DUALITY AND INVARIANTS FOR BUTLER GROUPS

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A duality is used to develop a complete set of numerical quasi-isomorphism invariants for the class of torsion-free abelian groups consisting of strongly indecomposable cokernels of diagonal embeddings $A_1 \cap \cdots \cap A_n \to A_1 \oplus \cdots \oplus A_n$ for $n$-tuples $(A_1, \ldots, A_n)$ of subgroups of the additive group of rational numbers.

A major theme in the theory of abelian groups is the classification of groups by numerical invariants. For the special case of torsion-free abelian groups of finite rank, one must first consider the decidedly non-trivial problem of classification up to quasi-isomorphism. To this end, we develop a contravariant duality on the quasi-homomorphism category of $T$-groups for a finite distributive lattice $T$ of types.

A Butler group is a finite rank torsion-free abelian group that is isomorphic to a pure subgroup of a finite direct sum of subgroups of $\mathbb{Q}$, the additive group of rationals. Isomorphism classes of subgroups of $\mathbb{Q}$, called types, form an infinite distributive lattice. For a finite distributive sublattice $T$ of types, a $T$-group is a Butler group $G$ with each element of the typeset of $G$ (the set of types of pure rank-1 subgroups of $G$) in $T$. Each Butler group is a $T$-group for some $T$, since Butler groups have finite typesets [BU1], but $T$ is not, in general, unique. There are various characterizations of Butler groups, as found in [AR2], [AR3], and [AV1], but a complete structure theory has yet to be determined. As E. L. Lady has pointed out in [LA1] and [LA2], the theory generalizes directly to Butler modules over Dedekind domains.

Define $B_T$ to be the category of $T$-groups with morphism sets $Q\otimes \mathbb{Z} \text{Hom}_\mathbb{Z}(G, H)$. Isomorphism in $B_T$ is called quasi-isomorphism and an indecomposable in $B_T$ is called strongly indecomposable. B. Jonsson in [JO] showed that direct sum decompositions in $B_T$ are unique up to order and quasi-isomorphism (see [AR1] for the categorical version). Thus, classification of $T$-groups up to quasi-isomorphism depends only on the classification of strongly indecomposable $T$-groups.

A complete set of numerical quasi-isomorphism invariants for strongly indecomposable $T$-groups of the form $G = G(A_1, \ldots, A_n)$,
the kernel of the map $A_1 \oplus \cdots \oplus A_n \to Q$ given by $(a_1, \ldots, a_n) \mapsto a_1 + \cdots + a_n$ for $(A_1, \ldots, A_n)$ an $n$-tuple of subgroups of $Q$, is given in [AV2]. Specifically, the invariants are $\{r_G[M] | M \subseteq T\}$, where $r_G[M] = \text{rank}(\bigcap \{G[\sigma] | \sigma \in M\})$.

Given an anti-isomorphism $\alpha : T \to T'$ of finite lattices of types, there is a contravariant duality $D(\alpha)$ from $B_T$ to $B_{T'}$ (Corollary 5). The duality $D(\alpha)$ coincides with a duality on $T$-valuated $Q$-vector spaces given by F. Richman in [RI1] and includes, as special cases, the duality for quotient divisible Butler groups (all types are isomorphism classes of subrings of $Q$) given in [AR5] and by E. L. Lady in [LA1], and the duality given for certain self-dual $T$ in [AV1]. The search for lattices anti-isomorphic to a given lattice is simplified by an observation in [RI1] that each finite distributive lattice is isomorphic to a sublattice of a Boolean algebra of subrings of $Q$.

Groups of the form $G = G(A_1, \ldots, A_n)$ are sent by the duality $D(\alpha)$ to groups of the form $G = G[A_1, \ldots, A_n]$, the cokernel of the embedding $\bigcap \{A_i | 1 \leq i \leq n\} \to A_1 \oplus \cdots \oplus A_n$ given by $a \mapsto (a, \ldots, a)$. This observation gives rise to an application of the duality $D(\alpha)$.

