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A duality is used to develop a complete set of numerical quasi-isomorphism invariants for the class of torsion-free abelian groups consisting of strongly indecomposable cokernels of diagonal embeddings $A_1 \cap \cdots \cap A_n \to A_1 \oplus \cdots \oplus A_n$ for n-tuples $(A_1, \ldots, A_n)$ of subgroups of the additive group of rational numbers.

A major theme in the theory of abelian groups is the classification of groups by numerical invariants. For the special case of torsion-free abelian groups of finite rank, one must first consider the decidedly non-trivial problem of classification up to quasi-isomorphism. To this end, we develop a contravariant duality on the quasi-homomorphism category of T-groups for a finite distributive lattice T of types.

A *Butler group* is a finite rank torsion-free abelian group that is isomorphic to a pure subgroup of a finite direct sum of subgroups of Q, the additive group of rationals. Isomorphism classes of subgroups of Q, called *types*, form an infinite distributive lattice. For a finite distributive sublattice T of types, a T*-group* is a Butler group G with each element of the *typeset of* G (the set of types of pure rank-1 subgroups of G) in T. Each Butler group is a T-group for some T, since Butler groups have finite typesets [**BU1**], but T is not, in general, unique. There are various characterizations of Butler groups, as found in [**AR2**], [**AR3**], and [**AV1**], but a complete structure theory has yet to be determined. As E. L. Lady has pointed out in [**LA1**] and [**LA2**], the theory generalizes directly to Butler modules over Dedekind domains.

Define B_T to be the category of T-groups with morphism sets $Q \otimes_Z \mathrm{Hom}_Z(G, H)$. Isomorphism in B_T is called *quasi-isomorphism* and an indecomposable in B_T is called *strongly indecomposable*. B. Joński in [**JO**] showed that direct sum decompositions in B_T are unique up to order and quasi-isomorphism (see [**AR1**] for the categorical version). Thus, classification of T-groups up to quasi-isomorphism depends only on the classification of strongly indecomposable T-groups.

A complete set of numerical quasi-isomorphism invariants for strongly indecomposable T-groups of the form $G = G(A_1, \ldots, A_n)$, the kernel of the map $A_1 \oplus \cdots \oplus A_n \to Q$ given by $(a_1, \ldots, a_n) \to a_1 + \cdots + a_n$ for $(A_1, \ldots, A_n)$ an n-tuple of subgroups of Q, is given in **[AV2]**. Specifically, the invariants are $\{r_G[M] | M \subseteq T\}$, where $r_G[M] = \operatorname{rank}(\bigcap\{G[\sigma] | \sigma \in M\})$.

Given an anti-isomorphism $\alpha : T \to T'$ of finite lattices of types, there is a contravariant duality $D(\alpha)$ from B_T to $B_{T'}$ (Corollary 5). The duality $D(\alpha)$ coincides with a duality on T-valuated Q-vector spaces given by F. Richman in **[RI1]** and includes, as special cases, the duality for *quotient divisible Butler groups* (all types are isomorphism classes of subrings of Q) given in **[AR5]** and by E. L. Lady in **[LA1]**, and the duality given for certain self-dual T in **[AV1]**. The search for lattices anti-isomorphic to a given lattice is simplified by an observation in **[RI1]** that each finite distributive lattice is isomorphic to a sublattice of a Boolean algebra of subrings of Q.

Groups of the form $G = G(A_1, \ldots, A_n)$ are sent by the duality $D(\alpha)$ to groups of the form $G = G[A_1, \ldots, A_n]$, the cokernel of the embedding $\bigcap\{A_i | 1 \leq i \leq n\} \to A_1 \oplus \cdots \oplus A_n$ given by $a \to (a, \ldots, a)$. This observation gives rise to an application of the duality $D(\alpha)$.

