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If all bounded linear operators from L_1 into a Banach space $\mathfrak{X}$ are *Dunford-Pettis* (i.e. carry weakly convergent sequences onto norm convergent sequences), then we say that $\mathfrak{X}$ has the *complete continuity property* (CCP). The CCP is a weakening of the Radon-Nikodým property (RNP). Basic results of Bourgain and Talagrand began to suggest the possibility that the CCP, like the RNP, can be realized as an internal geometric property of Banach spaces; the purpose of this paper is to provide such a realization. We begin by showing that $\mathfrak{X}$ has the CCP if and only if every bounded subset of $\mathfrak{X}$ is Bocce dentable, or equivalently, every bounded subset of $\mathfrak{X}$ is weak-norm-one dentable (§2). This internal geometric description leads to another; namely, $\mathfrak{X}$ has the CCP if and only if no bounded separated δ-trees grow in $\mathfrak{X}$, or equivalently, no bounded δ-Rademacher trees grow in $\mathfrak{X}$ (§3).

1. Introduction. Throughout this paper, $\mathfrak{X}$ denotes an arbitrary Banach space, $\mathfrak{X}^*$ the dual space of $\mathfrak{X}$, $B(\mathfrak{X})$ the closed unit ball of $\mathfrak{X}$, and $S(\mathfrak{X})$ the unit sphere of $\mathfrak{X}$. The triple (Ω, Σ, μ) refers to the Lebesgue measure space on $[0, 1]$, Σ^+ to the sets in Σ with positive measure, and L_1 to $L_1(\Omega, \Sigma, \mu)$. All notation and terminology, not otherwise explained, are as in **[DU]**. For clarity, known results are presented as Facts while new results are presented as Theorems, Lemmas, and Observations.

The following fact provides several equivalent formulations of the CCP.

Fact 1.1. For a bounded linear operator T from L_1 into $\mathfrak{X}$, the following statements are equivalent.

(1) T is Dunford-Pettis.

(2) T maps weak compact sets to norm compact sets.

(3) $T(B(L_\infty))$ is a relatively norm compact subset of $\mathfrak{X}$.

(4) The corresponding vector measure $F\colon \Sigma \to \mathfrak{X}$ given by $F(E) = T(\chi_E)$ has a relatively norm compact range in $\mathfrak{X}$.

(5) The adjoint of the restriction of T to L_∞ from $\mathfrak{X}^*$ into L_∞^* is a compact operator.

(6) As a subset of L_1, $T^*(B(\mathfrak{X}^*))$ is relatively L_1-norm compact.
(7) As a subset of L_1, $T^*(B(\mathfrak{X}^*))$ satisfies the Bocce criterion.

The equivalence of (2) and (3) follows from the fact that the subsets of L_1 that are relatively weakly compact are precisely those subsets that are bounded and uniformly integrable, which in turn, are precisely those subsets that can be uniformly approximated in L_1-norm by uniformly-bounded subsets. As for the equivalence of (6) and (7), **[G]** presents the two definitions below and shows that a relatively weakly compact subset of L_1 is relatively L_1-norm compact if and only if it satisfies the Bocce criterion.

DEFINITION 1.2. For f in L_1 and A in Σ, the *Bocce oscillation of f on A* is given by

$$\text{Bocce-osc}\, f|_A \equiv \frac{\int_A |f - [\int_A f\, d\mu/\mu(A)]|\, d\mu}{\mu(A)},$$

observing the convention that $0/0$ is 0.

DEFINITION 1.3. A subset K of L_1 satisfies the *Bocce criterion* if for each $\varepsilon > 0$ and B in Σ^+ there is a finite collection $\mathscr{F}$ of subsets of B each with positive measure such that for each f in K there is an A in $\mathscr{F}$ satisfying

$$\text{Bocce-osc}\, f|_A < \varepsilon .$$

The other implications in Fact 1.1 are straightforward and easy to verify. Because of (4), the CCP is also referred to as the compact range property (CRP).