**Corollary I.** Let $T$ be a finite distributive lattice of types. A complete set of numerical quasi-isomorphism invariants for strongly indecomposable $T$-groups of the form $G = G[A_1, \ldots, A_n]$ is given by $\{r_G(M) | M \text{ a subset of } T\}$, where $r_G(M) = \text{rank}(\Sigma G(\tau) | \tau \in M)$. Each such group has quasi-endomorphism ring isomorphic to $Q$.

Despite other options, we develop duality in terms of representations of finite posets (partially ordered sets) over an arbitrary field $k$. This choice is motivated by the fact that duality in this context is an easy consequence of vector space duality. Moreover, the quasi-isomorphism invariants given in Corollary I arise naturally when the groups are viewed as representations. As an added bonus, this duality is also applicable to classes of finite valued $p$-groups. Specifically, given any finite poset $S$ and prime $p$, there is an embedding from the category of $\mathbb{Z}/p\mathbb{Z}$-representations of $S$ to the category of finite valued $p$-groups that preserves isomorphism and indecomposability [AR4]. Implications of this embedding will be examined elsewhere.

Unexplained notation and terminology will be as in [AR1], [AR2] [AR4], and [AV1].

If $k$ is a field and $S$ is a finite poset, then a $k$-representation of $S$ is $X = (U, U_i | i \in S)$, where $U$ is a finite dimensional $k$-vector space, each $U_i$ is a subspace of $U$, and $i \leq j$ in $S$ implies that
Let $\operatorname{Rep}(k, S)$ denote the category of $k$-representations of a finite poset $S$, where a morphism $f: (U, U_i | i \in S) \to (U', U_i' | i \in S)$ is a $k$-linear transformation $f: U \to U'$ with $f(U_i) \subseteq U_i'$ for each $i$. This category is a pre-abelian category (as defined in [RIW]) with finite direct sums defined by

$$(U, U_i | i \in S) \oplus (U', U_i' | i \in S) = (U \oplus U', U_i \oplus U_i' | i \in S).$$

Direct sum decompositions into indecomposable representations exist and are unique, up to isomorphism and order, since endomorphism rings of indecomposable representations are local. A sequence in $\operatorname{Rep}(k, S)$, $0 \to (U, U_i) \to (U', U_i') \to (U'', U_i'') \to 0$, is exact if and only if $0 \to U \to U'' \to U'' \to 0$ and $0 \to U_i \to U_i' \to U_i'' \to 0$ are exact sequences of vector spaces for each $i \in S$.

For a poset $S$, let $S^{\text{op}}$ denote $S$ with the reverse ordering.

**Proposition 1** [DR]. Suppose that $S$ is a finite poset. There is an exact contravariant duality $\sigma: \operatorname{Rep}(k, S) \to \operatorname{Rep}(k, S^{\text{op}})$ defined by $\sigma(U, U_i: i \in S) = (U^*, U_i^*: i \in S^{\text{op}})$, where $U^* = \operatorname{Hom}_k(U, k)$ and $U_i^* = \{f \in U^*: f(U_i) = 0\}$.

**Proof.** A routine exercise in finite dimensional vector spaces, noting that if $f: X \to X'$ is a morphism of representations, then $\sigma(f) = f^*: \sigma(X') \to \sigma(X)$ is a morphism of representations and that $\sigma^2$ is naturally equivalent to the identity functor.

There are some extremal representations to be dealt with. A representation of the form $X = (U, U_i | i \in S)$ is called a simple representation of $S$ if $U = k$ and $U_i = 0$ for each $i$, and a co-simple representation if $U = k = U_i$ for each $i$. Simple representations are indecomposable projective and co-simple representations are indecomposable injective relative to exact sequences in $\operatorname{Rep}(k, S)$. The duality $\sigma$ carries simple representations into co-simple representations. It is easy to verify that a representation $X = (U, U_i | i \in S)$ has no simple summands if and only if $U = \Sigma\{U_i | i \in S\}$ and no co-simple summands if and only if $\bigcap\{U_i | i \in S\} = 0$.