COROLLARY I. *Let T be a finite distributive lattice of types. A complete set of numerical quasi-isomorphism invariants for strongly indecomposable T-groups of the form $G = G[A_1, \ldots, A_n]$ is given by $\{r_G(M) | M$ a subset of $T\}$, where $r_G(M) = \operatorname{rank}(\Sigma\{G(\tau) | \tau \in M\})$. Each such group has quasi-endomorphism ring isomorphic to Q.*

Despite other options, we develop duality in terms of representations of finite *posets* (partially ordered sets) over an arbitrary field k. This choice is motivated by the fact that duality in this context is an easy consequence of vector space duality. Moreover, the quasi-isomorphism invariants given in Corollary I arise naturally when the groups are viewed as representations. As an added bonus, this duality is also applicable to classes of finite valuated p-groups. Specifically, given any finite poset S and prime p, there is an embedding from the category of Z/pZ-representations of S to the category of finite valuated p-groups that preserves isomorphism and indecomposability **[AR4]**. Implications of this embedding will be examined elsewhere.

Unexplained notation and terminology will be as in **[AR1]**, **[AR2]** **[AR4]**, and **[AV1]**.

If k is a field and S is a finite poset, then a *k-representation of S* is $X = (U, U_i | i \in S)$, where U is a finite dimensional k-vector space, each U_i is a subspace of U, and $i \leq j$ in S implies that

$U_i \subseteq U_j$. Let $\mathrm{Rep}(k, S)$ denote the category of k-representations of a finite poset S, where a *morphism* $f: (U, U_i|i \in S) \to (U', U_i'|i \in S)$ is a k-linear transformation $f: U \to U'$ with $f(U_i) \subseteq U_i'$ for each i. This category is a pre-abelian category (as defined in **[RIW]**) with finite direct sums defined by

$$(U, U_i|i \in S) \oplus (U', U_i'|i \in S) = (U \oplus U', U_i \oplus U_i'|i \in S).$$

Direct sum decompositions into indecomposable representations exist and are unique, up to isomorphism and order, since endomorphism rings of indecomposable representations are local. A sequence in $\mathrm{Rep}(k, S)$, $0 \to (U, U_i) \to (U', U_i') \to (U'', U_i'') \to 0$, is exact if and only if $0 \to U \to U'' \to U'' \to 0$ and $0 \to U_i \to U_i' \to U_i'' \to 0$ are exact sequences of vector spaces for each $i \in S$.

For a poset S, let S^{op} denote S with the reverse ordering.

PROPOSITION 1 **[DR]**. *Suppose that S is a finite poset. There is an exact contravariant duality* σ: $\mathrm{Rep}(k, S) \to \mathrm{Rep}(k, S^{\mathrm{op}})$ *defined by* $\sigma(U, U_i: i \in S) = (U^*, U_i^{\perp}: i \in S^{\mathrm{op}})$, *where* $U^* = \mathrm{Hom}_k(U, k)$ *and* $U_i^{\perp} = \{f \in U^*: f(U_i) = 0\}$.

Proof. A routine exercise in finite dimensional vector spaces, noting that if $f: X \to X'$ is a morphism of representations, then $\sigma(f) = f^*: \sigma(X') \to \sigma(X)$ is a morphism of representations and that σ^2 is naturally equivalent to the identity functor.

There are some extremal representations to be dealt with. A representation of the form $X = (U, U_i|i \in S)$ is called a *simple representation* of S if $U = k$ and $U_i = 0$ for each i, and a *co-simple representation* if $U = k = U_i$ for each i. Simple representations are indecomposable projective and co-simple representations are indecomposable injective relative to exact sequences in $\mathrm{Rep}(k, S)$. The duality σ carries simple representations into co-simple representations. It is easy to verify that a representation $X = (U, U_i|i \in S)$ has no simple summands if and only if $U = \Sigma\{U_i|i \in S\}$ and no co-simple summands if and only if $\bigcap\{U_i|i \in S\} = 0$.

