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**OBSTRUCTION TO PRESCRIBED POSITIVE RICCI
CURVATURE**

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Obstruction to positive curvature is a phenomenon currently explored in global Riemannian geometry; the strongest results bear of course on the scalar curvature. Hereafter we consider the Ricci curvature and we adapt DeTurck and Koiso's device to non-compact manifolds. We also record a simple non-existence result on Kähler manifolds.

1. Statement of results. Let X be a connected non-compact C^3 n -manifold, $n > 2$, and \mathbf{h} be a fixed C^2 Riemannian metric on X . We are interested in finding conditions on \mathbf{h} which prevent it from being the Ricci tensor of any Riemannian metric on X . Following [5] we consider the largest eigenvalue $\lambda(\mathbf{h})$ of the curvature operator acting on covariant symmetric 2-tensors (see [1]). Given any C^2 metric \mathbf{g} on X , we let $e(\mathbf{g})$ denote the energy density of the identity map from (X, \mathbf{g}) to (X, \mathbf{h}) .

THEOREM 1. *Assume $\lambda(\mathbf{h}) \leq 1 - \varepsilon$ on X , for some positive real ε . Then there is no complete C^2 metric \mathbf{g} on X which admits \mathbf{h} as Ricci curvature.*

THEOREM 2. *Assume $\lambda(\mathbf{h}) \leq 1$ on X and \mathbf{h} complete. Then there is no C^2 metric \mathbf{g} on X , with $e(\mathbf{g})$ assuming a local maximum, which admits \mathbf{h} as Ricci curvature.*

THEOREM 3. *Assume $\lambda(\mathbf{h}) \leq 1$ on X . Then there is no C^2 metric \mathbf{g} on X , with $e(\mathbf{g})$ vanishing at infinity, which admits \mathbf{h} as Ricci curvature.*

2. Remarks and examples. Our results and methods of proof extend [5] from compact to non-compact manifolds. Related, though weaker, results, obtained by different techniques, are those of [0] (a reference kindly pointed out to us by the referee).

Theorem 1 may be viewed as the "true" extension of [5, Theorem 3.2-b]. Interestingly, Theorem 2 looks somewhat stronger than

[5, Theorem 3.2-b] due to the non-compactness of X ; an example here for (X, \mathbf{h}) is the Poincaré disk, since constant curvature -1 implies at once $\lambda(\mathbf{h}) \equiv 1$ by [1, Proposition 4.3]. Theorem 3 typically applies when (X, \mathbf{g}) is asymptotically flat; as such, it generalizes [8].

It is not possible to drop the completeness of both metrics and just assume $\lambda(\mathbf{h}) \leq 1$, as the following example shows: X is the euclidean n -space, \mathbf{h} the conformal metric $4(n-1)\sigma^{-4}E$, E denoting the standard euclidean metric and $\sigma := \sqrt{1 + |\mathbf{x}|^2}$. \mathbf{h} satisfies $\lambda(\mathbf{h}) \equiv 1$ and $\text{Ricci}(\mathbf{h}) = \mathbf{h}$ because it is constructed in the following way: start with the round n -sphere (S^n, \mathbf{g}_0) of radius $r = \sqrt{n-1}$ so that $\text{Ricci}(\mathbf{g}_0) = \mathbf{g}_0$. By [1, Proposition 4.3] we see at once that $\lambda(\mathbf{g}_0) \equiv 1$. Now \mathbf{h} is obtained as the pull-forward of \mathbf{g}_0 by a stereographic projection composed with the dilation of ratio $1/r$.

From the identity $\lambda(c\mathbf{h}) = \frac{1}{c}\lambda(\mathbf{h})$ valid for any positive constant c , one would like to infer that, given *any* C^2 metric \mathbf{h} on X , the preceding theorems hold with $c\mathbf{h}$ for suitable $c \gg 1$. This is what DeTurck and Koiso do on compact X . However, this cannot be done on non-compact X without *assuming* that $\lambda(\mathbf{h})$ is uniformly bounded from above (a mistake to be corrected in [8]). Keeping this in mind, one can formulate in an obvious way corollaries of our three theorems analogous to those of [5].

3. Proofs. For each theorem we argue by contradiction and assume the existence of a metric \mathbf{g} with the asserted properties. As observed in [5], the Bianchi identity thus satisfied by \mathbf{h} with respect to the metric \mathbf{g} means that the identity map from (X, \mathbf{g}) to (X, \mathbf{h}) is harmonic. Hence the energy density $e(\mathbf{g})$ satisfies on X the elliptic differential inequality

$$(1) \quad \Delta[e(\mathbf{g})] \leq -2\|T\|^2 - [1 - \lambda(\mathbf{h})]|\mathbf{h}|^2$$

deduced in [5] from an identity discovered by R. Hamilton [6]. Here Δ stands for the Laplacian (with *negative* symbol) of \mathbf{g} , T for the $(\frac{1}{2})$ -tensor difference between the Christoffel symbols of \mathbf{g} and \mathbf{h} , $|\cdot|$ for the norm in the metric \mathbf{g} , $\|\cdot\|$ for another norm (see [5]). Under the assumption $\lambda(\mathbf{h}) \leq 1$, made in all three theorems, $e(\mathbf{g})$ is thus C^2 positive subharmonic on (X, \mathbf{g}) .

