M_k) \to K_0(M_n \otimes B_1)$, where $K_0(M_k) = \mathbf{Z}[e_{11}^{(k)}]$. Then

$$\sigma_*[e_{11}^{(k)}] = a[e_{11}^{(n)} \otimes e] + b[e_{11}^{(n)} \otimes 1] + c(e_{11}^{(n)} \otimes x),$$

for some integers a, b, c. Now since $I_k = \sum_i e_{ii}^{(k)}$ is the sum of equivalent projections, we get from

$$I_n \otimes 1 = \sigma(I_k) = \sum_i \sigma(e_{ii}^{(k)})$$

that $[I_n \otimes 1] = k[\sigma(e_{ii}^{(k)})] \in K_0(M_n \otimes B_1)$. Thus

$$\begin{aligned} n[e_{11}^{(n)} \otimes 1] &= [I_n \otimes 1] = k\sigma_*[e_{11}^{(k)}] \\ &= ka[e_{11}^{(n)} \otimes e] + kb[e_{11}^{(n)} \otimes 1] + kc(e_{11}^{(n)} \otimes x), \end{aligned}$$

and therefore $n = kb$. □

REMARK. The argument in the above elementary proof can also be used to show a similar result for the rotation C^*-algebras $\mathscr{A}_\theta$. Recall that Rieffel **[14]** showed this for θ irrational, while in **[3]** and **[19]** it was shown for rational θ and $n = k = 1$.

5. Main Theorem. Before embarking on the main result let us introduce some notation and characterize the affine quasi-rotations. Later a partial result is given for the rational affine quasi-rotation algebras.

If $A \in \mathrm{GL}(2, \mathbf{Z})$, say

$$A = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$$

then its action on $\mathbf{T}^2$ is defined by $A(x, y) = (x^m y^n, x^p y^q)$. This actually gives the group isomorphism $\mathrm{Aut}(\mathbf{T}^2) \cong \mathrm{GL}(2, \mathbf{Z})$. It is easy to check that

$$A_1(A_2 z) = (A_1 A_2)(z),$$

for all $A_1, A_2 \in \mathrm{GL}(2, \mathbf{Z})$ and $z \in \mathbf{T}^2$.

Now if $X = [m\ n]$ is a 1×2 integral matrix, it induces a continuous function (actually a character) X: $\mathbf{T}^2 \to \mathbf{T}$ given by $X(x, y) = x^m y^n$. Clearly, $X(Az) = (XA)(z)$ for any X, and $A \in \mathrm{GL}(2, \mathbf{Z})$. Also, since X is a homomorphism, $X(zw) = X(z)X(w)$.

Let us suppose that $A \in \mathrm{GL}(2, \mathbf{Z})$ is such that $A \neq I_2$ and $\det(A - I_2) = 0$. Then the proof of Lemma 3.2 (uniqueness part) shows that there exists an integral matrix $X_A = [m\ n]$ having relatively prime entries such that

$$X_A(A - I_2) = [0\ 0],$$

and that X_A is unique up to sign. So $X_A A = X_A$.

Now let us determine the affine quasi-rotations of $\mathbf{T}^2$.

LEMMA 5.1. *Let* $\varphi(z) = aA(z)$ *be an affine transformation of* $\mathbf{T}^2$. *Then* φ *is a quasi-rotation if and only if the following conditions hold*:

(i) $A \neq I_2$,
(ii) $\det(A - I_2) = 0$,
(iii) $X_A(a) \neq 1$.

Proof. Suppose these three conditions hold. Then for $z \in \mathbf{T}^2$ one has

$$X_A \circ \varphi(z) = X_A(aA(z)) = X_A(a)X_A(A(z)) = X_A(a)X_A(z),$$

so that $X_A \circ \varphi = X_A(a)X_A$, where $X_A(a) \neq 1$ is a non-singular eigenvalue which is primitive as X_A has relatively prime entries. Since also $D(\varphi) = A$, φ is a quasi-rotation.

Conversely, suppose that φ is a quasi-rotation. By definition (i) holds, and by Lemma 3.1 condition (ii) holds. It remains to check (iii). By Lemma 3.2 φ has a primitive eigenvalue $\lambda \neq 1$ so that $f \circ \varphi = \lambda f$, where f is unitary with $D(f)$ having relatively prime entries. Taking D on both sides gives $D(f)(A - I_2) = 0$. By the uniqueness of X_A, we get that $D(f) = \pm X_A$. Replacing f by $\overline{f}$, if necessary, we may assume that $D(f) = X_A = [m\ n]$. So let us then write f as $f(x, y) = x^m y^n e^{2\pi i F(x,y)}$, where F is real-valued. This we may re-write as $f(z) = X_A(z)e^{2\pi i F(z)}$, where $z \in \mathbf{T}^2$. Thus the equation $f \circ \varphi = \lambda f$ becomes

$$X_A(\varphi(z))e^{2\pi i F(\varphi(z))} = \lambda X_A(z)e^{2\pi i F(z)}.$$

Now since $X_A \circ \varphi = X_A(a)X_A$, as we computed above, this equation reduces to

$$e^{2\pi i\{F(\varphi(z)) - F(z)\}} = \lambda\overline{X_A(a)},$$

which, by arguing as in the proof of Lemma 3.1, implies that $\lambda\overline{X_A(a)} = 1$. Hence $X_A(a) = \lambda \neq 1$. □

Let $\mathscr{B}(a, A)$ denote the crossed product C^*-algebra associated with the affine quasi-rotation corresponding to the pair (a, A) satisfying the conditions of the preceding lemma. The inverse of such a quasi-rotation can easily be checked to correspond to the pair $(A^{-1}(\overline{a}), A^{-1})$.

THEOREM 5.2 [**15**, *Theorem* 4.3.2]. *Let* (a_j, A_j) *be a pair corresponding to the irrational affine quasi-rotation* φ_j *of* $\mathbf{T}^2$, $j = 1, 2$. *Then the following are equivalent*:

(1) $\mathscr{B}(a_1, A_1) \cong \mathscr{B}(a_2, A_2)$,

(2) *φ_1 and φ_2 are topologically conjugate via an affine transformation,*

(3) *The following conditions hold:*

(i) $X_{A_2}(a_2) = X_{A_1}(a_1)^{\pm 1}$,

(ii) $\det(A_1) = \det(A_2)$,

(iii) $m(A_1) = m(A_2)$.

Proof. By Lemma 5.1, $A_j \neq I_2$ and $\det(A_j - I_2) = 0$, so that $X_j = X_{A_j}$, with relatively prime entries, exists such that $X_j A_j = X_j$, $j = 1, 2$.

In view of Corollary 4.2 condition (1) implies (3), as $X_j(a_j)$ is irrational. Clearly (2) implies (1). So we need to check that (3) implies (2).

Assuming that (i), (ii), (iii) hold we shall construct an affine transformation $\psi(z) = kK(z)$ which intertwines φ_1 and φ_2. By Corollary 2.6, $A_1 \sim A_2$ so choose $K \in \mathrm{GL}(2, \mathbf{Z})$ such that $KA_1K^{-1} = A_2$. The equation $X_2A_2 = X_2$ becomes $(X_2K)A_1 = X_2K$. Since X_2 has relatively prime entries then so does $X_2K = \pm X_1$. Replacing K by $-K$, if necessary, we may choose the $\pm$ in $X_2K = \pm X_1$ according to whether $X_2(a_2) = X_1(a_1)^{\pm 1}$, respectively.

We need to choose k so that $\psi \circ \varphi_1 = \varphi_2 \circ \psi$. The left-hand side of this is

$$\psi \circ \varphi_1(z) = kK(a_1A_1(z)) = kK(a_1)KA_1(z),$$

and the right side is

$$\varphi_2 \circ \psi(z) = a_2A_2(kK(z)) = a_2A_2(k)A_2K(z).$$

These expressions are equal if and only if

$$(*) \qquad kK(a_1) = a_2A_2(k),$$

and it suffices to show that this equation has a solution $k \in \mathbf{T}^2$.

To do this, first extend the equation $X_2K = \pm X_1$ to

$$\begin{pmatrix} X_2 \\ R_2 \end{pmatrix} K = \begin{pmatrix} \pm X_1 \\ R_1 \end{pmatrix},$$

for some 1×2 integral matrices R_1 and R_2 such that

$$T_j = \begin{pmatrix} X_j \\ R_j \end{pmatrix}$$

has determinant ± 1 (which is possible since X_2 has relatively prime entries). Now apply T_2 to both sides of $(*)$ to get

$$\begin{pmatrix} X_2 \\ R_2 \end{pmatrix}(k)\begin{pmatrix} X_2 \\ R_2 \end{pmatrix}K(a_1) = \begin{pmatrix} X_2 \\ R_2 \end{pmatrix}(a_2)\begin{pmatrix} X_2 \\ R_2 \end{pmatrix}A_2(k),$$

or

$$\begin{pmatrix} X_2 \\ R_2 \end{pmatrix}(k)\begin{pmatrix} \pm X_1 \\ R_1 \end{pmatrix}(a_1) = \begin{pmatrix} X_2 \\ R_2 \end{pmatrix}(a_2)\begin{pmatrix} X_2 \\ R_2' \end{pmatrix}(k),$$

where $R_2' = R_2 A_2$. Note that $R_2' \neq R_2$; for otherwise $R_2(A_2 - I_2) = 0$ which implies that $T_2(A_2 - I_2) = 0$ hence $A_2 - I_2 = 0$ as T_2 is invertible. Thus the above equation becomes

$$(X_2(k), R_2(k))(X_1(a_1)^{\pm 1}, R_1(a_1)) = (X_2(a_2), R_2(a_2))(X_2(k), R_2'(k)).$$

By condition (i) the first coordinates of both sides of this equation are equal for all k. The second coordinates become

$$R_2(k)R_1(a_1) = R_2(a_2)R_2'(k),$$

or

$$(R_2 - R_2')(k) = R_2(a_2)\overline{R_1(a_1)},$$

and this clearly has a solution k since $R_2 - R_2' \neq [0\ 0]$. $\square$

Therefore, the irrational affine quasi-rotation algebras $\mathscr{B}(a, A)$ are completely determined up to isomorphism by the triple $(X_A(a), \det(A), m(A))$, up to conjugacy of $X_A(a)$, where $X_A(a)$ is the primitive eigenvalue coming from the tracial range, $\det(A) = \pm 1$ is known from the K_0-group and $m(A)$ is known from the K_1-group.

COROLLARY 5.3. *For irrational affine quasi-rotations of* $\mathbf{T}^2$, *we have*: $M_n \otimes \mathscr{B}(a_1, A_1) \cong M_k \otimes \mathscr{B}(a_2, A_2)$ *if and only if* $k = n$, $X_{A_1}(a_1) = X_{A_2}(a_2)^{\pm 1}$, $\det(A_1) = \det(A_2)$, *and* $m(A_1) = m(A_2)$.

As a final remark let us note that an argument due to Yin **[19]** for the rational rotation algebras can be used to show Theorem 5.2 (and hence Corollary 5.3) for the rational case for the orientation reversing quasi-rotations. Condition (3) in Theorem 5.2 implies (2) in exactly the same way as in the proof. It only remains to check $(1) \Rightarrow (3)$. Let $X_{A_j}(a_j) = e^{2\pi i \theta_j}$, $j = 1, 2$. Clearly, (ii) and (iii) follow as before, so we need to check (i). An isomorphism $\sigma: \mathscr{B}(a_1, A_1) \to \mathscr{B}(a_2, A_2)$ induces one on their K_0-groups which on their generators (cf. Corollary 1.2(ii)) is of the form

$$\begin{aligned} \sigma_*[1] &= [1], \\ \sigma_*([P_1] - [1]) &= [P_2] - [1], \quad \text{being elements of order two}, \\ \sigma_*[e_{\theta_1}] &= r[1] + s([P_2] - [1]) + t[e_{\theta_2}], \end{aligned}$$

for some integers r, s, t. Taking traces of the last of these equations gives $\theta_1 = r + t\theta_2$. Since the matrix of σ_* is

$$\begin{pmatrix} 1 & 0 & r \\ 0 & 1 & s \\ 0 & 0 & t \end{pmatrix},$$

and is invertible over $\mathbf{Z}$ one has that $t = \pm 1$; hence $\theta_1 = r \pm \theta_2$ which yields (1).

This argument however fails for the orientation preserving case since the above gives us conditions on certain integers that do not necessarily imply that $\theta_1 = \pm\theta_2 \bmod \mathbf{Z}$. This we do not know how to prove since the role of the Bott projection here is not so clear.

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Received March 31, 1989 and in revised form July 18, 1989. The author wishes to thank Professor Edmond Granirer for supporting this research through NSERC grant #A-3016.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 148, No. 1, 1991

RANK-2 FANO BUNDLES OVER A SMOOTH QUADRIC Q_3

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In the present paper we examine rank-2 stable bundles over Q_3 with $c_1 = 0$ and $c_2 = 2$ or 4.

This paper is a continuation of [7] where rank-2 Fano bundles over $\mathbb{P}^3$ and Q_3 were studied. Let us recall that a bundle $\mathscr{E}$ is called Fano if its projectivization $\mathbb{P}(\mathscr{E})$ is a Fano manifold, i.e. a manifold with ample first Chern class $c_1(\mathbb{P}(\mathscr{E}))$. In the present paper we examine rank-2 stable bundles over Q_3 with $c_1 = 0$ and $c_2 = 2$ or 4. These are the cases whose knowledge was necessary to complete the classification of rank-2 Fano bundles over Q_3. They are very different: if $\mathscr{E}$ is stable with $c_1 = 0$, $c_2 = 2$ then its first twist $\mathscr{E}(1)$ is spanned by global sections (see Proposition 1), whereas if $c_2 = 4$ then for a general $\mathscr{E}$ from a component in the moduli $\mathscr{E}(1)$ has no section at all (Proposition 3). We complete the classification of rank-2 Fano bundles over Q_3. The results of §3 from [7] and of the present paper can be summarized in the following

THEOREM. *Let $\mathscr{E}$ be a rank-2 Fano bundle over Q_3. If $c_1\mathscr{E} = -1$ then $\mathscr{E}$ is either $\mathscr{O} \oplus \mathscr{O}(-1)$ or the spinor bundle $\underline{E}$. If $c_1\mathscr{E} = 0$ then $\mathscr{E}$ is either $\mathscr{O} \oplus \mathscr{O}$, or $\mathscr{O}(-1) \oplus \mathscr{O}(1)$, or any stable bundle with $c_2 = 2$* (*see a corollary in* §1 *for a complete description of such bundles*).

Let us recall that the spinor bundle $\underline{E}$ on an odd-dimensional quadric $Q_{2\nu+1}$ is the restriction of the universal 2^ν-bundle on the Grassmannian $\mathrm{Gr}(2^\nu, 2^{\nu+1})$. Then $\underline{E}^* = \underline{E}(1)$. On an even-dimensional quadric $Q_{2\nu}$, $\nu \geq 2$, there are two spinor bundles, corresponding to the two reguli of ν-planes. The following characterization of the bundles with no intermediate cohomology was proved in **[1]**:

THEOREM. *For a vector bundle F on Q_n, $n \geq 2$, it is $H^i(F(l)) = 0$ for all $0 < i < n$, $l \in \mathscr{Z}$, if and only if F is a direct sum of line bundles $\mathscr{O}(l)$ and of their tensor product with spinor bundles.*

1. Bundles with $c_1 = 0$, $c_2 = 2$. In this section we prove the following

PROPOSITION 1. *Let* $\mathscr{E}$ *be a stable bundle on* Q_3 *with* $c_1 = 0$, $c_2 = 2$. *Then* $\mathscr{E}(1)$ *is globally generated* (*and therefore is Fano*).

Then, in view of the Proposition (3.2) from [7] we have:

COROLLARY. *Any stable rank-*2 *bundle on* Q_3 *with* $c_1 = 0$, $c_2 = 2$ *is the pullback of a null correlation bundle on* $\mathbb{P}^3$ *via some double covering* $Q_3 \to \mathbb{P}^3$ (*see* [5] *for a definition of the null correlation bundle*).