Recall that types are ordered by $[X] \leq [Y]$ if and only if X is isomorphic to a subgroup of Y, where $[X]$ denotes the isomorphism class of a subgroup X of Q. The join of $[X]$ and $[Y]$ is $[X + Y]$, and the meet is $[X \cap Y]$.

Let G be a T-group and $0 \neq x \in G$. Then $\mathrm{type}_G(x)$ is the type of the pure rank-1 subgroup of G generated by x. Define $G(\tau) = \{x \in G|\, \mathrm{type}_G(x) \geq \tau\}$, the *$\tau$-socle of* G. Let $QG = Q \otimes_Z G$ denote the divisible hull of G, regard G as a subgroup of QG, and write $QG(\tau)$ for the Q-subspace of QG generated by $G(\tau)$.

Define $\mathrm{JI}(T)$ to be the set of *join-irreducible* elements of a finite lattice T of types. That is, $\mathrm{JI}(T) = \{\tau \in T \mid \text{if } \tau = \delta \text{ join } \gamma \text{ for } \delta, \gamma \in T, \text{ then } \tau = \gamma \text{ or } \tau = \delta\}$. The poset $\mathrm{JI}(T)^{\mathrm{op}}$ has a greatest element, namely the least element of T. In the correspondence of the following lemma, the simple indecomposables in $\mathrm{Rep}(Q, \mathrm{JI}(T)^{\mathrm{op}})$ have no non-zero group analogs. Thus, define $\mathrm{Rep}_0(Q, \mathrm{JI}(T)^{\mathrm{op}})$ to be $\mathrm{Rep}(Q, \mathrm{JI}(T)^{\mathrm{op}})$ subject to identifying a simple indecomposable representation with the indecomposable projective representation $(U, U_\tau \mid \tau \in \mathrm{JI}(T)^{\mathrm{op}})$ defined by $U = Q$, $U_\tau = Q$ if τ is the greatest element of $\mathrm{JI}(T)^{\mathrm{op}}$, and $U_\tau = 0$ otherwise. This guarantees that a simple indecomposable representation corresponds to a rank-1 group in B_T with type equal to the least element of T.

LEMMA 2 (a) **[BU2, BU3]**. *There is a category equivalence* $F_T\colon B_T \to \mathrm{Rep}_0(Q, \mathrm{JI}(T)^{\mathrm{op}})$ *given by* $F_T(G) = (QG, QG(\tau) \mid \tau \in \mathrm{JI}(T)^{\mathrm{op}})$.

(b) F_T *is an exact functor.*

Proof. (a) We observe only that the inverse of F_T sends $(U, U_\tau \mid \tau \in \mathrm{JI}(T)^{\mathrm{op}})$ to the subgroup of U generated by $\{G_\tau \mid \tau \in \mathrm{JI}(T)^{\mathrm{op}}\}$, where G_τ is a subgroup of torsion index in U_τ that is *τ-homogeneous completely decomposable* (isomorphic to a direct sum of rank-1 groups with types in τ). The proof is outlined in **[BU3]** with details in **[BU2]**.

(b) Note that B_T is also a pre-abelian category and that a sequence $0 \to G \xrightarrow{f} H \xrightarrow{g} K \to 0$ of T-groups is exact in B_T if and only if f is monic, $(\mathrm{kernel}\, g + \mathrm{image}\, f)/(\mathrm{kernel}\, g \cap \mathrm{image}\, f)$ is finite, and $(\mathrm{image}\, g + K)/(\mathrm{image}\, g \cap K)$ is finite. In particular, $0 \to QG \to QH \to QK \to 0$ is exact. Recall that, since we are working in a quasi-homomorphism category, equality in B_T is to be interpreted as *quasi-equality* of groups (G and H are quasi-equal if $QG = QH$ and there is a non-zero integer n with $nG \subseteq H$ and $nH \subseteq G$) and purity in B_T as *quasi-purity* (quasi-equal to a pure subgroup).