Recall that types are ordered by $[X] \leq [Y]$ if and only if $X$ is isomorphic to a subgroup of $Y$, where $[X]$ denotes the isomorphism class of a subgroup $X$ of $Q$. The join of $[X]$ and $[Y]$ is $[X + Y]$, and the meet is $[X \cap Y]$.

Let $G$ be a $T$-group and $0 \neq x \in G$. Then $\text{type}_G(x)$ is the type of the pure rank-1 subgroup of $G$ generated by $x$. Define $G(\tau) = \{x \in G| \text{type}_G(x) \geq \tau\}$, the \text{$\tau$-socle} of $G$. Let $QG = Q \otimes Z G$ denote the
divisible hull of $G$, regard $G$ as a subgroup of $QG$, and write $QG(\tau)$ for the $Q$-subspace of $QG$ generated by $G(\tau)$.

Define $JI(T)$ to be the set of join-irreducible elements of a finite lattice $T$ of types. That is, $JI(T) = \{ \tau \in T \mid \text{if } \tau = \delta \text{ join } \gamma \text{ for } \delta, \gamma \in T, \text{ then } \tau = \gamma \text{ or } \tau = \delta \}$. The poset $JI(T)^{op}$ has a greatest element, namely the least element of $T$. In the correspondence of the following lemma, the simple indecomposables in $\text{Rep}(Q, JI(T)^{op})$ have no non-zero group analogs. Thus, define $\text{Rep}_0(Q, JI(T)^{op})$ to be $\text{Rep}(Q, JI(T)^{op})$ subject to identifying a simple indecomposable representation with the indecomposable projective representation $(U, U_\tau | \tau \in JI(T)^{op})$ defined by $U = Q$, $U_\tau = Q$ if $\tau$ is the greatest element of $JI(T)^{op}$, and $U_\tau = 0$ otherwise. This guarantees that a simple indecomposable representation corresponds to a rank-1 group in $B_T$ with type equal to the least element of $T$.

**Lemma 2** (a) [BU2, BU3]. There is a category equivalence $F_T : B_T \rightarrow \text{Rep}_0(Q, JI(T)^{op})$ given by $F_T(G) = (QG, QG(\tau) | \tau \in JI(T)^{op})$.

(b) $F_T$ is an exact functor.

**Proof.** (a) We observe only that the inverse of $F_T$ sends $(U, U_\tau | \tau \in JI(T)^{op})$ to the subgroup of $U$ generated by $\{G_\tau | \tau \in JI(T)^{op}\}$, where $G_\tau$ is a subgroup of torsion index in $U_\tau$ that is $\tau$-homogeneous completely decomposable (isomorphic to a direct sum of rank-1 groups with types in $\tau$). The proof is outlined in [BU3] with details in [BU2].

(b) Note that $B_T$ is also a pre-abelian category and that a sequence $0 \rightarrow G \xrightarrow{f} H \xrightarrow{g} K \rightarrow 0$ of $T$-groups is exact in $B_T$ if and only if $f$ is monic, $(\text{kernel } g + \text{image } f)/(\text{kernel } g \cap \text{image } f)$ is finite, and $(\text{image } g + K)/(\text{image } g \cap K)$ is finite. In particular, $0 \rightarrow QG \rightarrow QH \rightarrow QK \rightarrow 0$ is exact. Recall that, since we are working in a quasi-homomorphism category, equality in $B_T$ is to be interpreted as quasi-equality of groups ($G$ and $H$ are quasi-equal if $QG = QH$ and there is a non-zero integer $n$ with $nG \subseteq H$ and $nH \subseteq G$) and purity in $B_T$ as quasi-purity (quasi-equal to a pure subgroup).

Let $0 \rightarrow G \xrightarrow{f} H \xrightarrow{g} K \rightarrow 0$ be an exact sequence in $B_T$. It is sufficient to show that if $\tau \in JI(T)^{op}$, then $QH(\tau) \xrightarrow{g} QK(\tau) \rightarrow 0$ is exact. In this case, $0 \rightarrow QG(\tau) \rightarrow QH(\tau) \rightarrow QK(\tau) \rightarrow 0$ is exact and $0 \rightarrow F_T(G) \rightarrow F_T(H) \rightarrow F_T(K) \rightarrow 0$ is exact in $\text{Rep}(Q, JI(T)^{op})$.