To prove the proposition we apply a technique of "killing H^1", developed by Horrocks, see the final acknowledgments in [2]. Namely, starting from a bundle $\mathscr{F}$ with, say, $H^1(\mathscr{F}(-1)) \neq 0$, we take a nontrivial extension of $\mathscr{F}(-1)$ by $\mathscr{O}$ which corresponds to this element of the cohomology. Then the middle bundle of the exact sequence that forms the extension has "simpler" cohomology than the initial one. Eventually, we obtain a bundle with no intermediate cohomology and we use classification theorems of such bundles, see [1]. The proof will be divided into several steps.

Step 1. Using the information on the spectrum of stable bundles, [3], we calculate the cohomology of $\mathscr{E}(1)$:

0	0	0	0	$\uparrow i$
1	1	0	0	$h^i(\mathscr{E}(j))$
0	0	1	1	
0	0	0	0	$\to$
	$j=-2$		$j=0$	j

Step 2. Let us take a nontrivial extension

$$0 \to \mathscr{E}(-1) \to B \to \mathscr{O} \to 0 \tag{1}$$

which corresponds to a non-zero element of $\operatorname{Ext}^1(\mathscr{O}, \mathscr{E}(-1)) = H^1(\mathscr{E}(-1))$. The extension is non-trivial; hence the connecting homomorphism $\delta\colon H^0(\mathscr{O}) \to H^1(\mathscr{E}(-1))$ is a non-zero map. Then we may fill out the cohomology diagram for $B(j)$ as follows:

$$
\begin{array}{|c|c|c|c|l}
\hline
0 & 0 & 0 & 0 & \quad h^i(B(j)) \\
\hline
1 & 1 & 0 & 0 & \\
\hline
0 & 0 & 0 & a & \\
\hline
0 & 0 & 0 & & \\
\hline
 & j=-2 & & j=0 & \quad j
\end{array}
$$

with $a \leq 1$.

Step 3. Let us take $B' = B^*(-1)$. The Chern classes of B' are the following: $c_1 = -1$, $c_2 = 2$, $c_3 = -2$. The cohomology of B' can be easily derived from that of B and the result is

$$
\begin{array}{|c|c|c|c|l}
\hline
 & 0 & 0 & 0 & \\
\hline
a & 0 & 0 & 0 & \quad h^i(B'(j)) \\
\hline
0 & 0 & 1 & 1 & \\
\hline
0 & 0 & 0 & 0 & \\
\hline
 & j=-2 & & j=0 & \quad j
\end{array}
$$

We see that $\dim H^1(B'(-1)) = \dim \operatorname{Ext}^1(\mathscr{O}, B'(-1)) = 1$, so that we consider an extension

$$0 \longrightarrow B'(-1) \longrightarrow C(-1) \longrightarrow \mathscr{O} \longrightarrow 0 \tag{2}$$

corresponding to a non-zero element of $H^1(B'(-1))$.

Step 4. We then calculate that C is a rank-4 vector bundle with all Chern classes zero and the cohomology

$$
\begin{array}{|c|c|c|c|l}
\hline
 & 0 & 0 & 0 & \\
\hline
a & 0 & 0 & 0 & \quad h^i(C(j)) \\
\hline
0 & 0 & 0 & b & \\
\hline
0 & 0 & 0 & & \\
\hline
 & j=-2 & & j=0 & \quad j
\end{array}
$$

with $a \leq 1$, $b \leq 1$.

Step 5. Let

$$0 \longrightarrow C \longrightarrow D \longrightarrow \mathscr{O} \longrightarrow 0 \tag{3}$$

be a non-trivial extension (if $b = 1$) or the splitting one (if $b = 0$). In both cases all Chern classes of D vanish and the cohomology of D

is

$$
\begin{array}{c|c|c|c|cc}
 & & & & i & \\
5-a & 0 & 0 & 0 & & \\
\hline
a & 0 & 0 & 0 & & h^i(D(j)) \\
\hline
0 & 0 & 0 & 0 & & \\
\hline
0 & 0 & 0 & 5 & & \\
\hline
 & j=-2 & & j=0 & & j
\end{array}
$$

Step 6. In a similar way we get rid of a (possibly) positive a. Let us take

$$0 \longrightarrow D^* \longrightarrow F \longrightarrow \mathscr{O} \longrightarrow 0, \tag{4}$$

corresponding to a generator of $H^1(D^*) = H^2(D(-3))$. The bundle F is uniquely determined up to proportionality in $\mathrm{Ext}^1(\mathscr{O}, D^*) = H^1(D^*)$. It is a 6-bundle on Q_3 with no intermediate cohomology, with $H^0(F(-1))$ vanishing and all Chern classes equal zero.

Claim. F is either $\mathscr{O}^6$ or $\mathscr{O}^2 \oplus \underline{E} \oplus \underline{E}^*$, where $\underline{E}$ is the spinor bundle on Q_3.

Proof. It follows easily from the characterization of bundles with no intermediate cohomology.

Step 7. If F is $\mathscr{O}^6$, then D and C in (4) and (3) must be trivial. Dualizing (2) then gives the sequence

$$0 \longrightarrow \mathscr{O}(-1) \longrightarrow \mathscr{O}^4 \longrightarrow B(1) \longrightarrow 0 \tag{5}$$

whose second exterior power is

$$0 \longrightarrow B \longrightarrow \mathscr{O}^6 \longrightarrow B^* \longrightarrow 0 \tag{6}$$

—notice that $B^* = \bigwedge^2[B(1)]$ because B is of rank 3 and $c_1(B) = -2$. Therefore $\mathscr{E}(1)$ is globally generated, because it is an image of B^* (see (1)).

Step 8. We now want to exclude the case $F = \mathscr{O}^2 \oplus \underline{E} \oplus \underline{E}^*$. Assume this is the case. Let us look at the epimorphism $F \longrightarrow \mathscr{O}$ in (4). Its dual is an embedding $\mathscr{O} \subset \mathscr{O} \oplus \mathscr{O} \oplus \underline{E} \oplus \underline{E}^*$. Because $H^0(\underline{E}) = 0$ and $\underline{E}^*$ has no non-vanishing sections, see **[1]**, then the embedding map sends $\mathscr{O}$ into $\mathscr{O} \oplus \mathscr{O}$. Hence the bundle D^* in (4) is equal to $\mathscr{O} \oplus \underline{E} \oplus \underline{E}^*$. In the same way we conclude that $C = \underline{E} \oplus \underline{E}^*$, so instead of (5) we get

$$0 \longrightarrow \mathscr{O}(-1) \longrightarrow \underline{E} \oplus \underline{E}^* \longrightarrow B(1) \longrightarrow 0. \tag{7}$$

Raising this sequence to the second symmetric power, making use of the identity $B^* = \bigwedge^2[B(1)]$ again and recalling that

$$\bigwedge^2(\underline{E} \oplus \underline{E}^*) = \bigwedge^2(\underline{E}) \oplus (\underline{E} \otimes \underline{E}^*) \oplus \bigwedge^2(\underline{E}^*)$$
$$= \mathscr{O}(-1) \oplus \mathscr{E}nd(\underline{E}) \oplus \mathscr{O}(1),$$

we obtain an analogue of (6):

$$(8) \qquad 0 \longrightarrow B \longrightarrow \mathscr{O}(1) \oplus \mathscr{E}nd(\underline{E}) \oplus \mathscr{O}(-1) \longrightarrow B^* \longrightarrow 0,$$

whose twist by -1 is

$$(9) \quad 0 \longrightarrow B(-1) \longrightarrow \mathscr{O} \oplus [\mathscr{E}nd(\underline{E})](-1) \oplus \mathscr{O}(-2) \longrightarrow B^*(-1) \longrightarrow 0,$$

which contradicts the cohomology tables from Step 2 and Step 3—namely that $B(-1)$ and $B^*(-1)$ have no sections.

2. Bundles with $c_1 = 0$, $c_2 = 4$. In view of the results of [7] the following completes the proof of the theorem stated at the beginning of the paper.

PROPOSITION 2. *A vector bundle $\mathscr{E}$ on Q_3 which has $c_1 = 0$, $c_2 = 4$ cannot be Fano.*

Proof. First let us note that an unstable $\mathscr{E}$ with $c_1 = 0$, $c_2 = 4$ cannot be Fano—this is proved at the beginning of §3 in [7]. So let us assume that $\mathscr{E}$ is stable. Using the spectrum technique [3], we calculate the cohomology of $\mathscr{E}(j)$ to be

$$\begin{array}{|c|c|c|c|l}
\hline
0 & 0 & 0 & 0 & \\
\hline
4 & 2 & 0 & 0 & \quad h^i(\mathscr{E}(j)) \\
\hline
0 & 0 & 2 & 4 & \\
\hline
0 & 0 & 0 & 0 & \\
\hline
 & j=-2 & & j=0 & \quad j
\end{array}$$

Consider the natural bilinear map

$$\vartheta\colon H^1(\mathscr{E}(-1)) \times H^0(\mathscr{O}(1)) \longrightarrow H^1(\mathscr{E}).$$

We see that $\dim H^1(\mathscr{E}(-1)) = 2$, $\dim H^0(\mathscr{O}(1)) = 5$ and moreover $\dim H^1(\mathscr{E}) = 4$. The bilinear lemma [4] gives the existence of s and h such that $(s, h) = 0$. Hence there is a section of $\mathscr{E}|Q_2$ over a (not necessarily smooth) hyperplane section of the quadric. The section vanishes at four points. These points are not necessarily distinct, but they are not collinear since otherwise the splitting type of $\mathscr{E}$ on this line would be $(-c, c)$ with $c \geq 2$, contradicting the ampleness of $\mathscr{E}(2)$. Let us take a conic C that passes through at least three of these points, counted with multiplicities. Then $\mathscr{E}|C = \mathscr{O}_C(-d) \oplus \mathscr{O}_C(d)$ with $d \geq 3$, because the section has at least triple zero. But this implies that there exists an effective 1-cycle C' associated to the section of $\mathscr{E}(-3)|C$; the cycle C' is numerically equivalent to $\xi_{\mathscr{E}(-3)} \cdot p^{-1}(C)$, where $\xi_{\mathscr{E}(-3)}$ is the relative hyperplane divisor on $\mathbb{P}(\mathscr{E})$ associated to $\mathscr{E}(-3)$ i.e. a class whose restriction to a fiber of the projection p; $\mathbb{P}(\mathscr{E}) \longrightarrow Q_3$ is a hyperplane and $p_*\mathscr{O}(\xi_{\mathscr{E}(-3)}) = \mathscr{E}(-3)$. Then $H \cdot C' = 2$, $\xi_{\mathscr{E}'} = -d$, where H is the pullback of the hyperplane divisor from Q_3 and $\xi_{\mathscr{E}}$ is equivalent to $\xi_{\mathscr{E}(-3)} + 3H$. Because the anticanonical divisor of $\mathbb{P}(\mathscr{E})$ is equivalent to $2\xi_{\mathscr{E}} + 3H$, we have

$$-K_{\mathbb{P}(\mathscr{E})} \cdot C' = (2\xi_{\mathscr{E}} + 3H) \cdot C' \leq 0,$$

so that $-K_{\mathbb{P}(\mathscr{E})}$ cannot be ample. □

REMARK. Although ruled out from our Fano list, the investigation of rank-2 vector bundles $\mathscr{E}$ with $c_1 = 0$, $c_2 = 4$ on Q_3 seems to be an interesting open problem. In particular:

does a general $\mathscr{E}(1)$ have a section?

We believe that the answer is no. So far we can only show

PROPOSITION 3. *In the moduli space of stable bundles with* $c_1 = 0$, $c_2 = 4$ *there is a component containing bundles with* $H^0(\mathscr{E}(1)) = 0$.

Proof. Assume Z is the zero set of a section of such an $\mathscr{E}(1)$. Because of stability, Z is not a surface while the indecomposability of $\mathscr{E}$ shows that Z is not empty. Hence Z must be a curve. By the adjunction formula we have

$$K_Z = \mathscr{O}_Q(-1)|Z; \tag{10}$$

hence no connected component of Z may be a single line.

Since $c_2(\mathscr{E}(1)) = 6$, we conclude that Z has at most three connected components. Let us consider the bundles given as extensions

$$0 \longrightarrow \mathscr{O} \longrightarrow \mathscr{E}(1) \longrightarrow J_C(2) \longrightarrow 0 \tag{11}$$

where C is the sum of three conics. Let us count how many bundles can be obtained in this way. The conics in Q_3 are in 1-1 correspondence with 2-planes in $\mathbb{P}^4$, hence the dimension of the family of triples of conics is equal to $3 \cdot \dim(\mathrm{Grass}(2, 4)) = 18$. The number of non-isomorphic extensions of the form (11) is equal to the dimension of

$$\mathrm{Ext}^1(J_C(2), \mathscr{O})) = H^0(\mathscr{E}xt(\mathscr{O}_C(2), \mathscr{O}) = H^0(\mathscr{O}_C),$$

i.e., to 3, see [5], Ch. I, §5.1. Because proportional extensions give rise to isomorphic bundles, altogether we have a bundle family of dimension $18 + 3 - 1 = 20$. On the other hand, using the obvious relation $\mathscr{E}nd(\mathscr{E}) = \mathscr{E} \otimes \mathscr{E}^*$ we calculate using (11) that

$$\dim(H^0(\mathscr{E}nd(\mathscr{E}))) = 1, \quad \dim(H^1(\mathscr{E}nd(\mathscr{E}))) = 21,$$
$$\dim(H^2(\mathscr{E}nd(E))) = 0, \quad \dim(H^3(\mathscr{E}nd(\mathscr{E}))) = 0.$$

Therefore a local deformation of a bundle given by (11) need not be such. The bundles that do not arise from deformations of those given by (11) must then come from curves C's having at least four components, which is not possible by (10). Hence $\mathscr{E}(1)$ has no section. Because of the semicontinuity, the same holds for a generic bundle in the same component. □

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Received January 22, 1989.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 148, No. 1, 1991

RICCI CURVATURE AND VOLUME GROWTH

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We give an example of a complete manifold M^m of nonnegative Ricci curvature for which the volume of distance tubes around a totally geodesic submanifold L^l divided by the corresponding volume in $L \times \mathbf{R}^{m-l}$ goes to infinity. Recall that in the case of nonnegative sectional curvature, this quotient is nonincreasing and bounded by 1.

1. Introduction. One of the fundamental tools in the study of Ricci curvature is the Bishop-Gromov volume inequality, which states that in a complete manifold M^m of Ricci curvature $\geq (m-1)\kappa$, the map

$$r \mapsto \frac{\operatorname{vol} B_r(p)}{\operatorname{vol}(D_r, \hat{g}_\kappa)}$$

is monotonically nonincreasing. Here, $B_r(p)$ is the ball of radius r around $p \in M$, and $(D_r, \hat{g}_\kappa)$ is a ball of same radius in the simply connected space of constant sectional curvature κ. Under somewhat different assumptions, this inequality still holds when p is replaced by a compact, totally geodesic submanifold L^l of M: The comparison space now becomes $(L \times D_r, g_\kappa)$, where for $x = (x_0, x_1)$ in the tangent space of $L \times D_r$ at (p, u), $g_\kappa(x, x) = c_\kappa^2(|u|)\, \check{g}(x_0, x_0) + \hat{g}_\kappa(x_1, x_1)$. (Here $\check{g}$ is the metric on L induced by the imbedding $L \hookrightarrow M$, and c_κ is the solution of the equation $c_\kappa'' + \kappa c_\kappa = 0$, with $c_\kappa(0) = 1$, $c_\kappa'(0) = 0$.) The volume inequality now reads (cf. **[4]**, **[3]**, **[6]**):

(*) If the radial sectional curvatures of M are $\geq \kappa$, then

$$q_L(r) \stackrel{\text{def}}{=} \frac{\operatorname{vol} B_r(L)}{\operatorname{vol}(L \times D_r, g_\kappa)}$$

is a nonincreasing function of r, with $q_L(0) = 1$. (A 2-plane $\sigma \subset M_q$ is said to be radial if it contains the tangent vector of some minimal geodesic from q to L.)