Towards a martingale characterization of the CCP, fix an increasing sequence $\{\pi_n\}_{n\geq 0}$ of finite positive interval partitions of Ω such that $\bigvee \sigma(\pi_n) = \Sigma$ and $\pi_0 = \{\Omega\}$. Let $\mathscr{F}_n$ denote the sub-σ-field $\sigma(\pi_n)$ of Σ that is generated by π_n. For f in $L_1(\mathfrak{X})$, let $E_n(f)$ denote the conditional expectation of f given $\mathscr{F}_n$.

DEFINITION 1.4. A sequence $\{f_n\}_{n\geq 0}$ in $L_1(\mathfrak{X})$ is an $\mathfrak{X}$-valued *martingale* with respect to $\{\mathscr{F}_n\}$ if for each n we have that f_n is $\mathscr{F}_n$-measurable and $E_n(f_{n+1}) = f_n$ in L_1. The martingale $\{f_n\}$ is *uniformly bounded* provided that $\sup_n \|f_n\|_{L_\infty}$ is finite. Often the martingale is denoted by $\{f_n, \mathscr{F}_n\}$ in order to display both the functions and the sub-σ-fields involved.

There is a one-to-one correspondence between the bounded linear operators T from L_1 into $\mathfrak{X}$ and the uniformly bounded $\mathfrak{X}$-valued martingales $\{f_n, \mathscr{F}_n\}$. This correspondence is obtained by taking

$$T(g) = \lim_{n\to\infty} \int_\Omega f_n(\omega) g(\omega)\, d\mu(\omega) \quad \text{if } \{f_n\} \text{ is the martingale,}$$

and

$$f_n(\omega) = \sum_{E\in\pi_n} \frac{T(\chi_E)}{\mu(E)} \chi_E(\omega) \quad \text{if } T \text{ is the operator.}$$

Fact 1.1.6 implies that a bounded linear operator T from L_1 into $\mathfrak{X}$ is Dunford-Pettis if and only if

$$\lim_{m,n\to\infty} \sup_{x^*\in B(\mathfrak{X}^*)} \|E_n(T^*x^*) - E_m(T^*x^*)\|_{L_1} = 0.$$

Since $E_n(T^*x^*) = x^*f_n$ in L_1, we have the following martingale characterization of Dunford-Pettis operators, and thus of the CCP.

FACT 1.5. A bounded linear operator from L_1 into $\mathfrak{X}$ is Dunford-Pettis if and only if the corresponding martingale is Cauchy in the Pettis norm. Consequently, a Banach space $\mathfrak{X}$ has the CCP if and only if all uniformly bounded $\mathfrak{X}$-valued martingales are Pettis-Cauchy.

Recall that a bounded linear operator $T\colon L_1 \to \mathfrak{X}$ is (*Bochner*) *representable* if there is g in $L_\infty(\mu, \mathfrak{X})$ such that for each f in $L_1(\mu)$

$$Tf = \int_\Omega fg\, d\mu.$$

A Banach space $\mathfrak{X}$ has the *Radon-Nikodým property* if all bounded linear operators from L_1 into $\mathfrak{X}$ are Bochner representable. It is clear that a representable operator from L_1 into $\mathfrak{X}$ is Dunford-Pettis. Thus, if $\mathfrak{X}$ has the RNP then $\mathfrak{X}$ has the CCP. Both the Bourgain-Rosenthal space **[BR]** and the dual of the James tree space **[J]** have the CCP yet fail the RNP.

2. Dentability. In this section, we examine in which Banach spaces bounded subsets have certain dentability properties.

Dentability characterizations of the RNP are well-known (cf. **[DU]** and **[GU]**).

FACT 2.1. The following statements are equivalent.

(1) $\mathfrak{X}$ has the RNP.

(2) Every bounded subset D of $\mathfrak{X}$ is dentable.

DEFINITION 2.2. D is *dentable* if for each $\varepsilon > 0$ there is x in D such that $x \notin \overline{\mathrm{co}}(D\backslash B_\varepsilon(x))$ where $B_\varepsilon(x) = \{y \in \mathfrak{X} : \|x - y\| < \varepsilon\}$.

(3) Every bounded subset D of $\mathfrak{X}$ is σ-dentable.

DEFINITION 2.3. D is *σ-dentable* if for each $\varepsilon > 0$ there is an x in D such that if x has the form $x = \sum_{i=1}^{n} \alpha_i z_i$ with $z_i \in D$, $0 \le \alpha_i$, and $\sum_{i=1}^{n} \alpha_i = 1$, then $\|x - z_i\| < \varepsilon$ for some i.