If $X$ is a pure rank-1 subgroup of $K$ of type $\geq \tau$, then $g^{-1}(X)$ is generated in $B_T$ by a finite set $L$ of pure rank-1 subgroups of $H$ whose types are in $T$ [BU1]. Thus, type($X$) is the join of the
elements in a set \( L' \) of types of groups in \( L \) with nonzero image under \( g \) in \( QX \). Also, \( \tau \) is the join of the elements in \( \{ \sigma \text{ meet } \tau | \sigma \in L' \} \). But \( \tau \) join irreducible in \( T \) implies that \( \sigma \geq \tau \) for some \( \sigma \in L' \), whence \( QX \) is in the image of \( QH(\tau) \xrightarrow{g} QK(\tau) \). Consequently, \( QH(\tau) \xrightarrow{g} QK(\tau) \to 0 \) is exact, as desired.

At this stage, it is tempting to try to define a duality from \( B_T \to B_T \), for anti-isomorphic lattices \( T \) and \( T' \) by using Lemma 2 and Proposition 1. This would require, however, that \( \text{JI}(T')^{\text{op}} \) be lattice isomorphic to \( \text{JI}(T) \), a rare occurrence as \( \text{JI}(T')^{\text{op}} \) has a greatest element but \( \text{JI}(T) \) need not. To overcome this difficulty, we need a functor from \( B_T \) to \( \text{Rep}(Q, S) \) for some other partially ordered set \( S \). A candidate for \( S \) is the opposite of \( \text{MI}(T) \), the set of meet irreducible elements of \( T \).

Note that \( \text{MI}(T)^{\text{op}} \) has a least element, the greatest element of \( T \). Define \( \text{Rep}^0(Q, \text{MI}(T)^{\text{op}}) \) to be \( \text{Rep}(Q, \text{MI}(T)^{\text{op}}) \) with a co-simple indecomposable representation identified with the indecomposable injective representation \( (U = Q, U_i | i \in S) \), where \( U_i = 0 \) if \( i \) is the least element of \( \text{MI}(T)^{\text{op}} \) and \( U_i = Q \) otherwise.

For a Butler group \( G \) and a type \( \tau \) the \( \tau \)-radical of \( G \), \( G[\tau] \), is defined to be \( \bigcap \{ \text{kernel } f | f : G \to Q, \text{type}(\text{image } f) \leq \tau \} \).

**Lemma 3 [LA2].** Let \( T \) be a finite lattice of types, \( G \) a \( T \)-group, and \( \tau \in T \).

(a) \( QG[\tau] = \Sigma \{ QG(\gamma) | \gamma \in T, \gamma \not\leq \tau \} \).

(b) \( QG(\tau) = \bigcap \{ QG(\gamma) | \tau \not\leq \gamma \in T \} \).

(c) If \( \tau \) is the meet of \( \gamma \) and \( \delta \), then \( QG[\tau] = QG[\gamma] + QG[\delta] \).

(d) If \( \tau \) is the join of \( \gamma \) and \( \delta \), then \( QG(\tau) = QG(\gamma) \cap QG(\delta) \).

**Proof.** Proofs of (a) and (b) are given in [AV1, Proposition 1.9]. (c) and (d) then follow.

**Theorem 4.** Assume that \( T \) is a finite lattice of types. There is an exact category equivalence \( E_T : B_T \to \text{Rep}^0(Q, \text{MI}(T)^{\text{op}}) \) given by \( E_T(G) = (QG, QG[\tau] | \tau \in \text{MI}(T)^{\text{op}}) \).