(**) If all sectional curvatures of M are $\geq \kappa$, then $q_L(r') = q_L(r)$ for some $0 < r' < r$ only if the normal bundle of $L \hookrightarrow M$ is flat with respect to the induced connection, and $B_r(L)$ is (locally) isometric to $(L \times D_r, g_\kappa)$.

In this note, we show that $(*)$ no longer holds in general if one only assumes $\mathrm{Ric}_M \geq (m-1)\kappa$ (see also [1] for a related result): In fact, the quotient $q_L(r)$ may go to infinity as $r \to \infty$. Moreover, even if the radial sectional curvatures are $\geq \kappa$—so that $(*)$ must hold—$(**)$ is no longer true if one replaces $K_M \geq \kappa$ by $\mathrm{Ric}_M \geq (m-1)\kappa$. More precisely, we have:

1.1. THEOREM. *Let $L = \mathbf{C}P^1$, and $M = \mathbf{C}P^2$. Then*

(a) *The normal bundle E of $L \hookrightarrow M$ admits a complete metric of nonnegative Ricci curvature such that*

$$q_L(r) \stackrel{\text{def}}{=} \frac{\operatorname{vol} B_r(L)}{\operatorname{vol}(L \times D_r, g_0)}$$

goes monotonically to infinity as $r \to \infty$.

(b) *There is a complete metric on M with the following properties:*

(1) *L is totally geodesically imbedded in M.*

(2) *$\mathrm{Ric}_M \geq 3$, and the radial sectional curvatures are ≥ 1.*

(3) *$q_L(r) \stackrel{\text{def}}{=} \frac{\operatorname{vol} B_r(L)}{\operatorname{vol}(L \times D_r, g_1)} \equiv 1$ for $r \leq \varepsilon$, provided ε is sufficiently small.*

2. Ricci curvature for connection metrics. Let $L = \mathbf{C}P^1 \hookrightarrow \mathbf{C}P^2$ with the standard metric of curvature $1 \leq K \leq 4$. As in [5], we identify a distance tube $B_r(L)$ around L with $[0, r] \times S^3/\sim$, where all the Hopf fibers are collapsed to a point at $\{0\} \times S^3$. Consider the class $d\sigma_r^2$ of metrics on S^3 obtained by multiplying the standard metric by $f^2(r)$ in the Hopf fiber direction, and by $h^2(r)$ on its orthogonal complement. If f is an odd smooth function with $f'(0) = 1$, and h is even and positive, then the metric $dr^2 + d\sigma_r^2$ on $(0, r] \times S^3$ extends to $B_r(L)$. The standard metric corresponds to $f(r) = (1/2)\sin 2r$ and $h(r) = \cos r$. Using the same vector fields X_i, $0 \leq i \leq 3$, as in [5] (where X_0 is radial, X_1 is tangent to the Hopf fiber, and X_2, X_3 are orthogonal to it), we obtain for $R_{ij} := \mathrm{Ric}(X_i/|X_i|, X_j/|X_j|)$:

$$R_{00} = -\frac{f''}{f} - 2\frac{h''}{h}, \tag{2-1}$$

$$R_{11} = -\frac{f''}{f} - 2\frac{f'h'}{fh} + 2\frac{f^2}{h^4}, \tag{2-2}$$

$$R_{22} = R_{33} = -\frac{h''}{h} - \frac{f'h'}{fh} + \frac{4h^2 - 2f^2 - h'^2h^2}{h^4}, \tag{2-3}$$

$$R_{ij} = 0, \qquad i \neq j. \tag{2-4}$$

The proof is straightforward and will be omitted.

This class of metrics is actually a special case of the following construction: Let $(L^l, \check{g})$ be a Riemannian manifold, and $\mathbf{R}^k \to E \xrightarrow{\pi} L$ a vector bundle with inner product $\langle\ ,\ \rangle$ and Riemannian connection ∇. Fix $0 < r_0 \leq \infty$, and consider the disk bundle $E^{r_0} = \{u \in E \mid \langle u, u\rangle < r_0\}$. If $\mathscr{V}$ denotes the vertical distribution defined by π, and $\mathscr{H}$ the horizontal distribution determined by the connection, define

$$g(x, x) = h^2(|u|)\, \check{g}(\pi_* x, \pi_* x) \qquad (x \in \mathscr{H} \cap T_u E),$$

where h is an even, smooth, positive function on $(-r_0, r_0)$. The fibers of E^{r_0} are endowed with a metric given in polar coordinates by

$$dr^2 + f^2(r)\, d\sigma^2,$$

where $d\sigma^2$ is the standard metric on the sphere, and f is an odd, smooth function with $f'(0) = 1$. We then obtain a metric g on E^{r_0} by declaring $\mathscr{H}$ and $\mathscr{V}$ to be mutually orthogonal. The fibers of the bundle are totally geodesic submanifolds in this metric, and the projection π restricted to a sphere bundle of radius r becomes a Riemannian submersion with base $(L, h^2(r)\,\check{g})$. One can easily compute the Ricci curvatures by using O'Neill's formula for Riemannian submersions and the Gauss equations (cf. also [2]): If ∂_r denotes the unit radial vector field (dual to dr), v a unit vertical vector orthogonal to ∂_r, and x a unit horizontal vector, then

$$\operatorname{Ric}(\partial_r, \partial_r) = -l\,\frac{h''}{h} - (k-1)\frac{f''}{f}, \tag{2-5}$$

$$\operatorname{Ric}(\partial_r, x) = \operatorname{Ric}(\partial_r, v) = 0, \tag{2-6}$$

$$\begin{aligned}\operatorname{Ric}(v, v) = &-\frac{f''}{f} + (k-2)\frac{1-f'^2}{f^2} - l\,\frac{f'h'}{fh}\\ &+ \sum_{i=1}^{l}\langle A_{x_i} v, A_{x_i} v\rangle,\end{aligned} \tag{2-7}$$

$$\begin{aligned}\operatorname{Ric}(x, x) = &-\frac{h''}{h} - (l-1)\frac{h'^2}{h^2} - (k-1)\frac{h'f'}{hf}\\ &+ \operatorname{Ric}^{\vee}(\pi_* x, \pi_* x) - 2\sum_{i=1}^{l}\langle A_x x_i, A_x x_i\rangle,\end{aligned} \tag{2-8}$$

$$\operatorname{Ric}(v, x) = \langle(\check{\delta} A)x, v\rangle. \tag{2-9}$$

Here, $\{x_i\}$ is an orthonormal basis of $\mathscr{H}$, A is the O'Neill tensor of the submersion with divergence $\check{\delta} A = \sum_{i=1}^{l} D_{x_i} A(x_i, \cdot)$ (D is the Levi-Civita connection of (E^{r_0}, g)), and $\mathrm{Ric}^{\vee}$ is the Ricci tensor of $(L, h^2(r)\check{g})$.

Moreover, if ∇ is a Yang-Mills connection, then (cf. **[2]**, p. 243):

$$\text{(2-9')} \qquad \mathrm{Ric}(v, x) = 0.$$

In the special case when E is the normal bundle of $\mathbf{C}P^1 \hookrightarrow \mathbf{C}P^2$, let ∇ denote the connection on E induced by the Levi-Civita connection of the symmetric space $\mathbf{C}P^2$. Then ∇ is Yang-Mills since the curvature tensor R^{∇} is parallel. In particular, (2-9') holds, and it is straightforward to check that (2-5)–(2-9) reduce to (2-1)–(2-4). Notice that the A-tensor can be expressed in terms of R^{∇}, cf. **[6]**.

3. Proof.

Proof of 1.1(a). The volume of a distance tube $B_r(L)$ with respect to the class of metrics described in §2 is given by:

$$\begin{aligned}\mathrm{vol}\, B_r(L) &= \int_0^r \mathrm{vol}\, S_t(L)\, dt \\ &= C \cdot \mathrm{vol}\,(L) \cdot h^{-l}(0) \cdot \int_0^r h^l(t) f^{k-1}(t)\, dt,\end{aligned}$$

where $S_t(L)$ is a distance sphere around L, $\mathrm{vol}\,(L) := \mathrm{vol}\,(L, h^2(0)\check{g})$, and C is the volume of the standard sphere $S^{k-1} \subset \mathbf{R}^k$. It thus suffices to find functions f and h such that (2-1)–(2-3) yield $\mathrm{Ric} \geq 0$, and $h^l(r) f^{k-1}(r)/r^{k-1} = h^2(r) f(r)/r \to \infty$ as $r \to \infty$. Let $f(r) := r/(1+r^2)^{1/2}$, and $h(r) := (r/f(r))^{\alpha}$, where α is any constant in the interval $[1/2, 1]$. Notice that $q_L(r) \to \infty$ as $r \to \infty$ if $\alpha > 1/2$, and $q_L(r) \equiv 1$ for $\alpha = 1/2$.

A straightforward calculation shows that (2-1)–(2-3) become:

$$\text{(3-1)} \qquad \begin{aligned} R_{0,0} &= \frac{-3(2\alpha-1)}{(1+r^2)^2} + \frac{2\alpha}{1+r^2}\left(2 - (\alpha+1)\frac{r^2}{1+r^2}\right) \\ &= \frac{\alpha}{1+r^2}(4 - \varphi_\alpha(r)),\end{aligned}$$

where $\varphi_\alpha(r) = \left(3(2\alpha-1) + 2\alpha(\alpha+1)r^2\right)/\alpha(1+r^2)$. Since φ_α is an increasing function on $[0, \infty)$ with $\lim_{r\to\infty} \varphi_\alpha(r) = 2(\alpha+1) \leq 4$, we conclude that $R_{0,0} \geq 0$.

$$\text{(3-2)} \qquad R_{1,1} = \frac{3-2\alpha}{(1+r^2)^2} + 2\frac{f^2}{h^4} \geq 0.$$

$$\text{(3-3)}\, R_{2,2} = R_{3,3} = \frac{-3\alpha}{(1+r^2)^2} + \frac{\alpha}{1+r^2}\left(1 - \alpha\frac{r^2}{1+r^2}\right)$$
$$+ 4\left(\frac{f(r)}{r}\right)^{2\alpha} - 2r^2\left(\frac{f(r)}{r}\right)^{2+4\alpha} - \frac{\alpha^2 r^2}{(1+r^2)^2}$$
$$\geq (1+r^2)^{-\alpha}(4 - (\psi_\alpha(r) + \theta_\alpha(r))),$$

where $\psi_\alpha(r) := 2r^2/(1+r^2)^{1+\alpha}$, and $\theta_\alpha(r) := (3\alpha + \alpha^2 r^2)/(1+r^2)^{2-\alpha}$.
One easily checks that the maximum of ψ_α equals

$$\eta(\alpha) = 2/\alpha(1 + 1/\alpha)^{1+\alpha} \leq \eta(1/2) = 4/3\sqrt{3},$$

for $\alpha \geq 1/2$. Moreover, θ_α is a decreasing function if $\alpha \leq 1$, with $\theta_\alpha(0) = 3\alpha$. Thus:

$$R_{2,2} = R_{3,3} \geq (1+r^2)^{-\alpha}(4 - (3 + 4/3\sqrt{3})) > 0,$$

thereby completing the proof of 1.1(a).

Proof of 1.1(b). When $h \equiv \cos$, (2-1)–(2-3) become:

(i) $$R_{0,0} = 2 - \frac{f''}{f},$$

(ii) $$R_{1,1} = -\frac{f''}{f} + 2\frac{f' \sin}{f \cos} + 2\frac{f^2}{\cos^4},$$

(iii) $$R_{2,2} = R_{3,3} = 1 + \frac{f' \sin}{f \cos} + \frac{4\cos^2 - 2f^2 - \sin^2\cos^2}{\cos^4}.$$

We will choose f so that $f(r) = \sin r$ for $r \leq \varepsilon$, $f(r) = \sin r \cos r$ for $r \geq \pi/4$, and $R_{i,i} \geq 3$. Define $k := f/\sin$. (i) and (ii) transform into:

(i') $$R_{0,0} = 3 - \frac{k''}{k} - 2\frac{k' \cos}{k \sin},$$

(ii') $$R_{1,1} = 3 - \frac{k''}{k} - 2\frac{k'}{k}\left(\frac{\cos}{\sin} - \frac{\sin}{\cos}\right) + 2k^2\frac{\sin^2}{\cos^4}.$$

If $\varepsilon > 0$ is sufficiently small, there exists a function k such that $k \equiv 1$ on $[0, \varepsilon]$, $k \equiv \cos$ on $[\pi/4, \pi/2]$, and $k'' \leq 0$. Then $R_{0,0}, R_{1,1} \geq 3$. To show that $R_{2,2} \geq 3$, observe that, since $f \leq \sin$,

$$F \stackrel{\text{def}}{=} (4\cos^2 - 2f^2 - \sin^2\cos^2)/\cos^4$$
$$\geq (4\cos^2 - 2\sin^2 - \sin^2\cos^2)/\cos^4 \stackrel{\text{def}}{=} G.$$

Now, the minimum value of $G = (5/\cos^2) - (2/\cos^4) + 1$ on the interval $[0, \pi/4]$ is $G(\pi/4) = 3$. Since $R_{2,2} - F = 2 + (k'\sin)/(k\cos) \geq 1$, the result follows.

We now proceed to show that the radial sectional curvatures are ≥ 1: Let $x \in T_pL$, and consider a unit-speed geodesic γ originating at p and orthogonal to L. If E denotes the parallel field along γ with $E(0) = x$, then $J := hE$ is a Jacobi field along γ, cf. [**3**]. Therefore, $R(E, \dot\gamma)\dot\gamma = -(h''/h)E$, so that $\langle R(E, \dot\gamma)\dot\gamma, E\rangle \equiv 1$. On the other hand, if v is orthogonal to both $\dot\gamma(0)$ and T_pL, and if F denotes the parallel field along γ with $F(0) = v$, then $R(F, \dot\gamma)\dot\gamma = -(f''/f)F$, and

$$\langle R(F, \dot\gamma)\dot\gamma, F\rangle = -f''/f = 1 - (k''/k) - 2(k'/k)(\cos/\sin).$$

This last expression is ≥ 1 and identically 1 on $[0, \varepsilon]$. The same is therefore true for all radial curvatures.

Finally, observe that the comparison space in [**4**] or [**3**] has the same volume growth as $(L \times D_r, g_\kappa)$. It follows that $q_L(r) \equiv 1$ for our choices of f and h when $r \leq \varepsilon$.

4. Remarks.

4.1. In 1.1(a), the maximal growth rate for the volume of $B_r(L)$ obtained by our method is of order r^3.

4.2. The maximal distance from L with respect to the metric g from 1.1(b) is $\pi/(2\sqrt{\kappa}) = \pi/2$, where κ is the infimum of the radial sectional curvatures and the Ricci curvature. Nevertheless, (M, g) is not symmetric, cf. the remark on p. 322 in [**3**].

4.3. As the general formulas of §2 show, one can produce similar examples on other vector bundles. It is, however, essential to have some information about the divergence of the A-tensor, cf. (2-9), (2-9').

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Received September 29, 1989. The first author was supported in part by the Heinrich Hertz Foundation, the second author by a grant from the National Science Foundation.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 148, No. 1, 1991

A RIESZ THEORY IN VON NEUMANN ALGEBRAS

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An operator T is called a Riesz operator relative to a von Neumann algebra $\mathscr{A}$ if $T - \lambda I$ is Fredholm relative to $\mathscr{A}$ for each $\lambda \neq 0$. Properties of Riesz operators are studied and a geometrical characterization of these operators are given. This characterization is used to show that a Riesz type of decomposition holds.