Let $0 \to G \xrightarrow{f} H \xrightarrow{g} K \to 0$ be an exact sequence in B_T. It is sufficient to show that if $\tau \in \mathrm{JI}(T)^{\mathrm{op}}$, then $QH(\tau) \xrightarrow{g} QK(\tau) \to 0$ is exact. In this case, $0 \to QG(\tau) \to QH(\tau) \to QK(\tau) \to 0$ is exact and $0 \to F_T(G) \to F_T(H) \to F_T(K) \to 0$ is exact in $\mathrm{Rep}(Q, \mathrm{JI}(T)^{\mathrm{op}})$.

If X is a pure rank-1 subgroup of K of type $\geq \tau$, then $g^{-1}(X)$ is generated in B_T by a finite set L of pure rank-1 subgroups of H whose types are in T **[BU1]**. Thus, $\mathrm{type}(X)$ is the join of the elements in a set L' of types of groups in L with nonzero image under g in QX. Also, τ is the join of the elements in $\{\sigma \text{ meet } \tau | \sigma \in L'\}$. But τ join irreducible in T implies that $\sigma \geq \tau$ for some $\sigma \in L'$, whence QX is in the image of $QH(\tau) \xrightarrow{g} QK(\tau)$. Consequently, $QH(\tau) \xrightarrow{g} QK(\tau) \to 0$ is exact, as desired.

At this stage, it is tempting to try to define a duality from $B_T \to B_T$, for anti-isomorphic lattices T and T' by using Lemma 2 and Proposition 1. This would require, however, that $\mathrm{JI}(T')^{\mathrm{op}}$ be lattice isomorphic to $\mathrm{JI}(T)$, a rare occurrence as $\mathrm{JI}(T')^{\mathrm{op}}$ has a greatest element but $\mathrm{JI}(T)$ need not. To overcome this difficulty, we need a functor from B_T to $\mathrm{Rep}(Q, S)$ for some other partially ordered set S. A candidate for S is the opposite of $\mathrm{MI}(T)$, the set of meet irreducible elements of T.

Note that $\mathrm{MI}(T)^{\mathrm{op}}$ has a least element, the greatest element of T. Define $\mathrm{Rep}^0(Q, \mathrm{MI}(T)^{\mathrm{op}})$ to be $\mathrm{Rep}(Q, \mathrm{MI}(T)^{\mathrm{op}})$ with a co-simple indecomposable representation identified with the indecomposable injective representation $(U = Q, U_i | i \in S)$, where $U_i = 0$ if i is the least element of $\mathrm{MI}(T)^{\mathrm{op}}$ and $U_i = Q$ otherwise.

For a Butler group G and a type τ the *τ-radical of* G, $G[\tau]$, is defined to be $\bigcap\{\text{kernel} f | f \colon G \to Q,\ \text{type(image} f) \leq \tau\}$.

Lemma 3 **[LA2]**. *Let T be a finite lattice of types, G a T-group, and $\tau \in T$.*

(a) $QG[\tau] = \Sigma\{QG(\gamma) | \gamma \in T, \gamma \nleq \tau\}$.
(b) $QG(\tau) = \bigcap\{QG[\gamma] | \tau \nleq \gamma \in T\}$.
(c) *If τ is the meet of γ and δ, then* $QG[\tau] = QG[\gamma] + QG[\delta]$.
(d) *If τ is the join of γ and δ, then* $QG(\tau) = QG(\gamma) \cap QG(\delta)$.

Proof. Proofs of (a) and (b) are given in **[AV1**, Proposition 1.9**]**. (c) and (d) then follow.

Theorem 4. *Assume that T is a finite lattice of types. There is an exact category equivalence $E_T \colon B_T \to \mathrm{Rep}^0(Q, \mathrm{MI}(T)^{\mathrm{op}})$ given by $E_T(G) = (QG, QG[\tau] | \tau \in \mathrm{MI}(T)^{\mathrm{op}})$.*

Proof. Clearly, E_T is a functor where if $q \otimes f \in Q \otimes \mathrm{Hom}_Z(G, H)$, then $E_T(q \otimes f) = q(1 \otimes f) \colon QG \to QH$. Also, E_T is well defined, since $\gamma \leq \tau$ in $\mathrm{MI}(T)^{\mathrm{op}}$ implies that $G[\gamma] \subseteq G[\tau]$.