Proof of Theorem 1. By Schwarz inequality $e(\mathbf{g}) \leq \sqrt{n}|\mathbf{h}|$; so (1) implies that $e(\mathbf{g})$ solves on X the inequality

$$(2) \quad \Delta u \leq -f(u)$$

where

$$f(t) := (2\varepsilon/n)t^2.$$

The function f is positive strictly increasing on $(0, \infty)$ and it readily satisfies the following condition: for all $a < b$ in $(0, \infty)$,

$$(3) \quad \int_b^\infty \left(\int_a^s f(t) dt \right)^{-1/2} ds < \infty.$$

Assume provisionally that (X, \mathbf{g}) is of class C^3 . Since $\mathbf{h} = \text{Ricci}(\mathbf{g})$ is non-negative, (X, \mathbf{g}) and f fulfill all the conditions required for the proof of Calabi's extension of Hopf's maximum principle [2] (Theorem 4). Fixing $a \in (0, \min_X[e(\mathbf{g})])$ in (3) and arguing as in [2] yields an impossibility for $e(\mathbf{g})$ to satisfy (2) on X . So we get the desired contradiction.

We are left with the C^3 regularity of (X, \mathbf{g}) . It follows basically from local elliptic regularity, as a repeated use of [4] now shows. Fix α in $(0, 1)$. Since \mathbf{g} is $C^{1,\alpha}$, X admits a $C^{2,\alpha}$ atlas of coordinates harmonic for \mathbf{g} [4] (Lemma 1.2). Being $C^{1,\alpha}$ in the original atlas, \mathbf{h} remains so in the harmonic atlas [4] (Corollary 1.4). Since $\text{Ricci}(\mathbf{g}) = \mathbf{h}$, \mathbf{g} is $C^{3,\alpha}$ in the harmonic atlas [4] (Theorem 4.5-b) and the atlas itself actually is $C^{4,\alpha}$ [4] (Lemma 1.2). \square

Proof of Theorem 2. By Hopf's maximum principle [7], $e(\mathbf{g})$ is necessarily constant on X . It follows from (1) that $T \equiv 0$ hence $\text{Ricci}(\mathbf{h}) = \mathbf{h}$ on X . Moreover, the regularity argument above, now applied to \mathbf{h} , combined with a bootstrap argument, provides a harmonic atlas in which (X, \mathbf{h}) is a C^∞ Riemannian manifold. So Myers' theorem [10] holds for (X, \mathbf{h}) , contradicting the noncompactness of X . \square

Proof of Theorem 3. Since $e(\mathbf{g})$ vanishes at infinity, it assumes a positive global maximum M . Fix μ in $(0, M)$ and let K be a compact subdomain of X outside which $e(\mathbf{g}) \leq \mu$. Hopf's maximum principle [7] applied to $e(\mathbf{g})$ inside K implies that either $e(\mathbf{g})$ is constant on K , or $e(\mathbf{g}) \leq \mu$ on K . In both cases it contradicts $\mu < M$. \square

4. A non-existence result on Kähler manifolds. Let X be a connected complex manifold, of complex dimension $n \geq 1$, admitting a C^2 Kähler metric \mathbf{h} . Denote by $|\mathbf{h}|$ the Riemannian density of \mathbf{h} .

THEOREM 4. *Assume that the scalar curvature of \mathbf{h} is bounded above by n , but not identical to n . Then there exists no C^2 Kähler metric \mathbf{g}*

on X , with relative density $|\mathbf{g}|/|\mathbf{h}|$ assuming a local minimum, which admits \mathbf{h} as Ricci curvature.

Proof. Again by contradiction; let \mathbf{g} be such a metric. Then the C^2 function $f := \text{Log}(|\mathbf{g}|/|\mathbf{h}|)$ satisfies on X the equation $\Delta f = n - S$, S standing for the scalar curvature of \mathbf{h} , Δ for its (complex) Laplacian. From the assumption, f is superharmonic on (X, \mathbf{h}) ; moreover, it assumes a local minimum, so it must be *constant* according to Hopf's maximum principle [7]. It implies that $S \equiv n$, contradicting the assumption. \square

For non-compact X , Theorem 4 typically applies when (X, \mathbf{g}) is Kähler asymptotically C^n [3]. For compact X , recalling that $S(c\mathbf{h}) = S(\mathbf{h})/c$ for any positive constant c , we obtain a simple proof of the following

COROLLARY. *Let (X, \mathbf{h}) be a C^2 compact Kähler manifold. Then there exists a positive real $c(\mathbf{h})$ such that, for any real $c > c(\mathbf{h})$, no C^2 Kähler metric on X admits $c\mathbf{h}$ as Ricci curvature.*

Of course, as emphasized by J.-P. Bourguignon (in a letter to us), the classical cohomological constraint bearing on Ricci tensors of compact Kähler manifolds makes Theorem 4 rather relevant for *non-compact* simply connected X .

Acknowledgment. This work originated from a question posed to me by Albert Jeune, about the contradiction between [3] and Jeune's Corollary 1 in [8]; as pointed out in §2, the latter turns out to be incorrect without a boundedness assumption on $\lambda(\mathbf{h})$.

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