Introduction. The main theme of this paper is to introduce Riesz operators relative to a von Neumann algebra and to obtain a Riesz type of decomposition for these operators.

The theory of compact and Fredholm operators relative to a von Neumann algebra has been studied in detail by various authors (cf. [3], [4], [7], [8], [10], etc.). In the present paper Riesz operators are defined in a natural way via the Fredholm operators relative to a von Neumann algebra $\mathscr{A}$, i.e. T will be called Riesz relative to $\mathscr{A}$ if $T - \lambda I$ is Fredholm relative to $\mathscr{A}$ for every $\lambda \neq 0$.

After some preliminaries in §1 we develop the basic results on Riesz operators in §2. These results are similar to results known for the classical case and will be used in the sequel. Section 3 contains a geometrical characterization of the Riesz operators. This may be considered as the main result of this paper, since it allows one to use the techniques of [4] and [5] to obtain the required Riesz decomposition in §4.

Whereas in the classical case the theory of Riesz operators has an intimate connection with spectral theory, it should be noted that in our representation we do not use spectral theory at all. Actually one cannot hope to obtain any results on the spectrum of a Riesz operator relative to a von Neumann algebra. In finite von Neumann algebras for instance all operators are Riesz. One can thus find Riesz operators with spectral properties very different from the classical case.

1. Preliminaries. Let $L(H)$ be the algebra of all bounded linear operators on a Hilbert space H. Throughout the paper $\mathscr{A}$ will denote a concrete von Neumann algebra on H. We denote by $\mathscr{F}$ the ideal generated by the projections which are finite relative to $\mathscr{A}$ (cf. [11], Chapter V for properties of the projection lattice $\mathscr{P}(\mathscr{A})$ on $\mathscr{A}$). The ideal of compact operators $\mathcal{K}$ relative to $\mathcal{A}$ is the uniform closure of $\mathcal{F}$. Let $\pi: \mathcal{A} \to \mathcal{A}/\mathcal{K}$ be the canonical quotient map. An operator $T \in \mathcal{A}$ is called a Fredholm operator (relative to $\mathcal{A}$) if $\pi(T)$ is invertible. For any $T \in \mathcal{A}$ we shall denote by N_T the null projection and R_T the range projection (cf. [3], (3.1) and (3.2)). It follows from [4], Theorem 1 and [7], Theorem 2.2 that T is Fredholm iff N_T is finite and R_T is cofinite relative to $\mathcal{A}$ (cf. §4 for the definition of a cofinite range projection). The set of all Fredholm operators in $\mathcal{A}$ will be denoted by Φ. We refer to [3] and [4] for the definition of the index function on Φ with values in a partially ordered abelian group. Let Φ_0 denote the class of Fredholm operators T with index zero (i.e. $N_T \sim N_{T^*}$).

For $T \in \mathcal{A}$ we denote the spectral radius by $r(T)$ and we shall call the spectrum of $\pi(T)$ in $\mathcal{A}/\mathcal{K}$ the *(Wolf) essential spectrum* of T and denote it by $\sigma^e(T)$.

2. **Riesz operators.** An operator $T \in \mathcal{A}$ will be called a Riesz operator (relative to $\mathcal{A}$) if $\lambda I - T \in \Phi$ for every $\lambda \neq 0$. It is clear that T is Riesz iff $\sigma^e(T) = \{0\}$, which is also equivalent to

$$\lim_{n\to\infty} \left(\inf_{K \in \mathcal{K}} \|T^n - K\|\right)^{1/n} = r(\pi(T)) = 0$$

(cf. [7], 3.10). Since $\mathcal{F}$ is dense in $\mathcal{K}$ we may replace $\mathcal{K}$ with $\mathcal{F}$ in the last characterization. We shall denote the set of all Riesz operators by $\mathcal{R}$ and if the reference to $\mathcal{A}$ is necessary we denote this set by $\mathcal{R}(\mathcal{A})$.

REMARKS. 1. Since for a finite von Neumann algebra $\mathcal{A}$ we know that $\Phi = \mathcal{A}$ it is clear that then also $\mathcal{R} = \mathcal{A}$. The theory of Riesz operators in this case is trivial.

2. For any compact $K \in \mathcal{A}$ one has $r(\pi(K)) = 0$ from which it follows that $\mathcal{K} \subseteq \mathcal{R}$. There are many cases where this inclusion is strict.

3. In purely infinite von Neumann algebras the Riesz operators coincide with the quasinilpotent operators (recall that in this case $\mathcal{K} = \{0\}$).

We denote by $[S, T]$ the commutator of S and T, i.e. $[S, T] = ST - TS$. By using the well-known property that in any Banach algebra the relations $r(TS) \leq r(T)r(S)$ and $r(T + S) \leq r(T) + r(S)$ hold for any two commuting S and T, one easily obtains the following proposition.

2.1. PROPOSITION. (a) *If* $S \in \mathscr{R}$, $T \in \mathscr{A}$ *and* $[S, T] \in \mathscr{K}$, *then* $ST, TS \in \mathscr{R}$.

(b) *If* $S, T \in \mathscr{R}$ *and* $[S, T] \in \mathscr{K}$ *then* $T + \alpha S \in \mathscr{R}$ *for any* $\alpha \in \mathbb{C}$.

(c) *If a sequence* (T_n) *of Riesz operators is uniformly convergent to* $T \in \mathscr{A}$ *and if* $[T_n, T] \in \mathscr{K}$ *for all* $n \in \mathbb{N}$ *then* $T \in \mathscr{R}$.

It follows from 2.1 that the closed algebra generated by a Riesz operator is contained in $\mathscr{R}$.

2.2. PROPOSITION. *For* $T \in \mathscr{A}$ *we have that* $T \in \mathscr{R}$ *iff* $T^n \in \mathscr{R}$ *for any (and hence for all)* $n \in \mathbb{N}$.

Proof. If $T \in \mathscr{R}$ then $T^n \in \mathscr{R}$ for any $n \in \mathbb{N}$ follows trivially from 2.1. Conversely if $T^n \in \mathscr{R}$ it follows by definition that

$$\lim_{k \to \infty} \inf_{K \in \mathscr{K}} \|T^{nk} - K\|^{1/nk} = 0.$$

Since

$$r(\pi(T)) = \lim_{k \to \infty} \inf_{K \in \mathscr{K}} \|T^k - K\|^{1/k}$$

is finite one clearly has

$$\lim_{k \to \infty} \inf_{K \in \mathscr{K}} \|T^k - K\|^{1/k} = 0. \qquad \square$$

From the fact that $\mathscr{K}$ is a two-sided *-ideal in $\mathscr{A}$ we have for any $T \in \mathscr{A}$ and $K \in \mathscr{K}$ that $r(\pi(T)) = r(\pi(T^*))$ and $r(\pi(T + K)) = r(\pi(T))$. Hence we obtain:

2.3. PROPOSITION. (a) *Let* $T \in \mathscr{A}$ *and* $K \in \mathscr{K}$. *Then* $T \in \mathscr{R}$ *iff* $T^* \in \mathscr{R}$.

(b) $\mathscr{R}$ *is stable under compact perturbations.*

If T is a normal operator in $\mathscr{A}$ it follows that $r(\pi(T)) = \|\pi(T)\|_{\mathscr{A}/\mathscr{K}}$. Hence we have:

2.4. PROPOSITION. *For a normal operator* $T \in \mathscr{R}$ *iff* $T \in \mathscr{K}$.

It seems that the following result is not known even for the classical case.

2.5. PROPOSITION. *If* $T \in \mathscr{A}$, $S \in \mathscr{R}$ *and* $[S, T] \in \mathscr{K}$ *then* $\sigma^e(T + S) = \sigma^e(T)$.

Proof. For any two commuting elements a, b in a Banach algebra one knows that $\sigma(a+b) \subseteq \sigma(a)+\sigma(b)$, in particular $\sigma(\pi(T)+\pi(S)) \subseteq \sigma(\pi(T))+\sigma(\pi(S))$. By assumption $\sigma(\pi(S)) = \{0\}$. Hence $\sigma^e(T+S) \subset \sigma^e(T)$. Similarly $\sigma^e(T) = \sigma^e(T+S-S) \subseteq \sigma^e(T+S)$. □

The above-mentioned proposition may be used to prove a characterization of Riesz operators in von Neumann algebras which is similar to a result due to Schechter (cf. **[9]**, Theorem 12).

2.6. Corollary. *$T \in \mathscr{R}$ iff $T+S \in \Phi$ for all $S \in \Phi$ for which $[S, T] \in \mathscr{K}$.*

Proof. Let $T \in \mathscr{R}$ and $S \in \Phi$ with the property that $[S, T] \in \mathscr{K}$, then we know that $0 \notin \sigma^e(S) = \sigma^e(T+S)$, so $T+S \in \Phi$. Since $[\lambda I, T] = 0$ the converse is trivial. □

For Riesz operators one obtains the following functional calculus.

2.7. Proposition. *Let f be a holomorphic function on an open set U containing $\sigma(T)$ with $f(0) = 0$. Then*

(a) *If $T \in \mathscr{R}$ then $f(T) \in \mathscr{R}$*

(b) *If $f(T) \in \mathscr{R}$ and f does not vanish on $\sigma(T) \setminus \{0\}$ it follows that $T \in \mathscr{R}$.*

Proof. (a) From our assumptions it follows that $f(T) = Tg(T)$ where g is holomorphic on U and $[T, g(T)] = 0$. Then (a) follows directly from 2.1(a).

(b) Since $\sigma^e(T) \subseteq \sigma(T)$ the functional calculus in $\mathscr{A}/\mathscr{K}$ shows that $\pi(f(T)) = f(\pi(T))$, and by the spectral mapping theorem $f(\sigma^e(T)) = \sigma^e(f(T)) = \{0\}$. By hypothesis f does not vanish on $\sigma(T) \setminus \{0\}$, leaving $\sigma^e(T) = \{0\}$ as the only possibility. □

In any unital $\mathscr{C}^*$-algebra $\mathscr{A}$ it is known that $\sigma(xy) \setminus \{0\} = \sigma(yx) \setminus \{0\}$ and $\sigma(uxu^*) \setminus \{0\} = \sigma(x) \setminus \{0\}$ hold for $x, y \in \mathscr{A}$ and $u \in \mathscr{A}$ unitary. The following proposition therefore follows:

2.8. Proposition. (a) *$TS \in \mathscr{R}$ if and only if $ST \in \mathscr{R}$.*

(b) *If S and T are unitary equivalent, then $S \in \mathscr{R}$ iff $T \in \mathscr{R}$.*

One can easily see from the next proposition that if a von Neumann algebra contains non-compact quasinilpotent operators, then $\mathscr{K}$ is properly contained in $\mathscr{R}$.

2.9. PROPOSITION. *If $K \in \mathscr{K}$ and $Q \in \mathscr{A}$ is quasinilpotent, then $K + Q \in \mathscr{R}$.*

Proof. This clearly follows from

$$\|\pi(Q+K)^n\|^{1/n} = \|\pi(Q)^n\|^{1/n} \le \|Q^n\|^{1/n} \quad \text{for all } n \in \mathbb{N}. \qquad \square$$

By the well-known West decomposition theorem (cf. [6], 3.33) the converse of 2.9 holds in the case where $\mathscr{A} = L(H)$. It is an open problem whether this is true in general von Neumann algebras. A partial converse can be obtained by using a result of Akemann and Pedersen [1]: If $T \in \mathscr{A}$ with $T^n \in \mathscr{K}$ for some $n \in \mathbb{N}$ (note that in this case $T \in \mathscr{R}$ by 2.2), then $T = K + Q$ where $K \in \mathscr{K}$ and Q is nilpotent. This follows from the fact that [1] 4.3 implies that there exists a $K \in \mathscr{K}$ such that $(T - K)^n = 0$.

2.10. PROPOSITION (*Generalized Fredholm alternative*). *Let $T \in \mathscr{R}$. Then $(I - T) \in \Phi_0$.*

Proof. By definition $(I - \lambda T) \in \Phi$ for all $\lambda \neq 0$. Since the index map on Φ is locally constant (cf. [4], Lemma 6), $\{I - \lambda T | \lambda \in [0, 1]\}$ is contained in the same connected component of Fredholm operators and the result follows. $\square$

For any subset $B \subseteq \mathscr{A}$ we define the perturbation class of B by

$$P(B) = \{T \in \mathscr{A} | T + S \in B \text{ for all } S \in B\}.$$

In 2.3(b) we have seen that $\mathscr{K}$ is contained in the perturbation class of $\mathscr{R}$. The next proposition shows that one actually has equality:

2.11. PROPOSITION. *The perturbation class of $\mathscr{R}$ is the ideal $\mathscr{K}$ of compact operators.*

Proof. Let $Q(\mathscr{A}/\mathscr{K})$ be the class of quasinilpotent elements of $\mathscr{A}/\mathscr{K}$. From a theorem due to Zemånek (cf. [2], BA2.8) we have:

$$\begin{aligned} \operatorname{rad}(\mathscr{A}/\mathscr{K}) &= \{\pi(T) | T \in \mathscr{A} \text{ and } \pi(T) + Q(\mathscr{A}/\mathscr{K}) \subset Q(\mathscr{A}/\mathscr{K})\} \\ &= \{\pi(T) | T \in \mathscr{A} \text{ and } T + S \in \mathscr{R} \text{ for all } S \in \mathscr{R}\}. \end{aligned}$$

Since $\mathscr{A}/\mathscr{K}$ is a $\mathscr{C}^*$-algebra $\operatorname{rad}(\mathscr{A}/\mathscr{K}) = \{0\}$, it then follows that

$$\begin{aligned} \mathscr{K} &= \pi^{-1}(\operatorname{rad}(\mathscr{A}/\mathscr{K})) \\ &= \{T \in \mathscr{A} | T + S \in \mathscr{R} \text{ for all } S \in \mathscr{R}\} \\ &= P(\mathscr{R}). \end{aligned} \qquad \square$$

2.12. THEOREM. *The ideal $\mathscr{K}$ of compact operators is the largest two-sided ideal consisting of Riesz operators only.*

Proof. We first show that every Riesz projection is finite. Let E be a Riesz projection. Then $\lim_{n\to\infty}(\inf_{K\in\mathscr{K}} \|E-K\|)^{1/n}=0$ and hence $E\in\mathscr{K}$. Since any compact projection is finite we have $E\in\mathscr{F}$. Let $\mathscr{T}$ be any two-sided ideal contained in $\mathscr{R}$. From the first part of the proof it then follows that $\overline{\mathscr{T}}\subset\overline{\mathscr{F}}=\mathscr{K}$. □

In the last two results of this section we show that the class of Riesz operators behaves well under reduction with respect to central projections as well as under decompositions of the von Neumann algebra. These results will be needed later in an important counterexample.

Similar results for the class of compact operators in $\mathscr{A}$ were obtained by Kaftal (cf. [**8**], 2.1, 2.2).

Let E be a central projection in the von Neumann algebra $\mathscr{A}$. We shall then use the following notation: $\mathscr{A}_E:=\mathscr{A}E$ and $\mathscr{R}_E:=\mathscr{R}E$.