The fact that $E_T: Q\operatorname{Hom}(G, H) \to \operatorname{Hom}(E_T(G), E_T(H))$ is an isomorphism is proved in [**LA2**, Theorem 1.5]. Also E_T has a well defined inverse, since G can be recovered, up to quasi-isomorphism, from $(QG, QG(\tau)|\tau \in \mathrm{JI}(T)^{\mathrm{op}})$ by Lemma 2 and the $QG(\tau)$'s can be recovered from $(QG, QG[\gamma]|\gamma \in \mathrm{MI}(T)^{\mathrm{op}})$ by Lemma 3.

It remains to show exactness of E_T. Assume that $0 \to G \to H \xrightarrow{g} K \to 0$ is exact in B_T, and let X be a pure rank-1 subgroup of K in B_T of type not less than or equal to γ. As noted in the proof of Lemma 2, $g^{-1}(X)$ is generated in B_T by a finite number of pure rank-1 subgroups of H in B_T such that type(X) is the join of the types of those groups having non-zero image under g in QX. Therefore, at least one of these types is not less than or equal to γ. It follows from Lemma 3.a that QX is contained in $g(QH[\gamma])$. Thus, $QH[\gamma] \xrightarrow{g} QK[\gamma] \to 0$ is exact, since $g(QH[\gamma]) \subseteq QK[\gamma]$ is immediate. Note that this part of the proof does not require γ to be meet irreducible.

Next, $QG \cap QH[\gamma] \supseteq QG[\gamma]$ for each γ. To show that $QG[\gamma] \supseteq QG \cap QH[\gamma]$ for $\gamma \in \mathrm{MI}(T)$, let X be a pure rank-1 subgroup of G in B_T and assume that $X \cap G[\gamma] = 0$. Then type$(X) \leq \gamma$, by Lemma 3.a. As H is a pure subgroup in B_T of a finite rank completely decomposable T-group, type(X) is the meet of the elements in a subset L of types of rank-1 torsion-free quotients of H in B_T such that the image of X in each of these quotients is non-zero [**AV1**]. In view of the distributivity of T, γ is the meet of the elements in $\{\gamma \text{ join } \alpha | \alpha \in L\}$. Since γ is meet irreducible, $\alpha \leq \gamma$ for some $\alpha \in L$. Hence, $X \cap H[\gamma] = 0$, as X is not in the kernel of a homomorphism from H to a rank-1 torsion-free quotient of H with type $= \alpha \leq \gamma$. Consequently, if X is a pure rank-1 subgroup of $G \cap H[\gamma]$, then $X \subseteq G[\gamma]$, since $X \cap G[\gamma] = 0$ implies that $X \cap H[\gamma] = 0$, as desired.

An exact sequence $0 \to G \to H \to K \to 0$ in B_T is *balanced* if $0 \to G(\tau) \to H(\tau) \to K(\tau) \to 0$ is exact in B_T for each type $\tau \in T$ and *cobalanced* if $0 \to G/G[\tau] \to H/H[\tau] \to K/K[\tau] \to 0$ is exact in B_T for each type $\tau \in T$.

COROLLARY 5. *Let $\alpha: T \to T'$ be a lattice anti-isomorphism of finite distributive lattices of types. There is a contravariant exact category equivalence $D = D(\alpha): B_T \to B_{T'}$ defined by $D(G) = H$, $QH = QG^* = \operatorname{Hom}_Q(QG, Q)$, and $QH[\alpha(\tau)] = QG(\tau)^{\perp}$ for each $\tau \in T$, with the following properties:*

(a) $D(\alpha^{-1})D(\alpha)$ *is naturally equivalent to the identity functor on* B_T, $\operatorname{rank}(D(G)) = \operatorname{rank}(G)$, *and* $QH(\alpha(\tau)) = QG[\tau]^{\perp}$ *for each* $\tau \in T$.