**Proof.** Clearly, \( E_T \) is a functor where if \( q \otimes f \in Q \otimes \text{Hom}_Z(G, H) \), then \( E_T(q \otimes f) = q(1 \otimes f) : QG \to QH \). Also, \( E_T \) is well defined, since \( \gamma \leq \tau \) in \( \text{MI}(T)^{\text{op}} \) implies that \( G[\gamma] \subseteq G[\tau] \).
The fact that \( E_T : \text{QHom}(G, H) \to \text{Hom}(E_T(G), E_T(H)) \) is an isomorphism is proved in [LA2, Theorem 1.5]. Also \( E_T \) has a well defined inverse, since \( G \) can be recovered, up to quasi-isomorphism, from \( (QG, QG(\tau) | \tau \in \text{MI}(T)^{\text{op}}) \) by Lemma 2 and the \( QG(\tau) \)'s can be recovered from \( (QG, QG[\gamma] | \gamma \in \text{MI}(T)^{\text{op}}) \) by Lemma 3.

It remains to show exactness of \( E_T \). Assume that \( 0 \to G \xrightarrow{g} H \xrightarrow{f} K \to 0 \) is exact in \( B_T \), and let \( X \) be a pure rank-1 subgroup of \( K \) in \( B_T \) of type not less than or equal to \( \gamma \). As noted in the proof of Lemma 2, \( g^{-1}(X) \) is generated in \( B_T \) by a finite number of pure rank-1 subgroups of \( H \) in \( B_T \) such that \( \text{type}(X) \) is the join of the types of those groups having non-zero image under \( g \) in \( QX \).

Therefore, at least one of these types is not less than or equal to \( \gamma \). It follows from Lemma 3.a that \( QX \) is contained in \( g(QH[\gamma]) \). Thus, \( QH[\gamma] \xrightarrow{g} QK[\gamma] \to 0 \) is exact, since \( g(QH[\gamma]) \subseteq QK[\gamma] \) is immediate. Note that this part of the proof does not require \( \gamma \) to be meet irreducible.

Next, \( QG \cap QH[\gamma] \supseteq QG[\gamma] \) for each \( \gamma \). To show that \( QG[\gamma] \supseteq QG \cap QH[\gamma] \) for \( \gamma \in \text{MI}(T) \), let \( X \) be a pure rank-1 subgroup of \( G \) in \( B_T \) and assume that \( X \cap G[\gamma] = 0 \). Then \( \text{type}(X) \leq \gamma \), by Lemma 3.a. As \( H \) is a pure subgroup in \( B_T \) of a finite rank completely decomposable \( T \)-group, \( \text{type}(X) \) is the meet of the elements in a subset \( L \) of types of rank-1 torsion-free quotients of \( H \) in \( B_T \) such that the image of \( X \) in each of these quotients is non-zero [AV1].

In view of the distributivity of \( T \), \( \gamma \) is the meet of the elements in \( \{ \gamma | \gamma \text{ join } \alpha | \alpha \in L \} \). Since \( \gamma \) is meet irreducible, \( \alpha \leq \gamma \) for some \( \alpha \in L \). Hence, \( X \cap H[\gamma] = 0 \), as \( X \) is not in the kernel of a homomorphism from \( H \) to a rank-1 torsion-free quotient of \( H \) with type = \( \alpha \leq \gamma \).

Consequently, if \( X \) is a pure rank-1 subgroup of \( G \cap H[\gamma] \), then \( X \subseteq G[\gamma] \), since \( X \cap G[\gamma] = 0 \) implies that \( X \cap H[\gamma] = 0 \), as desired.

An exact sequence \( 0 \to G \to H \to K \to 0 \) in \( B_T \) is balanced if \( 0 \to G(\tau) \to H(\tau) \to K(\tau) \to 0 \) is exact in \( B_T \) for each type \( \tau \in T \) and cobalanced if \( 0 \to G/G[\tau] \to H/H[\tau] \to K/K[\tau] \to 0 \) is exact in \( B_T \) for each type \( \tau \in T \).

**Corollary 5.** Let \( \alpha : T \to T' \) be a lattice anti-isomorphism of finite distributive lattices of types. There is a contravariant exact category equivalence \( D = D(\alpha) : B_T \to B_{T'} \) defined by \( D(G) = H \), \( QH = QG^* = \text{Hom}_Q(QG, Q) \), and \( QH[\alpha(\tau)] = QG(\tau) \downarrow \) for each \( \tau \in T \), with the following properties:
(a) $D(\alpha^{-1})D(\alpha)$ is naturally equivalent to the identity functor on $B_T$, $\operatorname{rank}(D(G)) = \operatorname{rank}(G)$, and $QH(\alpha(\tau)) = QG[\tau]^\perp$ for each $\tau \in T$.