2.13. LEMMA. *With the above notation one has that $\mathscr{R}_E=\mathscr{R}(\mathscr{A}_E)$.*

Proof. Let $T\in\mathscr{R}_E$ and $\lambda\neq 0$ be given. There exists an $S\in\mathscr{R}$ such that $T=SE$. Then $S_\lambda:=\lambda I-S$ is invertible modulo $\mathscr{K}$, i.e. there exists an S'_λ such that $S_\lambda S'_\lambda\in I+\mathscr{K}$ and $S'_\lambda S_\lambda\in I+\mathscr{K}$ hence $ES_\lambda ES'_\lambda\in E+\mathscr{K}_E$ and $ES'_\lambda ES_\lambda\in E+\mathscr{K}_E$. We know that $\mathscr{K}(\mathscr{A}_E)=\mathscr{K}_E$ and therefore $\lambda E-T$ $(=ES_\lambda)$ is invertible modulo $\mathscr{K}(\mathscr{A}_E)$. Hence $T\in\mathscr{R}(\mathscr{A}_E)$. Conversely, suppose $T\in\mathscr{R}(\mathscr{A}_E)$ and $\lambda\neq 0$. Then $S_\lambda:=E-1/\lambda T\in\Phi(\mathscr{A}_E)$. Thus there exists $S'_\lambda\in\mathscr{A}_E$ such that

$$S_\lambda S'_\lambda\in E+\mathscr{K}_E \quad\text{and}\quad S'_\lambda S_\lambda\in E+\mathscr{K}_E.$$

Let $A_\lambda=S_\lambda+I-E$ and $B_\lambda=S'_\lambda+I-E$. Then A_λ, $B_\lambda\in\mathscr{A}$ and

$$A_\lambda B_\lambda\in I+\mathscr{K} \quad\text{and}\quad B_\lambda A_\lambda\in I+\mathscr{K}.$$

Thus $I=1/\lambda T=A_\lambda\in\Phi(\mathscr{A})$ for all $\lambda\neq 0$ and therefore $T\in\mathscr{R}$ and since $T=TE$, we have that $T\in\mathscr{R}_E$. □

Let $\mathscr{A}=\sum_{i\in I}^{\oplus}\mathscr{A}_i$ be the direct sum of von Neumann algebras $\mathscr{A}_i$. We may identify the identity of $\mathscr{A}_i$ with a central projection $E_i\in\mathscr{A}$ and $\mathscr{A}_i$ with $\mathscr{A}E_i$. Denote $\{T\in\mathscr{A}: TE_i\in\mathscr{R}(\mathscr{A}_i)\}$ by $\sum_{i\in I}^{\oplus}\mathscr{R}(\mathscr{A}_i)$.

For $L\subseteq I$ we may identify $\sum_{i\in L}^{\oplus}\mathscr{A}_i$ with a closed subalgebra of $\sum_{i\in I}^{\oplus}\mathscr{A}_i$ in an obvious way.

2.14. PROPOSITION. *Let* $\mathscr{A} = \sum_{i\in I}^{\oplus} \mathscr{A}_i$. *Then* $\mathscr{R}(\mathscr{A}) \subseteq \sum_{i\in I}^{\oplus} \mathscr{R}(\mathscr{A}_i)$ *and equality holds if at most finitely many* E_i *are infinite.*

Proof. The inclusion follows directly by application of the previous lemma. Suppose then that E_i is finite for all $i \notin J$, where J is some finite subset $J \subseteq I$. Let $T \in \sum_{i\in I}^{\oplus} \mathscr{R}(\mathscr{A}_i)$ and $\lambda \neq 0$. Then if $T = \sum_{i\in I}^{\oplus} T_i$, $S_{i,\lambda} := E_i - 1/\lambda T_i \in \Phi(\mathscr{A}_i)$ from which it follows that there exist $S'_{i,\lambda}$ and $K_{i,\lambda}$, $K'_{i,\lambda} \in \mathscr{K}(\mathscr{A}_i)$ such that

$$S_{i,\lambda}S'_{i,\lambda} = E_i + K_{i,\lambda} \quad \text{and} \quad S'_{i,\lambda}S_{i,\lambda} = E_i + K'_{i,\lambda} \quad \text{for every } i \in I.$$

For $i \notin J$ we may choose $S'_{i,\lambda} = 0$ and $K'_{i,\lambda} = -E_i$.

Let $S_\lambda = \sum_{i\in I}^{\oplus} S_{i,\lambda}$ and $S'_\lambda = \sum_{i\in I}^{\oplus} S'_{i,\lambda}$. The last series is an element of $\mathscr{A}$ since it actually reduces to a finite sum by our choice of $S'_{i,\lambda}$.

Then clearly, $S_\lambda S'_\lambda \in I + \sum_{i\in I}^{\oplus} \mathscr{K}(\mathscr{A}_i) = I + \mathscr{K}(\mathscr{A})$ (cf. [**8**], 2.2) and similarly $S'_\lambda S_\lambda \in I + \mathscr{K}(\mathscr{A})$. Since $S_\lambda = I - 1/\lambda T$ it follows that $T \in \mathscr{R}(\mathscr{A})$. □

3. Characterizations of Riesz operators relative to a von Neumann algebra. Smyth obtained a geometric characterization for Riesz operators on a general Banach space (cf. [**2**], 0.3.5). In proving this result a somewhat laborious machinery of vector sequence spaces was needed. We shall prove a similar result for general von Neumann algebras which gives an elegant proof of Smyth's result for the $L(H)$-case.

For an operator T in a von Neumann algebra $\mathscr{A}$ the following property (referred to as property A) will be used to characterize the Riesz operators relative to $\mathscr{A}$.

A. For every $\varepsilon > 0$ there exists an $n \in \mathbb{N}$, a finite projection $P_\varepsilon \in \mathscr{P}(\mathscr{A})$ and a bounded set $N_\varepsilon \subseteq P_\varepsilon(H)$ such that for each $x \in U_H$ there exists a $y \in N_\varepsilon$ such that $\|T^n x - y\| < \varepsilon^n$.

(Here and in the following U_H will denote the unit ball of H.)

3.1. LEMMA. *If* $T \in \mathscr{A}$ *has property* A *then* T^m *also has property* A *for all* $m \in \mathbb{N}$.

Proof. Without loss of generality we may assume $T^m \neq 0$. For $\varepsilon > 0$, put $\delta = \varepsilon/\|T^{m-1}\|$. By assumption there exist an $n \in \mathbb{N}$, a finite projection P_δ and a bounded set $N_\delta \subseteq P_\delta(H)$ such that for each $w \in U_H$ there exists a $z \in N_\delta$ such that $\|T^n w - z\| < \delta^n$. Let $N_\varepsilon = \|T^{m-1}\|^n N_\delta$ and $P_\varepsilon = P_\delta$. Then for $x \in U_H$ it follows that $\|T^{mn} x - y\| < \varepsilon^n$ for some $y \in N_\varepsilon$. □

3.2. THEOREM. *Let $T \in \mathscr{A}$. Then $T \in \mathscr{R}$ if and only if T has property* A.

Proof. Let $T \in \mathscr{R}$ and $\varepsilon > 0$. Then since

$$\lim_{n\to\infty}\left(\inf_{F\in\mathscr{F}}\|T^n - F\|\right)^{1/n} = 0$$

there is an $n \in \mathbb{N}$ and an $F_\varepsilon \in \mathscr{F}$ such that

$$\|T^n - F_\varepsilon\| < \varepsilon^n. \tag{1}$$

Let $P_\varepsilon = R_{F_\varepsilon}$ and $N_\varepsilon = F_\varepsilon(U_H)$, then P_ε is a finite projection and N_ε is a bounded subset of $P_\varepsilon(H)$. By (1) $\|T^n x - F_\varepsilon x\| < \varepsilon^n$, for all $x \in U_H$. This proves property A.

Conversely let T have property A. We are going to show that there exists a subsequence of $\{(\inf_{K\in\mathscr{K}}\|T^n - K\|)^{1/n}\}_n$ which converges to zero, implying that the spectral radius of $\pi(T)$ vanishes.

Let $\varepsilon > 0$. Then there exist an $n \in \mathbb{N}$, a finite projection $P_\varepsilon \in \mathscr{P}(\mathscr{A})$ and a bounded set $N_\varepsilon \subseteq P_\varepsilon(H)$ such that for every $x \in U_H$ there exists a $y \in N_\varepsilon$ such that

$$\|T^n x - y\| < \varepsilon^n.$$

Thus $\|T^n x - P_\varepsilon T^n x\| = \inf_{w\in P_\varepsilon(H)}\|T^n x - w\| < \varepsilon^n$. This holds for every $x \in U_H$, hence

$$\|T^n - P_\varepsilon T^n\| \leq \varepsilon^n.$$

Since P_ε is finite and therefore $P_\varepsilon T^n \in \mathscr{K}(\mathscr{A})$, it follows that for any $\varepsilon > 0$ there exists an $n \in \mathbb{N}$ such that $(\inf_{K\in\mathscr{K}}\|T^n - K\|)^{1/n} \leq \varepsilon$.

We now find the zero converging subsequence recursively. There exists an $n_1 \in \mathbb{N}$ such that $(\inf_{K\in\mathscr{K}}\|T^{n_2} - K\|)^{1/n_1} < 1$. Since T^{n_1+1} has property A by 3.1, there exists an $m_1 \in \mathbb{N}$ such that

$$\left(\inf_{K\in\mathscr{K}}\|(T^{n_1+1})^{m_1} - K\|\right)^{1/m_1} < (1/2)^{n_1+1}. \tag{2}$$

Let $n_2 = (n_1 + 1)m_1$. Then clearly $n_1 < n_2$ and from (2) it follows that

$$\left(\inf_{K\in\mathscr{K}}\|T^{n_2} - K\|\right)^{1/n_2} < 1/2.$$

Repeating this argument one finds a monotone increasing sequence of positive integers $n_1 < n_2 < \cdots < n_k < \cdots$ such that

$$\left(\inf_{K\in\mathscr{K}}\|T^{n_k} - K\|\right)^{1/n_k} < 1/k \quad \text{for every } k \in \mathbb{N}. \qquad \square$$

REMARK. It should be noted that in the case where $\mathscr{A} = L(H)$, property A coincides with the notion of a finite ε^n-net for $T^n(U_H)$ (cf. [2], §0.3 for the definition of an ε-net).

From the proof of 3.2 we have:

3.3. COROLLARY. *$T \in \mathscr{R}$ if and only if for every $\varepsilon > 0$ there exist an $n \in \mathbb{N}$ and a projection $Q \in \mathscr{P}(\mathscr{A})$ such that $\|QT^n\| \leq \varepsilon^n$ and $I - Q$ is finite.*

Proof. If $T \in \mathscr{R}$ it has property A. Now if we put $Q = I - P_\varepsilon$ in the converse part of the proof of 3.2 the condition holds. Clearly the condition implies property A and the result follows. □

3.4. COROLLARY. *Let $S, T \in \mathscr{A}$ be commuting. If $T \in \mathscr{R}$ and $S(H) \subseteq T(H)$ then $S \in \mathscr{R}$.*

Proof. Let $\varepsilon > 0$ be given. Under the conditions of the theorem there exists an $\alpha > 0$ such that for any $n \in \mathbb{N}$ one has

$$S^n(U_H) \subseteq \alpha^n \overline{T^n(U_H)} \tag{1}$$

(cf. [2], 0.4.1, 0.4.3).

Since $T \in \mathscr{R}$ there exist an $n \in \mathbb{N}$, a finite projection $P_\varepsilon \in \mathscr{P}(\mathscr{A})$ and a bounded set $N_\varepsilon \subseteq P_\varepsilon(H)$ such that for each $x \in U_H$ there exists a $y \in N_\varepsilon$ with

$$\|T^n x - y\| < (\varepsilon/2\alpha)^n. \tag{2}$$

Let $x \in U_H$, then it follows from (1) that there exists a $z \in U_H$ such that $\|S^n x - \alpha^n T^n z\| < \varepsilon^n/2$.

By(2) there exists a $w \in N_\varepsilon$ such that $\|\alpha^n T^n z - \alpha^n w\| < \varepsilon^n/2^n$. Thus $\|S^n x - \alpha^n w\| < \varepsilon^n$. By noting that the set $\alpha^n N_\varepsilon \subset P_\varepsilon(H)$ is bounded the corollary follows. □

4. Riesz decomposition. In [5] a Riesz type of decomposition was obtained for compact operators in a von Neumann algebra. With our characterization 3.2 and its Corollary 3.3 in hand we can now use the techniques of [3] and [5] to obtain a Riesz type of decomposition for Riesz operators. It should be noted that all proofs are similar to the proofs in [3], [5]. Hence we shall only give attention to the essential differences.

For $T \in \mathscr{A}$ let

$$N_n := N_{(I-T)^n}; \quad F_n := N_{n+1} - N_n, \qquad n = 0, 1, 2, \ldots,$$

$$R_n := R_{(I-T)^n}; \quad G_n := R_n - R_{n+1}, \qquad n = 0, 1, 2, \ldots.$$

Note that (N_n) is non-decreasing (i.e. $N_{n+r}N_n = N_n$ for all $r \in \mathbb{N}$) and (R_n) is non-increasing (i.e. $R_n R_{n+r} = R_{n+r}$ for all $r \in \mathbb{N}$).

The range projection R_T will be called (relatively) cofinite if $I - R_T$ is finite and if there exists a projection $Q \in \mathscr{A}$ such that $Q(H) \subseteq T(H)$ with $R_T - Q$ finite. In $L(H)$ this coincides with the classical definition of cofiniteness.

4.1. LEMMA. *With the above notation*
(a) $N_{n+r}T^k N_n = T^k N_n$,
(b) $F_n T^k F_n = F_n$,
(c) $R_n T^k R_{n+r} = T^k R_{n+r}$,
(d) $G_n T^k G_n = G_n$,
for $n = 0, 1, 2, \ldots;\ r = 0, 1, \ldots;\ k = 1, 2, \ldots$.

Proof. (a) and (b) follow by induction (on k) and by using the relation $(I - N_n)TN_{n+1} = F_n$ which follows from the properties of the projection N_n (cf. [3]). Similarly (c) and (d) follow by using the relation $(I - R_{n+1})TR_n = G_n$ which follows from the properties of the range projection. □

4.2. THEOREM. *Let* $T \in \mathscr{R}$. *Then the following hold*:
(a) N_n *is relatively finite and* R_n *relatively cofinite.*
(b) *If* $N_\infty = \sup_{n\in\mathbb{N}} N_n$ *and* $R_\infty = \inf_{n\in\mathbb{N}} R_n$ *then both* $N_\infty(H)$ *and* $R_\infty(H)$ *are invariant under* T^k *for any* $k \in \mathbb{N}$.
(c) N_∞ *is relatively finite and* $N_\infty \sim I - R_\infty$.
(d) $\inf\{N_\infty, R_\infty\} = 0$ *and* $\sup\{N_\infty, R_\infty\} = I$.

Proof. (a) Clearly $(I - T) \in \Phi$. By ([7], 2.2) N_1 is relatively finite and R_1 is relatively cofinite. For $n \in \mathbb{N}$, $n > 1$ it follows from 2.1 that $(I - T)^n = I - T_0$ where $T_0 \in \mathscr{R}$ and as before it follows that N_n is relatively finite and R_n relatively cofinite.

(b) This follows from 4.1(a) and (c) for $r = 0$ and taking the strong operator limit on both sides.

(c) By using 3.3 and 4.1 the proof for the relative finiteness of N_∞ for the compact case may be carried over virtually word for word by only replacing T with T^n. From 2.10 and the fact that Φ_0 is a semigroup, it follows that $N_n \sim I - R_n$ and hence $N_\infty \sim I - R_\infty$ follows similarly as for the compact case, cf. [3], Theorem 2, (i).

(d) This again follows along the lines of [3], Theorem 2 (iv) and [5], Theorem 3 (ii) by only noting that $(I - T)^k = I - T_{(k)}$ where $T_{(k)} \in \mathscr{R}$. □

It is well-known that both the sequences (N_n) and (R_n) eventually become stationary in the classical case. The following example shows that this is not always the case in general von Neumann algebras.