(b) $D(G(\tau))$ *is quasi-isomorphic to* $D(G)/D(G)[\alpha(\tau)]$ *and* $D(G/G(\tau))$ *is quasi-isomorphic to* $D(G)[\alpha(\tau)]$ *for each* $\tau \in T$.

(c) *If* X *is a* rank-1 *T-group with* $\operatorname{type}(X) =$ *the join of the elements in a subset* $\{\tau_1, \dots, \tau_n\}$ *of* $\mathrm{JI}(T)$, *then* $\operatorname{type}(D(X))$ *is the meet of the elements in* $\{\alpha(\tau_1), \dots, \alpha(\tau_n)\} \subset \mathrm{MI}(T')$.

(d) D *sends balanced sequences to cobalanced sequences and conversely.*

(e) $D(G(A_1, \dots, A_n))$ *is quasi-isomorphic to* $G[D(A_1), \dots, D(A_n)]$ *for each n-tuple* $(A_1, \dots, A_n)$ *of subgroups of* Q *with types in* T.

Proof. (a) Define $D = D(\alpha) = E_{T'}^{-1}, \sigma\alpha F_T$, where F_T and $E_{T'}$, are as defined in Lemma 2 and Theorem 4, respectively;

$$\alpha : \mathrm{Rep}_0(Q, \mathrm{JI}(T)^{\mathrm{op}}) \to \mathrm{Rep}_0(Q, \mathrm{MI}(T'))$$

is a relabelling; and

$$\sigma : \mathrm{Rep}_0(Q, \mathrm{MI}(T')) \to \mathrm{Rep}_0(Q, \mathrm{MI}(T')^{\mathrm{op}})$$

is as given in Proposition 1. Note that D is contravariant, since σ is, and that D is exact since each of the defining functors are exact. Unravelling the definition of D shows that $D(G) = H$, where $QH = (QG)^*$ and $QH[\alpha(\tau)] = (QG(\tau))^{\perp}$ for $\tau \in \mathrm{JI}(T)$. In fact, $QH[\alpha(\tau)] = QG(\tau)^{\perp}$ for each $\tau \in T$. To see this, note that τ is the join of elements in a subset M of $\mathrm{JI}(T)$. Therefore,

$$QG(\tau) = \bigcap\{QG(\delta) | \delta \in M\},$$

by Lemma 3.d, and

$$\begin{aligned} QG(\tau)^{\perp} &= \Sigma\{QG(\delta)^{\perp} | \delta \in M\} \\ &= \Sigma\{QH[\alpha(\delta)] | \delta \in M\} = QH[\alpha(\tau)], \end{aligned}$$

by Lemma 3.c, since $\alpha(\tau)$ is the meet of the elements in $\{\alpha(\delta) | \delta \in M\}$.

Now G is naturally quasi-isomorphic to $D(\alpha^{-1})D(\alpha)(G)$, via the natural vector space isomorphism $QG \to QG^{**}$, as a consequence of Lemma 3. Clearly, $\operatorname{rank}(D(G)) = \operatorname{rank}(G)$. An argument using Lemma 3, analogous to that of the preceding paragraph, shows that if $H = D(G)$, then $QH(\alpha(\tau)) = QG[\tau]^{\perp}$ for each $\tau \in T$.

(a) is now clear; (c) and (e) follow from (a) and the exactness of D; and (d) is a consequence of (b).