(b) $D(G(\tau))$ is quasi-isomorphic to $D(G)/D(G)[\alpha(\tau)]$ and $D(G/G(\tau))$ is quasi-isomorphic to $D(G)[\alpha(\tau)]$ for each $\tau \in T$.

(c) If $X$ is a rank-1 $T$-group with $\operatorname{type}(X) = \text{the join of the elements in a subset } \{\tau_1, \ldots, \tau_n\}$ of $\text{JI}(T)$, then $\operatorname{type}(D(X))$ is the meet of the elements in $\{\alpha(\tau_1), \ldots, \alpha(\tau_n)\} \subset \text{MI}(T')$.

(d) $D$ sends balanced sequences to cobalanced sequences and conversely.

(e) $D(G(A_1, \ldots, A_n))$ is quasi-isomorphic to $G[D(A_1), \ldots, D(A_n)]$ for each $n$-tuple $(A_1, \ldots, A_n)$ of subgroups of $Q$ with types in $T$.

Proof. (a) Define $D = D(\alpha) = E_T^{-1}, \sigma \alpha F_T$, where $F_T$ and $E_T'$, are as defined in Lemma 2 and Theorem 4, respectively;

$$\alpha : \text{Rep}_0(Q, \text{JI}(T)^{\text{op}}) \to \text{Rep}_0(Q, \text{MI}(T'))$$

is a relabelling; and

$$\sigma : \text{Rep}_0(Q, \text{MI}(T')) \to \text{Rep}_0(Q, \text{MI}(T')^{\text{op}})$$

is as given in Proposition 1. Note that $D$ is contravariant, since $\sigma$ is, and that $D$ is exact since each of the defining functors are exact. Unravelling the definition of $D$ shows that $D(G) = H$, where $QH = (QG)^*$ and $QH[\alpha(\tau)] = (QG(\tau))^\perp$ for $\tau \in \text{JI}(T)$. In fact, $QH[\alpha(\tau)] = QG(\tau)^\perp$ for each $\tau \in T$. To see this, note that $\tau$ is the join of elements in a subset $M$ of $\text{JI}(T)$. Therefore,

$$QG(\tau) = \bigcap \{QG(\delta) | \delta \in M \},$$

by Lemma 3.d, and

$$QG(\tau)^\perp = \Sigma \{QG(\delta)^\perp | \delta \in M \}$$

$$= \Sigma \{QH[\alpha(\delta)] | \delta \in M \} = QH[\alpha(\tau)],$$

by Lemma 3.c, since $\alpha(\tau)$ is the meet of the elements in $\{\alpha(\delta) | \delta \in M \}$.

Now $G$ is naturally quasi-isomorphic to $D(\alpha^{-1})D(\alpha)(G)$, via the natural vector space isomorphism $QG \rightarrow QG^{**}$, as a consequence of Lemma 3. Clearly, $\operatorname{rank}(D(G)) = \operatorname{rank}(G)$. An argument using Lemma 3, analogous to that of the preceding paragraph, shows that if $H = D(G)$, then $QH(\alpha(\tau)) = QG[\tau]^\perp$ for each $\tau \in T$.

(a) is now clear; (c) and (e) follow from (a) and the exactness of $D$; and (d) is a consequence of (b).
As for (b), observe that $QD(G/G(\tau)) = \text{Hom}(QGQG(\tau), Q)$ can be identified with $QG(\tau)^\perp = QD(G)[\alpha(\tau)]$. Under this identification, $QD(G/G(\tau))[\alpha(\delta)] = Q(G/G(\tau))(\delta)^\perp$ corresponds to $QG(\tau)^\perp[\alpha(\delta)] = QD(G)[\alpha(\tau)][\alpha(\delta)]$ for each $\delta \in \text{JI}(T)$. Therefore, $D(G/G(\tau))$ is quasi-isomorphic to $D(G)[\alpha(\tau)]$, as desired. The other part of (b) now follows from the fact that $D$ is a contravariant exact duality.