EXAMPLE. Let

$$\mathscr{A} = \sum_{n=1}^{\infty}{}^{\oplus} L(H_n)$$

where $H_n = H$ is a separable Hilbert space. Let $T_k \in L(H)$ be defined by

$$T_k\left(\sum_{i=1}^{\infty} x_i\phi_i\right) = x_1\phi_1 + \sum_{i=2}^{k+1}(x_i - x_{i-1})\phi_i,$$

where $\{\phi_i | i \in \mathbb{N}\}$ is any orthonormal basis for H. It is easy to see that

$$N(I - T_k) \neq N(I - T_k)^2 \neq \cdots \neq N(I - T_k)^{k+1} = N(I - T_k)^{k+r}$$

for all $k, r \in \mathbb{N}$. Let

$$\bar{I} = \sum_{n=1}^{\infty}{}^{\oplus} I_n \quad \text{where } I_n = I \text{ for all } n \in \mathbb{N} \text{ and } T := \sum_{n=1}^{\infty}{}^{\oplus} T_n.$$

Then T is compact and hence Riesz relative to $\mathscr{A}$. However

$$N(\bar{I} - T)^k \subsetneqq N(\bar{I} - T)^{k+r} \quad \text{for all } k, r \in \mathbb{N}.$$

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Received October 15, 1988.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 148, No. 1, 1991

THE CLASSIFICATION OF FLAT COMPACT COMPLETE SPACE-FORMS WITH METRIC OF SIGNATURE (2, 2)

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Those flat compact complete space-forms with metric of signature (2,2) are classified up to finite covers. The simply transitive subgroups of $R^4 \rtimes \mathrm{SO}(2, 2)$ are classified up to conjugation.

1. Introduction.

(1.1) If $\Gamma \subseteq R^4 \rtimes \mathrm{SO}(2, 2)$ and Γ acts on R^{p+q} freely and properly discontinuously with compact quotient, then $X = R^{p+q}/\Gamma$ is a flat compact complete space-form with metric of signature (p, q). Recently D. Fried **[3]** has classified those flat compact complete space-forms with metric of signature (1,3) upto finite covers. Ravi S. Kulkarni pointed out that Fried's method can be applied to the case $(p, q) = (2, 2)$. The basic idea of Fried's method is in the following theorem:

(1.2) THEOREM. *Suppose X is a flat compact complete space-form with fundamental group $\Gamma \subseteq R^4 \rtimes \mathrm{SO}(2, 2)$. Then there is a uniquely determined subgroup H of $R^4 \rtimes \mathrm{SO}(2, 2)$ that acts simply transitively on R^4 and $H \cap \Gamma = \pi$ has finite index in Γ.*

(1.3) In §2 we classify those subgroups of $R^4 \rtimes \mathrm{SO}(2, 2)$ that act on R^4 simply transitively, up to the conjugacy of $R^4 \rtimes \mathrm{O}(2, 2)$. Every such subgroup, as a Lie group, is isomorphic to one of the following:

$$R^4, \quad R \times \mathrm{Nil}^3, \quad \mathrm{Nil}^4, \quad R \times \left\{ R^2 \rtimes \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} ; t \in R \right\},$$
$$R \times \{R^2 \rtimes \mathrm{SO}(2)\}.$$

All of them, except the last one, correspond to Γ's. Their uniform lattices are known, cf. **[3]** and **[7]**.

(1.4) To prove Theorem (1.2), we first prove in §3 that Γ is virtually solvable. This result confirms a conjecture by Milnor in a special case. In **[6]**, it is conjectured that the fundamental group of a complete affinely flat manifold is virtually polycyclic. Our result, combined with

Fried's result, shows that this conjecture is true for compact pseudo-Riemannian 4-manifolds.

(1.5) In §4 we complete the proof of Theorem 1.2, using the theory of crystallographic hull developed by Fried and Goldman, cf. [**4**]. In §5, we give our classification. By comparing our list with Fried's, we obtain an interesting fact: as differential manifolds, they are the same coset spaces of the form H/Γ, where H is a Lie group isomorphic to R^4, $R \times \mathrm{Nil}^3$, Nil^4 or $R \times \{R^2 \rtimes \left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right); t \in R\}$ and Γ is a uniform lattice of H. These Lie groups have simply transitive representations as affine motions and when the signature is (2,2) (resp. (3,1)), the images of the representations are $R^4 \rtimes \mathrm{SO}(2,2)$ (resp. $R^4 \rtimes \mathrm{SO}(3,1)$).

(1.6) *Notations and some properties of* $\mathrm{SO}(2,2)$ *and* $\mathrm{so}(2,2)$. Throughout this paper we will call $\{e_i\}$, $1 \le i \le 4$, a standard basis s.t. the metric Q, w.r.t. this basis, has the form

$$Q(v, v) = v_1 v_3 + v_2 v_4,$$

where $v = \sum_{i=1}^4 v_i e_i$. The full group of orientation-preserving isometries is $R^4 \rtimes \mathrm{SO}(2,2)$ and

$$\mathrm{SO}(2,2) = \left\{ g \in \mathrm{SL}_4(R);\, {}^t g \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right\},$$

where $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The infinitesimal isometries are $R^4 \rtimes \mathrm{so}(2,2)$ and

(1.6.1)
$$\begin{aligned} \mathrm{so}(2,2) &= \left\{ X \in \mathrm{gl}_4(R);\, {}^t X \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} X = 0 \right\} \\ &= \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & d \\ a_{21} & a_{22} & -d & 0 \\ 0 & c & -a_{11} & -a_{21} \\ -c & 0 & -a_{12} & -a_{22} \end{pmatrix};\, a_{ij}, d, c \in R \right\}. \end{aligned}$$

(1.6.2) $\mathrm{so}(2,2) = L_1 \oplus L_2$, where $L_i \simeq \mathrm{sl}_2(R)$, $i = 1, 2$; and

$$L_1 = \left\{ \begin{pmatrix} a & b & & \\ c & -a & & \\ & & -a & -c \\ & & -b & a \end{pmatrix};\, a, b, c \in R \right\},$$

$$L_2 = \left\{ \begin{pmatrix} a' & 0 & 0 & d' \\ 0 & a' & -d' & 0 \\ 0 & c' & -a' & 0 \\ -c' & 0 & 0 & -a' \end{pmatrix};\, a', d', c' \in R \right\}.$$

L_1, L_2 are permuted by an element of $\mathrm{O}(2,2)$.

(1.6.3) It is easy to show that any Cartan subalgebra of so$(2, 2)$ is conjugate under O$(2, 2)$ to one of the following:

(1)
$$\left\{\begin{pmatrix} a & & & \\ & b & & \\ & & -a & \\ & & & -b \end{pmatrix} ; a, b, \in R\right\},$$

(2)
$$\left\{\begin{pmatrix} 0 & a & 0 & b \\ -a & 0 & -b & 0 \\ 0 & b & 0 & a \\ -b & 0 & -a & 0 \end{pmatrix} ; a, b, \in R\right\},$$

(3)
$$\left\{\begin{pmatrix} a & b & & \\ -b & a & & \\ & & -a & b \\ & & -b & -a \end{pmatrix} ; a, b, \in R\right\}.$$

An immediate corollary is

(1.6.4) *If* X *is in a Cartan subalgebra of* so$(2, 2)$ *and* $\det X = 0$, *then* X *must conjugate under* O$(2, 2)$ *to*

(4)
$$\left\{\begin{pmatrix} a & & & \\ & 0 & & \\ & & -a & \\ & & & 0 \end{pmatrix}\right\},$$

or

(5)
$$\left\{\begin{pmatrix} 0 & a & 0 & a \\ -a & 0 & -a & 0 \\ 0 & a & 0 & a \\ -a & 0 & -a & 0 \end{pmatrix}\right\}.$$

(1.7) We identify Aff(n), resp. aff(n), with

$$\left\{\begin{pmatrix} A & v \\ 0 & a \end{pmatrix} ; A \in \mathrm{GL}_4(R), v \in R^4\right\},$$

resp.

$$\left\{\begin{pmatrix} X & v \\ 0 & 0 \end{pmatrix} ; X \in \mathrm{gl}_4(R), v \in R^4\right\},$$

w.r.t. a given basis. Let P_l be the natural homomorphism taking an affine transformation (or an infinitesimal affine transformation) to its linear part. Let $L(G)$ be the Lie algebra of a Lie group G and $A(G)$ be the algebraic hull of G. We will need the following well-known lemma.

(1.7.1) LEMMA. *If $G \subseteq \mathrm{Aff}(n)$ s.t. G acts freely on R^n, then every $A \in P_l(G)$ has 1 as an eigenvalue.*

(1.7.2) LEMMA (*Kostant and Sullivan, cf.* [**5**]). *If G is as in* (1.7.1), *then every $A \in P_l(A(G))$ has 1 as an eigenvalue.*

(1.7.3) COROLLARY. *If G is an in (1.7.1), then every $X \in P_l(L(A(G))$ or $X \in L(A(P_l(G))$ has 0 as an eigenvalue.*

2. Simply transitive subgroups. We will classify subgroups of $R^4 \rtimes \mathrm{SO}(2, 2)$ that act simply transitively on R^4. Our classification is up to the conjugation under $R^4 \rtimes \mathrm{O}(2, 2)$. It is well known that a simply transitive group of affine motions must be solvable, connected, simply connected and of dimension 4, cf. [**1**]. We will start from a special case when the groups are unipotent. The following lemma from Auslander and Scheuneman plays the key role in this section.

(2.1) LEMMA. *Let U be a nilpotent Lie group which has a faithful representation $\rho\colon U \to \mathrm{Aff}(n)$, let ρ_* be the induced monomorphism of Lie algebras*

$$\rho_* L(U) \to \left\{ \begin{pmatrix} X & v \\ 0 & 0 \end{pmatrix} ; X \in \mathrm{gl}_n(R), v \in R^n \right\} = \mathrm{aff}(n),$$

and let P_l be as in (1.7), *let P_t be the projection from an element in* $\mathrm{aff}(n)$ *to its translation part. Then $\rho(U)$ acts on R^n simply transitively if and only if*

(1) *$P_l \circ \rho_*(L(U))$ is nilpotent, and*
(2) *$P_t \circ \rho_*(L(U))$ is a linear isomorphism of $L(U)$ onto R^n.*

For a proof, cf. [**1**]. So unipotent simply transitive subgroups are exactly the following U's s.t.

$$L(U) = \left\{ \begin{pmatrix} X(v) & v \\ 0 & 0 \end{pmatrix} ; v \in R^n \right\}, \tag{2.2}$$

where $X(v)$ is a linear function of v and $P_l(L(U)) = \{X(v) ; v \in R^n\}$ is nilpotent.

(2.3) LEMMA. *There is a vector $v_0 \in R^4$ such that*

(i) $P_l(L(U))(v_0) = 0$,
(ii) $Q(v_0, v_0) = 0$.

Proof. If $V = \{v \in R^4 ; P_l(L(U))v = 0\}$, then $V^\perp$ is invariant. By Engel's Theorem on $V^\perp$, $V^\perp$ meets V. □

Let $\{e_i\}$ be our standard basis. Then we choose $v_0 = e_1$ since $O(2, 2)$ is transitive on $\{v ; Q(v, v) = 0\}/v \sim tv$, where $t \in R - \{0\}$.

(2.4) COROLLARY. *W.r.t. the above standard basis, $X \in P_l(L(U))$ has the form*

$$X(v) = \begin{pmatrix} 0 & a & 0 & b \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 \end{pmatrix},$$

where $a = a(v)$ and $b = b(v)$ are linear functions of v.

To find $a(v)$ and $b(v)$, we compute the commutator of $L(U)$.

$$\left[\begin{pmatrix} X(v) & v \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} X(v') & v' \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} X(v'') & v'' \\ 0 & 0 \end{pmatrix}, \tag{2.5}$$

where $v'' = X(v)v' - X(v')v$, $X(v'') = X(v)X(v') - X(v')X(v) = 0$. So

$$a(v'') = b(v'') \equiv 0.$$

Write

$$a(v) = \sum_{i=1}^{4} a_i v_i, \qquad b(v) = \sum_{i=1}^{4} b_i v_i. \tag{2.6}$$

Then we have

$$0 = \sum_{i=1}^{4} a_i v_i'', \qquad 0 = \sum_{i=1}^{4} b_i v_i'', \tag{2.7}$$

where v_i'''s are linear functions of a_i, b_i and $v_i v_j'$, $1 \leq i, j \leq 4$, and all coefficients of $v_i v_j'$ must be zero. We obtain

(2.8) LEMMA.
(i) $a_1 = b_1 = 0$,
(ii) $a_2 b_4 + a_4^2 = 0$,
(iii) $a_2 b_2 + a_4 a_2 = 0$,
(iv) $b_2 b_4 + b_4 a_4 = 0$,
(v) $b_2^2 + b_4 a_2 = 0$.

(2.9) COROLLARY.

(i) $b_4(b_2+a_4)=0$,
(ii) $a_2(b_2+a_4)=0$,
(iii) $(b_2-a_4)(b_2+a_4)=0$.

(2.10) Now we can get some necessary conditions for the nontranslation unipotent simply transitive subgroups. If $b_2+a_4\neq 0$, then $b_4=a_2=0$. By (2.8) (ii) and (v), $b_2^2=a_4^2=0$ and we get a contradiction. So $b_2+a_4=0$, and we have three subcases:

(2.10.1) $b_2=a_4=b_4=a_2=0$, but $(a_3, b_3)\neq(0, 0)$, i.e.,

$$\begin{cases} a(v)=a_3v_3 \\ b(v)=b_3v_3. \end{cases}$$

(2.10.2) $b_2+a_4=0$ but $b_2\neq 0$, $a_4\neq 0$. Then by (2.8) $b_4\neq 0$, $a_2\neq 0$, i.e.

$$\begin{cases} a(v)=a_2v_2+a_3v_3+a_4v_4 \\ b(v)=b_2v_2+b_3v_3+b_4v_4. \end{cases}$$

(2.10.3) $b_2=0$, $a_4=0$, $(a_2, b_4)\neq(0, 0)$. By (2.8), $b_4a_2=0$, so

$$\begin{cases} a(v)=a_2v_2+a_3v_3 \\ b(v)=b_3v_3, \end{cases}$$

or

$$\begin{cases} a(v)=a_3v_3 \\ b(v)=b_3v_3+b_4v_4. \end{cases}$$

(2.11) THEOREM. *Up to conjugacy under $R^4\rtimes O(2,2)$, the nontranslation unipotent simply transitive groups U of $R^4\rtimes SO(2,2)$, have the following Lie algebras*:

$$L(U)=\left\{\begin{pmatrix} X(v) & v \\ 0 & 0 \end{pmatrix} ; v\in R^4\right\},$$

where

$$X(v)=\begin{pmatrix} 0 & a(v) & 0 & b(v) \\ 0 & 0 & -b(v) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a(v) & 0 \end{pmatrix},$$

$a(v)$ and $b(v)$ are listed in the following table:

Type of $L(U)$	$a(v)$	$b(v)$	isomorphism type as an abstract Lie algebra
I-1	v_3	v_3	$N_3 \oplus R$
I-2	v_3	$-v_3$	$N_3 \oplus R$
I-3	v_3	0	$N_3 \oplus R$
II-1	$v_2 + v_4 + tv_3$, $(t \geq 0)$	$-v_2 - v_4$	N_4
II-2	$-v_2 + v_4 + tv_3$, $(t \geq 0)$	$-v_2 + v_4$	N_4
II-3	v_2	v_3	N_4

The equivalence classes are uniquely determined by the type of $L(U)$ and the parameter t (in Type II).

Proof. The discussion of the conjugacy under $R^4 \rtimes \mathrm{O}(2, 2)$ is long and tedious. We will only write down a brief one for subcase (2.10.2). We give the following lemma without proof.