As for (b), observe that $QD(G/G(\tau)) = \mathrm{Hom}(QGQG(\tau), Q)$ can be identified with $QG(\tau)^{\perp} = QD(G)[\alpha(\tau)]$. Under this identification, $QD(G/G(\tau))[\alpha(\delta)] = Q(G/G(\tau))(\delta)^{\perp}$ corresponds to $QG(\tau)^{\perp}[\alpha(\delta)] = QD(G)[\alpha(\tau)][\alpha(\delta)]$ for each $\delta \in \mathrm{JI}(T)$. Therefore, $D(G/G(\tau))$ is quasi-isomorphic to $D(G)[\alpha(\tau)]$, as desired. The other part of (b) now follows from the fact that D is a contravariant exact duality.

The proof of Corollary 5 shows that if G has rank one with type τ, then $D(G)$ is rank one with type $\alpha(\tau)$. This observation, together with Corollary 5.c, shows that $D = D(\alpha)$ is the duality induced by the duality of T-valuated vector spaces given in **[RI1]**. In case T is a locally free lattice, as defined in **[AV1]**, then T' and D may be chosen with D representable as $\mathrm{Hom}_Z(*, X)$ for X a rank-1 group with type equal to the greatest element in T. This special case of Corollary 5 follows from Warfield duality **[WA]**.

As noted earlier, given a finite lattice T of types, there is a quotient divisible T' anti-isomorphic to T **[RI1]**. If, for example, T is quotient divisible, then T' and $\alpha : T \to T'$ may be chosen by $\alpha(\tau) = \tau'$, where the p-component of τ' is 0 if and only if the p-component of τ is ∞ and the p-component of τ' is ∞ if and only if the p-component of τ is 0. Thus, D induces a duality, independent of T, on the quasi-homomorphism category of quotient divisible Butler groups. This duality coincides with the duality functor A on quotient divisible Butler groups given in **[LA1]** and the restriction of the functor F given in **[AR5]** to quotient divisible Butler groups.

For a T-group G and a subset M of T, define

$$G(M) = \Sigma\{G(\tau) | \tau \in M\} \quad \text{and} \quad G[M] = \bigcap\{G[\tau] | \tau \in M\}.$$

Then $r_G(M) = \mathrm{rank}(G(M))$ and $r_G[M] = \mathrm{rank}(G[M])$, as defined in the introduction. Lemma 3 can be applied to see that the $r_G(M)$'s or the $r_G[M]$'s appear as the dimensions of associated subspaces of QG generated by $\{QG(\tau) | \tau \in \mathrm{JI}(T)^{\mathrm{op}}\}$ or $\{QG[\tau] | \tau \in \mathrm{MI}(T)^{\mathrm{op}}\}$.

Proof of Corollary I. Since T is a finite distributive lattice of types there is a (quotient divisible) lattice T' of types and an anti-isomorphism $\alpha : T \to T'$. Let $D = D(\alpha)$ be as defined in Corollary 5. If G and H are T-groups both of the form $G[B_1, \ldots, B_n]$ and $r_G(M) = r_H(M)$, then $QG(M)^{\perp}$ and $QH(M)^{\perp}$ have the same Q-dimension. But $D(G)[\alpha(M)] = QG(M)^{\perp}$ and $D(H)[\alpha(M)] = QH(M)^{\perp}$ via Corollary 5 and Lemma 3. Consequently, if $r_G(M) = r_H(M)$ for each subset M of T, then $r_{D(G)}[M'] = r_{D(H)}[M']$ for each subset M' of

T'. Now $D(G)$ and $D(H)$ are both of the form $G(A_1, \dots, A_n)$, by Corollary 5.e, so that $D(G)$ and $D(H)$ are quasi-isomorphic **[AV2]**. This implies that, by applying the duality $D(\alpha^{-1})$, G and H are quasi-isomorphic as desired. Finally, each strongly indecomposable group of the form $G(A_1, \dots, A_n)$ has endomorphism ring isomorphic to Q in B_T [**AV2**], and D is a category equivalence. The last statement of the corollary follows.

Corollary I includes a complete set of quasi-isomorphism invariants for the proper-subclass, co-CT-groups, of T-groups of the form $G[A_1, \dots, A_n]$ studied by W. Y. Lee in **[LE]**.

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