The proof of Corollary 5 shows that if $G$ has rank one with type $\tau$, then $D(G)$ is rank one with type $\alpha(\tau)$. This observation, together with Corollary 5.c, shows that $D = D(\alpha)$ is the duality induced by the duality of $T$-valued vector spaces given in [RI1]. In case $T$ is a locally free lattice, as defined in [AV1], then $T'$ and $D$ may be chosen with $D$ representable as $\text{Hom}_Z(\ast, X)$ for $X$ a rank-1 group with type equal to the greatest element in $T$. This special case of Corollary 5 follows from Warfield duality [WA].

As noted earlier, given a finite lattice $T$ of types, there is a quotient divisible $T'$ anti-isomorphic to $T$ [RI1]. If, for example, $T$ is quotient divisible, then $T'$ and $\alpha : T \to T'$ may be chosen by $\alpha(\tau) = \tau'$, where the $p$-component of $\tau'$ is 0 if and only if the $p$-component of $\tau$ is $\infty$ and the $p$-component of $\tau'$ is $\infty$ if and only if the $p$-component of $\tau$ is 0. Thus, $D$ induces a duality, independent of $T$, on the quasi-homomorphism category of quotient divisible Butler groups. This duality coincides with the duality functor $A$ on quotient divisible Butler groups given in [LA1] and the restriction of the functor $F$ given in [AR5] to quotient divisible Butler groups.

For a $T$-group $G$ and a subset $M$ of $T$, define

$$G(M) = \Sigma \{G(\tau) | \tau \in M\} \quad \text{and} \quad G[M] = \bigcap \{G[\tau] | \tau \in M\}.$$ 

Then $r_G(M) = \text{rank}(G(M))$ and $r_G[M] = \text{rank}(G[M])$, as defined in the introduction. Lemma 3 can be applied to see that the $r_G(M)$'s or the $r_G[M]$'s appear as the dimensions of associated subspaces of $QG$ generated by $\{QG(\tau) | \tau \in \text{JI}(T)\}^{\text{op}}$ or $\{QG[\tau] | \tau \in \text{MI}(T)\}^{\text{op}}$.

**Proof of Corollary I.** Since $T$ is a finite distributive lattice of types there is a (quotient divisible) lattice $T'$ of types and an anti-isomorphism $\alpha : T \to T'$. Let $D = D(\alpha)$ be as defined in Corollary 5. If $G$ and $H$ are $T$-groups both of the form $G[B_1, \ldots, B_n]$ and $r_G(M) = r_H(M)$, then $QG(M)^\perp$ and $QH(M)^\perp$ have the same $Q$-dimension. But $D(G)[\alpha(M)] = QG(M)^\perp$ and $D(H)[\alpha(M)] = QH(M)^\perp$ via Corollary 5 and Lemma 3. Consequently, if $r_G(M) = r_H(M)$ for each subset $M$ of $T$, then $r_{D(G)}[M'] = r_{D(H)}[M']$ for each subset $M'$ of...
Now $D(G)$ and $D(H)$ are both of the form $G(A_1, \ldots, A_n)$, by Corollary 5.e, so that $D(G)$ and $D(H)$ are quasi-isomorphic $[AV2]$. This implies that, by applying the duality $D(\alpha^{-1})$, $G$ and $H$ are quasi-isomorphic as desired. Finally, each strongly indecomposable group of the form $G(A_1, \ldots, A_n)$ has endomorphism ring isomorphic to $Q$ in $B_T$ $[AV2]$, and $D$ is a category equivalence. The last statement of the corollary follows.

Corollary 1 includes a complete set of quasi-isomorphism invariants for the proper-subclass, co-$CT$-groups, of $T$-groups of the form $G[A_1, \ldots, A_n]$ studied by W. Y. Lee in $[LE]$.

**References**


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