(2.11.1) LEMMA. *If $a(v) \not\equiv 0$, $b(v) \not\equiv 0$, $a'(v') \not\equiv 0$, $b'(v') \not\equiv 0$, and if there is a matrix $A = (a_{ij}) \in \mathrm{O}(2, 2)$ such that*

$$A^{-1}\begin{pmatrix} 0 & a(v) & 0 & b(v) \\ 0 & 0 & -b(v) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a(v) & 0 \end{pmatrix} A = \begin{pmatrix} 0 & a'(v') & 0 & b'(v') \\ 0 & 0 & -b'(v') & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a'(v') & 0 \end{pmatrix},$$

then either

$$(1) \quad \begin{cases} a'(v') = \dfrac{a_2 a_{22}^2}{a_{11}} v_2' + \left\{ \dfrac{a_{22}a_{23}}{a_{11}} a_2 + \dfrac{a_{22}}{a_{11}^2} a_3 + \dfrac{a_{22}a_{43}}{a_{11}} a_4 \right\} v_3' \\ \qquad + \dfrac{a_4}{a_{11}} v_4' \\ b'(v') = \dfrac{b_2}{a_{11}} v_2' + \left\{ \dfrac{a_{23}}{a_{11}a_{22}} b_2 + \dfrac{1}{a_{11}^2 a_{22}} b_3 + \dfrac{a_{43}}{a_{11}a_{22}} b_4 \right\} v_3' \\ \qquad + \dfrac{b_4}{a_{11}a_{22}^2} v_4' \end{cases}$$

where $a_{11}a_{22} \neq 0$; *or*

$$(2)\quad \begin{cases} a'(v') = \dfrac{b_4 a_{42}^2}{a_{11}} v_2' + \left\{ \dfrac{a_{42}a_{23}}{a_{11}} b_2 + \dfrac{a_{42}}{a_{11}^2} b_3 + \dfrac{a_{42}a_{43}}{a_{11}} b_4 \right\} v_3' \\ \qquad + \dfrac{b_2}{a_{11}} v_4' \\ b'(v') = \dfrac{a_4}{a_{11}} v_2' + \left\{ \dfrac{a_{23}}{a_{11}a_{42}} a_2 + \dfrac{1}{a_{11}^2 a_{42}} a_3 + \dfrac{a_{43}}{a_{11}a_{42}} a_4 \right\} v_3' \\ \qquad + \dfrac{a_4}{a_{11}a_{41}^2} v_4' \end{cases}$$

where $a_{11}a_{42} \neq 0$.

Write $a'(v') = \sum_{i=2}^4 a_i' v_i'$ and $b'(v') = \sum_{i=2}^4 b_i' v_i'$, then from (2.11.1)

$$a_2' b_4' = a_4' b_2' = \frac{a_4 b_2}{a_{11}^2} = -\frac{a_4^2}{a_{11}^2} < 0,$$

since $a_4 = -b_2 \neq 0$. So we can choose a_{11} such that $a_2'b_4' = a_4'b_2' = -1$, i.e. $a_4/a_{11} = \pm 1$. Next we use (1) (resp. (2)) if $a_4/a_{11} = 1$ (resp. -1), and choose a_{22} (resp. a_{42}) to reduce

$$\begin{pmatrix} a_2' & a_4' \\ b_2' & b_4' \end{pmatrix}$$

to

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{if } a_2a_4 > 0,$$

or

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{if } a_2a_4 < 0.$$

Now a_3', b_3' have the form

$$\begin{cases} a_3' = z_1 + \dfrac{a_{22}}{a_{11}^2} a_3 \\ b_3' = \pm z_1 + \dfrac{1}{a_{11}^2 a_{22}} b_3, \end{cases}, \quad \text{or} \quad \begin{cases} a_3' = z_2 + \dfrac{a_{42}}{a_{11}^2} b_3 \\ b_3' = \pm z_2 + \dfrac{1}{a_{11}^2 a_{42}} a_3, \end{cases},$$

where z_1 (resp. z_2) depends on a_{23}, a_{43} (resp. a_{23}, a_{43}) and z_i, $i = 1, 2$ can assume any real number. We can choose z_i so that $b_3' = 0$ and we can choose the sign of a_{22} (resp. a_{42}) so that $a_3' \geq 0$. So we can find an $A \in \mathrm{O}(2, 2)$ such that

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X & v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

is of Type II-1 or Type II-2. We can replace $\left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$ by $\left(\begin{smallmatrix} A & w \\ 0 & 1 \end{smallmatrix}\right)$ and show that the translation part doesn't contribute to the classification.

We omit the rest of the proof. □

(2.12) To handle the general case, namely when the simply transitive group of affine motion is non-unipotent solvable, we need the following lemma from Auslander, cf. **[1]**.

(2.12.1) Lemma. *Let H be an n-dimensional, connected, simply connected, solvable Lie group acting simply transitively as affine motions on R^n. Let $A(H)$ be the algebraic hull of H and let U be the unipotent radical of $A(H)$. Then U operates simply transitively as affine motions on R^n.*

Now all such nontranslation U's are known from (2.11), and we'll study them first.

(2.12.2) Lemma. *Let H, U be as in* (2.12.1) *and assume that U is not the translation group T. Then $H = U$.*

Proof. W.r.t. the standard basis $\{e_i\}$, $1 \le i \le 4$, we know

$$L(P_l(U)) = \left\{ \begin{pmatrix} 0 & a(v) & 0 & b(v) \\ 0 & 0 & -b(v) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a(v) & 0 \end{pmatrix} ; v \in R^4 \right\}.$$

Notice that $A(H)$ is contained in the normalizer of U, we have

$$[L(A(H)), L(U)] \subseteq L(U), \qquad [L(P_l(A(H))), L(P_l(U))] \subseteq L(P_l(U)).$$

Since for

$$Y = \begin{pmatrix} a_{11} & a_{12} & 0 & d \\ a_{21} & a_{22} & -d & 0 \\ 0 & c & -a_{11} & -a_{21} \\ -c & 0 & -a_{12} & -a_{22} \end{pmatrix}, \quad X = \begin{pmatrix} 0 & a(v) & 0 & b(v) \\ 0 & 0 & -b(v) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a(v) & 0 \end{pmatrix},$$

we have

$$[Y, X] = \begin{pmatrix} -a(v)a_{21} + b(v)c & a(v)(a_{11} - a_{22}) & 0 & b(v)(a_{11} + a_{22}) \\ 0 & a(v)a_{21} + b(v)c & -b(v)(a_{11} + a_{22}) & 0 \\ 0 & 0 & a(v)a_{21} - b(v)c & 0 \\ 0 & 0 & -a(v)(a_{11} - a_{22}) & -a(v)a_{21} - b(v)c \end{pmatrix}.$$

So

$$\begin{cases} -a(v)a_{21} + b(v)c = 0, \\ a(v)a_{21} + b(v)c = 0, \end{cases}$$

i.e.

$$\begin{cases} a(v)a_{21} = 0 \\ b(v)c = 0 \end{cases}$$

for any a_{21}, c, $a(v)$ and $b(v)$, $v \in R^4$.

By (2.11), we can always find a v so that $a(v) \neq 0$, so we must have $a_{21} = 0$. Similarly $c = 0$, unless $b(v) \equiv 0$. So we have two cases.

Case 1. Type of $L(U)$ is I-1, I-2 or II.
$L(P_l(A(H)))$ is contained in

$$\left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & d \\ 0 & a_{22} & -d & 0 \\ 0 & 0 & -a_{11} & 0 \\ 0 & 0 & -a_{12} & -a_{22} \end{pmatrix} ; a_{11}, a_{12}, a_{22}, d \in R \right\}.$$

Case 2. Type of $L(U)$ is I-3.
$L(P_l(A(H)))$ is contained in

$$\left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & \\ 0 & a_{22} & 0 & 0 \\ 0 & c & -a_{11} & 0 \\ -c & 0 & -a_{12} & -a_{22} \end{pmatrix} ; a_{11}, a_{12}, a_{22}, c \in R \right\}.$$

It's easy to show that matrices in Case 1 and Case 2 are conjugate under O(2, 2). We will only write down a proof for Case 1; a proof for Case 2 can be obtained similarly.

Again let $Y \in L(P_l(A(H)))$. Then by (1.7.3) $\det Y = 0$, so $a_{11}a_{22} = 0$, i.e. $a_{11} = 0$ or $a_{22} = 0$.

If $a_{22} = 0$, then an element in $L(A(H))$ has the form

$$\begin{pmatrix} Y & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 & d(v) & v_1 \\ 0 & 0 & -d & 0 & v_2 \\ 0 & 0 & -a_{11} & 0 & v_3 \\ 0 & 0 & -a_{12} & 0 & v_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{for some } v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}.$$

By subtracting an element $\begin{pmatrix} X(v) & v \\ 0 & 0 \end{pmatrix} \in L(U)$, we have

$$\begin{pmatrix} Y - X(v) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} - a(v) & 0 & d - b(v) & 0 \\ 0 & 0 & -d + b(v) & 0 & 0 \\ 0 & 0 & -a_{11} & 0 & 0 \\ 0 & 0 & -a_{12} + a(v) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in L(A(H)).$$

For any $\begin{pmatrix} X(v') & v' \\ 0 & 0 \end{pmatrix} \in L(U)$, we have

$$\left[\begin{pmatrix} Y - X(v) & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} X(v') & v' \\ 070 & \end{pmatrix} \right]$$
$$= \begin{pmatrix} 0 & a_{11}a(v') & 0 & a_{11}b(v') & a_{11}v_1' + (a_{12} - a(v))v_2' + (d - b(v))v_4' \\ 0 & 0 & -a_{11}b(v') & 0 & -(d - b(v))v_3' \\ 0 & 0 & 0 & 0 & -a_{11}v_3' \\ 0 & 0 & -a_{11}a(v') & 0 & -(a_{12} - a(v))v_3' \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in L(U).$$

But we know that

$$\begin{pmatrix} 0 & a_{11}a(v') & 0 & a_{11}b(v') & a_{11}v_1' \\ 0 & 0 & -a_{11}b(v') & 0 & a_{11}v_2' \\ 0 & 0 & 0 & 0 & a_{11}v_3' \\ 0 & 0 & -a_{11}a(v') & 0 & a_{11}v_4' \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in L(U).$$

So we have

(1) $a_{11}v_1' = a_{11}v_1' + (a_{12} - a(v))v_2' + (d - b(v))v_4'$;
(2) $a_{11}v_2' = -(d - b(v))v_3'$;
(3) $a_{11}v_3' = -a_{11}v_3'$;
(4) $a_{11}v_4' = -(a_{12} - a(v))v_3'$.

From (3) we get $a_{11} = 0$. Then (2), resp. (4), implies $d = b(v)$, resp. $a_{12} = a(v)$, i.e. $Y = X(v)$. So $\begin{pmatrix} Y & v \\ 0 & 0 \end{pmatrix} \in L(U)$.

If $a_{11} = 0$, let $\begin{pmatrix} Y & v \\ 0 & 0 \end{pmatrix} \in L(A(H))$. By subtracting an element $\begin{pmatrix} X(v) & v \\ 0 & 0 \end{pmatrix} \in L(U)$, we have

$$\begin{pmatrix} Y - X(v) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a - 12 - a(v) & 0 & d - b(v) & 0 \\ 0 & a_{22} & -(d - b(v)) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(a_{12} - a(v)) & -a_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in L(A(H)).$$

Then for any $\left(\begin{smallmatrix} X(v') & v' \\ 0 & 0 \end{smallmatrix}\right) \in L(U)$, we have

$$\left[\begin{pmatrix} Y - X(v) & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} X(v') & v' \\ 0 & 0 \end{pmatrix}\right]$$
$$= \begin{pmatrix} 0 & -a_{22}a(v') & 0 & a_{22}b(v') & (a_{12} - a(v))v_2' + (d - b(v))v_4' \\ 0 & 0 & -a_{22}b(v') & 0 & a_{22}v_2' - (d - b(v))v_3' \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{22}a(v') & 0 & -(a_{12} - a(v))v_3' - a_{22}v_4' \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in L(U).$$

Let $a(v) = \sum a_i v_i$, $b(v) = \sum b_i v_i$, $2 \le i \le 4$ as in (2.6) $(a_1 = b_1 = 0)$ and let

$$v'' = \begin{pmatrix} (a_{12} - a(v))v_2' + (d - b(v))v_4' \\ a_{22}v_2' - (d - b(v))v_3' \\ 0 \\ -(a_{12} - a(v))v_3' - a_{22}v_4' \end{pmatrix}.$$

Then

$$\begin{cases} -a_{22}a(v') = a(v'') = a_2(a_{22}v_2' - (d - b(v))v_3') \\ \qquad\qquad + a_4(-(a_{12} - a(v))v_3' - a_{22}v_4'), \\ a_{22}b(v') = b(v'') = b_2(a_{22}v_2' - (d - b(v))v_3') \\ \qquad\qquad + b_4(-(a_{12} - a(v))v_3' - a_{22}v_4'), \end{cases}$$

i.e.

$$\begin{cases} -a_{22}(a_2v_2' + a_3v_3' + a_4 + v_4') = a_2(a_{22}v_2' - (d - b(v))v_3') \\ \qquad\qquad + a_4(-(a_{12} - a(v))v_3' - a_{22}v_4'), \\ a_{22}(b_2v_2' + b_3v_3' + b_4v_4') = b_2(a_{22}v_2' - (d - b(v))v_3') \\ \qquad\qquad + b_4(-(a_{12} - a(v))v_3' - a_{22}v_4'), \end{cases}$$

i.e.

$$\begin{cases} 2a_{22}a_2v_2' + (a_3a_{22} - a_4(a_{12} - a(v)) - a_2(d - b(v)))v_3' = 0, \\ 2a_{22}b_4v_4' + (b_3a_{22} + b_4(a_{12} - a(v)) + b_2(d - b(v)))v_3' = 0. \end{cases}$$

Letting v_i''s vary, we have

(1) $a_{22}a_2 = 0$;
(2) $a_{22}b_4 = 0$;
(3) $a_3a_{22} - a_4(a_{12} - a(v)) - a_2(d - b(v)) = 0$;
(4) $b_3a_{22} + b_4(a_{12} - a(v)) + b_2(d - b(v)) = 0$.

If $a_{22} \neq 0$, we must have $a_2 = b_4 = 0$ by (1) and (2). According to (2.10), this implies $b_2 = a_4 = b_4 = a_2 = 0$. Then (3) and (4) lead to

$$\begin{cases} a_3 a_{22} = 0, \\ b_3 a_{22} = 0, \end{cases}$$

i.e. $a_3 = b_3 = 0$. So $U = T$, and we have a contradiction. So $a_{22} = 0$. We always have $a_{11} = a_{22} = 0$, i.e. $A(H)$ is unipotent; so H is unipotent. But any unipotent connected Lie group is Zariski closed, so $H = A(H)$. U, as the unipotent radical of H must be H itself. □

(2.12.3) Now consider the case when the unipotent radical $A(H)$ is precisely the group T of translations of R^4. Suppose $H \neq T$, i.e. H is not unipotent.

(2.12.3.1) LEMMA. *$P_l(H)$ is abelian.*

Proof. $P_l(H) \simeq H/\operatorname{Ker}(P_l|_H)) = H/(H \cap T) \subseteq A(H)/T$, but $A(H)/T$ is abelian (cf. [2], $A(H)/U(H)$ is abelian, since $A(H)$ is solvable and algebraic). □

(2.12.3.2) LEMMA. $\dim P_l(H) = 1$; *$P_l(H)$ is diagnolizable in* $\mathbf{C}$.

Proof. $P_l(H)$ is a connected abelian subgroup of $\mathrm{SO}_0(2,2)$, so $\dim P_l(H) \leq 2$. By (1.7.3) $\det X = 0$ for every $X \in L(P_l(H))$, i.e. 0 is an eigenvalue of X. Since $X \in \mathrm{so}(2,2)$, so

$$X = \begin{pmatrix} a_{11} & a_{12} & 0 & d \\ a_{21} & a_{22} & -d & 0 \\ 0 & c & -a_{11} & -a_{21} \\ -c & 0 & -a_{12} & -a_{22} \end{pmatrix},$$

and

$$\begin{aligned} \det(X - \lambda I) &= \lambda^4 + (2dc - 2a_{12}a_{21} - a_{11}^2 - a_{22}^2)\lambda^2 \\ &\quad + (-a_{11}a_{22} + a_{12}a_{21} + dc)^2 \\ &= \lambda^4 + \{-4a_{12}a_{21} - (a_{11} - a_{12})^2\}\lambda^2, \end{aligned}$$

since 0 is an eigenvalue. So the eigenvalues of X are $\{0, 0, 0, 0\}$ or $\{0, 0, \lambda, -\lambda\}$, $\lambda \neq 0$, $\lambda \in R$ or $\sqrt{-1}R$. If $\dim P_l(H) = 2$, then by (1.6.2) $\mathrm{so}(2,2) = L_1 \oplus L_2$, $L_i \simeq \mathrm{sl}_2(R)$. So $L(P_l(H)) = RX_1 + RX_2$ where $X_i \in L_i$, $i = 1, 2$. But by (1.6.2)

$\det(X_1 - \lambda I) = \lambda^4 + 2(a^2 + bc)\lambda^2 + (a^2 + bc)^2$, and

$$\det(X_2 - \lambda I) = \lambda^4 + 2(b'c' - a'^2)\lambda^2 + (b'c' - a'^2)^2.$$

So zero is an eigenvalue of X_i, $i = 1, 2$, if and only if all the eigenvalues of X_i are zero. This means $P_l(H)$ is unipotent and leads to a contradiction. So $\dim P_l(H) = 1$, $L(P_l(H)) = RX$ and X has eigenvalues $\{0, 0, \lambda, -\lambda\}$, $\lambda \neq 0$, $\lambda \in R$ or $\sqrt{-1}R$. Since X is an infinitesimal isometry, it is diagnolizable. □

(2.12.3.3) Corollary. *$L(P_l(H))$ is contained in a Cartan subalgebra of* $\mathrm{so}(2, 2)$ *and is conjugate under* $\mathrm{O}(2, 2)$ *to*

(1)
$$\begin{pmatrix} a & & & 0 \\ & 0 & & \\ & & -a & \\ 0 & & & 0 \end{pmatrix}$$

(2)
$$\begin{pmatrix} 0 & a & 0 & a \\ -a & 0 & -a & 0 \\ 0 & a & 0 & a \\ -a & 0 & -a & 0 \end{pmatrix}.$$

Proof. By (1.6.4). □

Since H is simply transitive, the map $P_t \colon L(H) \to R^4$ is a linear isomorphism, so in (2.12.3.3) we have $a = \sum_{i=1}^4 a_i v_i$, where

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

is the corresponding translation part. Since T is the unipotent radical of $A(H)$, we have $[L(H), L(H)] \subseteq L(T) = R^4$. By computing the commutator and using the fact that H is simply transitive, we must have $a(v) = a_2 v_2 + a_4 v_4$, $(a_2, a_4) \neq (0, 0)$ in Case (1) and $a(v) = a_1(v_1 - v_3) + a_2(v_2 - v_4)$, $(a_1, a_2) \neq (0, 0)$ in Case (2). Finally, by considering the conjugation under $R^4 \rtimes \mathrm{O}(2, 2)$, we get

(2.12.4) Theorem. *If $H \subseteq R^4 \rtimes \mathrm{SO}(2, 2)$ acts simply transitively on R^4 and H is not unipotent, then H is conjugate under $R^4 \rtimes \mathrm{O}(2, 2)$*

to one of the following:

(i) *Type* III-1:

$$\begin{pmatrix} a(v) & & & 0 \\ & 0 & & \\ & & -a(v) & \\ 0 & & & 0 \end{pmatrix} \begin{pmatrix} \\ v \\ \\ \\ \end{pmatrix},$$

where $a(v) = tv_2 + v_4$, $t > 0$ *and* $L(H) \simeq R \oplus \{R^2 \rtimes R\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\}$.

(ii) *Type* III-2:

$$\begin{pmatrix} 0 & a(v) & 0 & a(v) \\ -a(v) & 0 & -a(v) & 0 \\ 0 & a(v) & 0 & a(v) \\ -a(v) & 0 & -a(v) & 0 \end{pmatrix} \begin{pmatrix} \\ v \\ \\ \\ \end{pmatrix},$$

where $a(v) = t(v_1 - v_3)$, $t > 0$ *and* $L(H) = R \oplus \{R^2 \rtimes R\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\}$. *The type and the parameter* t *determine the equivalence classes uniquely.*

(2.13) Combining (2.11) with (2.12.4) and denoting $H = T_4$ as Type 0, we complete the classification of simply transitive subgroups of $R^4 \rtimes \mathrm{SO}(2, 2)$. We summarize our result in the following table. We denote

$$A(a, b, v) = \left\{ \begin{pmatrix} \begin{pmatrix} 0 & a(v) & 0 & b(v) \\ 0 & 0 & -b(v) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a(v) & 0 \end{pmatrix} & v \\ 0 & 0 \end{pmatrix} ; v \in R^4 \right\},$$

$$B(a, v) = \left\{ \begin{pmatrix} \begin{pmatrix} a(v) & & & 0 \\ & 0 & & \\ & & -a(v) & \\ 0 & & & 0 \end{pmatrix} & v \\ 0 & 0 \end{pmatrix}, v \in R^4 \right\},$$

$$C(a, v) = \left\{ \begin{pmatrix} \begin{pmatrix} 0 & a(v) & 0 & a(v) \\ -a(v) & 0 & -a(v) & 0 \\ 0 & a(v) & 0 & a(v) \\ -a(v) & 0 & -a(v) & 0 \end{pmatrix} & v \\ 0 & 0 \end{pmatrix} ; v \in R^4 \right\}.$$

Table of equivalence classes of simply transitive subgroups of $R^4 \rtimes \mathrm{SO}(2,2)$ (given in the form of subalgebras of $\mathrm{aff}(n)$ w.r.t. a standard basis).

type of $L(H)$	affine form of $L(H)$	isomorphism type as abstract Lie algebra
0	$\{\left(\begin{smallmatrix} 0 & v \\ 0 & 0 \end{smallmatrix}\right) ; v \in R^4\}$;	R^4
I-1	$A(a, b, v), \begin{cases} a(v) = v_3, \\ b(v) = v_3 \end{cases}$	$R \oplus N_3$
I-2	$A(a, b, v), \begin{cases} a(v) = v_3, \\ b(v) = -v_3 \end{cases}$	$R \oplus N_3$
I-3	$A(a, b, v), \begin{cases} a(v) = v_3, \\ b(v) = 0 \end{cases}$	$R \oplus N_3$
II-1	$A(a, b, v), \begin{cases} a(v) = v_2 + v_4 + tv_3, \\ b(v) = -v_2 - v_4, \ t \geq 0, \end{cases}$	N_4
II-2	$A(a, b, v), \begin{cases} a(v) = -v_2 + v_4 + tv_3, \\ b(v) = -v_2 + v_4, \ t \geq 0 \end{cases}$	N_4
II-3	$A(a, b, v), \begin{cases} a(v) = v_2, \\ b(v) = v_3 \end{cases}$	N_4
III-1	$B(a, v), a(v) = tv_2 + v_4, t \in R$	$R \oplus \{R^2 \rtimes R\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\}$
III-2	$C(a, v), a(v) = t(v_1 - v_3), t > 0$	$R \oplus \{R^2 \rtimes R\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\}$

The type of $L(H)$ and the parameter t determine the equivalence classes uniquely.

3. Γ is virtually solvable. A group with a solvable subgroup of finite index is called virtually solvable.

(3.1) THEOREM. *If $\Gamma \subset R^4 \rtimes \mathrm{SO}(2,2)$ and Γ acts freely and properly discontinuously on R^4 with compact quotient, then Γ is virtually solvable.*

Proof. Let $\pi = P_l(\Gamma)$ and $A(\pi)$ be the algebraic hull of Γ. The identity component A_0 is of finite index in $A(\pi)$. We will show A_0 is solvable. The following lemma is due to D. Fried.

(3.2) LEMMA. *If A_0 fixes a vector $v \in R^4$ s.t. $Q(v, v) \neq 0$, then A_0 is solvable.*

For a proof, cf. [**3**].

Assume that A_0 is not solvable. As in (1.7.2), for every $g \in A(\pi)$, $\det(g - I) = 0$. This shows $\det = 0$ on $L(A_0)$ and $\dim A_0 < \dim \mathrm{SO}(2,2)$. So A_0 contains a semisimpleconnected subgroup S

such that $\dim S = 3$ and $L(S) \simeq \mathrm{sl}_2(R)$. By (1.6.2) $\det \neq 0$ on l_i so $L(S) \neq L_i$, $i = 1, 2$. So $L(S)$ must be a maximal subalgebra of $\mathrm{so}(2,2)$, so $A_0 = S$. Let $P_i\colon L(S) \to L_i$, $i = 1, 2$ be the projection map, then $P_i(L(S)) = L_i$, $i = 1, 2$.

(3.3) *Claim.* There is a nonzero vector $v \in R^4$ such that

(i) $Q(v, v) \neq 0$;
(ii) $A_0(v) = v$.

To prove the claim, let $0 \neq X \in L(A_0)$ such that RX is a split Cartan subalgebra of $L(A_0)$. Then $h = P_1(RX) \oplus P_2(RX)$ is a split Cartan subalgebra of $\mathrm{so}(2,2)$. By (1.6.3) h is conjugate under $\mathrm{O}(2,2)$ to $\{\mathrm{diag}\cdot(a, b, -a, -b); a, b \in R\}$. Since $\det X = 0$ we can rescale and permute coordinates so $X = \mathrm{diag}\cdot(1, 0, -1, 0)$. Let $\{X, Y, Z\}$ be the basis of $L(A)$ such that $[X, Y] = 2Y$, $[X, Z] = -2Z$, $[Y, Z] = X$ and $X = \mathrm{diag}\cdot(1, 0, -1, 0)$. Then $\mathrm{ad}\, X$ has three real eigenvalues on $\mathrm{so}(2,2)$: $\{2, 0, -2\}$. Let E_λ be the corresponding eigenspaces, then

$$E_2 = \left\{ \begin{pmatrix} 0 & c & 0 & e \\ 0 & 0 & -e & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -c & 0 \end{pmatrix} ; c, e \in R \right\},$$

$$E_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 \\ 0 & f & 0 & -d \\ -f & 0 & 0 & 0 \end{pmatrix} ; d, f, \in R \right\}, \quad \text{and}$$

$$[E_2, E_{-2}] = \left\{ \begin{pmatrix} cd - ef & & & 0 \\ & -cd - ef & & \\ & & -cd + ef & \\ 0 & & & cd + ef \end{pmatrix} ; \quad c, d, e, f \in R \right\}.$$

So there are $c, e, d, f \in R$ such that

$$Y = \begin{pmatrix} 0 & c & 0 & e \\ 0 & 0 & -e & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -c & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 \\ 0 & f & 0 & -d \\ -f & 0 & 0 & 0 \end{pmatrix}$$

and $[X, Z] = X$ implies

$$\begin{cases} cd - ef = 1, \\ cd + ef = 0, \end{cases}$$

i.e. $cd = -ef = \frac{1}{2}$, $cdef \neq 0$. Let $v = \frac{1}{c}e_2 - \frac{1}{e}e_4$. It's easy to check that $Q(v, v) = \frac{1}{ce} \neq 0$, $A_0(v) = v$.

Combining (3.3) with Lemma (3.2), we have a contradiction, so A_0 must be solvable. □

4. Proof of Theorem (1.2). The principal tool is the following theorem from [**4**].

(4.1) THEOREM (*Fried and Goldman*). *Let* $\Gamma \subseteq \mathrm{Aff}(n)$ *be virtually polycyclic and suppose that* Γ *acts properly discontinuously on* R^n. *Then there exists at least one subgroup* $H \subseteq \mathrm{Aff}(n)$ *containing* Γ *such that*:

(a) *H has finitely many components and each component meets Γ*;

(b) *H/Γ is compact*;

(c) *H and Γ have the same algebraic hull in* $\mathrm{Aff}(n)$;

(d) *if Γ has a subgroup Γ_1 of finite index such that every element of $P_l(\Gamma_1)$ has all real eigenvalues, then H is uniquely determined by the above conditions*;

(e) *the identity component H_0 of H acts simply transitively on R^n and $H_0 \cap \Gamma$ is a discrete cocompact subgroup of H_0 and is of finite index in Γ.*

Such a subgroup H in (4.1) is called a crystallographic hull for Γ. Since a discrete solvable subgroup of Lie with finitely many components is polycyclic and we proved in §3 that Γ in (1.2) is virtually solvable, by (4.1) we need only to check for the uniqueness of H. By (4.1)-(d), we need only to show that $P_l(\Gamma)$ has a subgroup of finite index with real eigenvalues only. Since H_0 must occur in our table of simply transitive motions and all these simply transitive motions, except Type III-2, have linear parts with only real eigenvalues, we need only to check Type III-2. By Bieberbach's theorem (cf. [**8**]), any discrete subgroup of Type III-2 meets T in a subgroup of finite index. □

5. Classification of Γ.

(5.1) LEMMA. *Let Γ be a uniform lattice in a simply transitive group $H \subseteq R^4 \rtimes \mathrm{SO}(2,2)$. Then H is the identity component of the crystallographic hull of Γ if and only if H is not of Type* III-2.

Proof. If H is of Type III-2, then Γ has a subgroup of finite index, say Γ_1, such that $\Gamma_1 \subset T$. So Γ is virtually abelian. By **[4]**, the crystallographic hull of a virtually abelian affine polycyclic group is itself virtually abelian, so H doesn't arise from any Γ.

In the unipotent cases, the algebraic hull of H is H itself. So $A(\Gamma)$, the algebraic hull of Γ, is contained in H. Since H_0', the identity component of the crystallographic hull H' of Γ, acts simply transitively on R^4, the dimension of H_0' must be four, and then by (4.1)-(C) we have

$$H_0' \subseteq H' \subseteq A(H') = A(\Gamma) \subseteq H.$$

So $H = H_0'$; then $H' = H$.

The only remaining case is Type III-1. Since Γ is not unipotent, H_0', the identity component of the crystallographic hull H' of Γ, must be nonunipotent solvable, i.e. H_0' is of Type III-1 and $\Gamma \subseteq H \cap H_0'$. Then it's easy to show that $H_0' = H$. □

(5.2) COROLLARY. *Up to finite covers, every flat compact complete space-form with metric of signature* (2,2) *is of the form H/Γ, where H is a simply transitive subgroup of $R^4 \times \mathrm{SO}(2,2)$ of Type* 0, *Type* I, *Type* II *or Type* III-1 *and Γ is a uniform lattice of H.*

(5.3) *Uniform lattices.* The uniform lattices depend only on the structure of H as a Lie group and do not depend on its embedding in $R^4 \rtimes \mathrm{SO}(2,2)$. Since Type 0 $\simeq R^4$, Type I $\simeq R \times \mathrm{Nil}^3$, Type II $\simeq \mathrm{Nil}^4$ and Type III-1 $\simeq R \times \{R^2 \rtimes \left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right); t \in R\}$, as Lie groups, they are exactly the same group as that listed in **[8]**, and D. Fried gave a list of their uniform lattices there. C. T. C. Wall also studied them, cf. **[7]**. Here we only write them down to complete our classification.

(5.3.1) The uniform lattices of H are semidirect products $Z^3 \rtimes Z_A$, where $A \in \mathrm{SL}(Z)$ has a characteristic polynomial

$$\det(t - A) = (t - 1)(t^2 - bt + 1),$$

where $b \geq 2$ is an integer, and A and b satisfy:

(i) Type 0: $A = I$, $b = 2$;

(ii) Type I: $(A - I)^2 = 0$, $A \neq I$, $b = 2$;

(iii) Type II: $(A - I)^2 \neq 0$, $(A - I)^3 = 0$, $b = 2$;

(iv) Type III-1: $b \geq 3$.

(Cf. [3] and [7] for a proof.)

Acknowledgment. The author wishes to thank Professor Ravi S. Kulkarni whose guidance and encouragement made the work of this paper possible.

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Received September 13, 1988 and in revised form February 27, 1989. Partially supported by the Max-Planck-Institut fur Mathematik, Bonn, West Germany and an NSF grant.

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Pacific Journal of Mathematics
Vol. 148, No. 1 March, 1991

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