The natural question to explore next is what dentability condition characterizes the CCP. Towards this, the next definition is a weakening of Definition 2.2.

DEFINITION 2.4. A subset D of $\mathfrak{X}$ is *weak-norm-one dentable* if for each $\varepsilon > 0$ there is a finite subset F of D such that for each x^* in $S(\mathfrak{X}^*)$ there is x in F satisfying

$$x \notin \overline{\mathrm{co}}\{z \in D : |x^*(z - x)| \ge \varepsilon\} \equiv \overline{\mathrm{co}}(D \backslash V_{\varepsilon, x^*}(x)) .$$

Petrakis and Uhl **[PU]** showed that if $\mathfrak{X}$ has the CCP then every bounded subset of $\mathfrak{X}$ is weak-norm-one dentable. For our characterization of the CCP, we introduce the following variations of Definition 2.3 that are useful in showing the converse of the above implication of **[PU]**.

DEFINITION 2.5. A subset D of $\mathfrak{X}$ is *Bocce dentable* if for each $\varepsilon > 0$ there is a finite subset F of D such that for each x^* in $S(\mathfrak{X}^*)$ there is x in F satisfying: if $x = \sum_{i=1}^{n} \alpha_i z_i$ with $z_i \in D$, $0 \le \alpha_i$, and $\sum_{i=1}^{n} \alpha_i = 1$, then $\sum_{i=1}^{n} \alpha_i |x^*(x - z_i)| < \varepsilon$.

DEFINITION 2.6. A subset D of $\mathfrak{X}$ is *midpoint Bocce dentable* if for each $\varepsilon > 0$ there is a finite subset F of D such that for each x^* in $S(\mathfrak{X}^*)$ there is x in F satisfying: if $x = \frac{1}{2}z_1 + \frac{1}{2}z_2$ with $z_i \in D$ then $|x^*(x - z_1)| \equiv |x^*(x - z_2)| < \varepsilon$.

We obtain equivalent formulations of the above definitions by replacing $S(\mathfrak{X}^*)$ with $B(\mathfrak{X}^*)$.

The next theorem, this section's main result, shows that these dentability conditions provide an internal geometric characterization of the CCP.

THEOREM 2.7. *The following statements are equivalent.*

(1) *$\mathfrak{X}$ has the CCP.*
(2) *Every bounded subset of $\mathfrak{X}$ is weak-norm-one dentable.*
(3) *Every bounded subset of $\mathfrak{X}$ is midpoint Bocce dentable.*
(4) *Every bounded subset of $\mathfrak{X}$ is Bocce dentable.*

The remainder of this section is devoted to the proof of Theorem 2.7. Because of its length and complexity and also for the sake of clarity of the exposition, we present the implications as separate theorems. It is clear from the definitions that (2) implies (3) and that (4) implies (3). [**PU**, Theorem II.7] shows that (1) implies (2) by constructing, in a bounded non-weak-norm-one dentable subset D, a $(\overline{\mathrm{co}}\, D)$-valued martingale that is not Cauchy in the Pettis norm. Using Fact 1.1.7, Theorem 2.10 shows that (3) implies (1). That (1) implies (4) follows from Theorem 2.8 and the martingale characterization of the CCP (Fact 1.5).

THEOREM 2.8. *If a subset D of $\mathfrak{X}$ is not Bocce dentable, then there is an increasing sequence $\{\pi_n\}$ of partitions of $[0, 1)$ and a D-valued martingale $\{f_n, \sigma(\pi_n)\}$ that is not Cauchy in the Pettis norm. Moreover, $\{\pi_n\}$ can be chosen so that $\bigvee \sigma(\pi_n) = \Sigma$, $\pi_0 = \{\Omega\}$, and each π_n partitions $[0, 1)$ into a finite number of half-open intervals.*

Proof. Let D be a subset of $\mathfrak{X}$ that is not Bocce dentable. Accordingly, there is an $\varepsilon > 0$ satisfying:

(∗) for each finite subset F of D there is x_F^* in $S(\mathfrak{X}^*)$ such that each x in F has the form $x = \sum_{i=1}^m \alpha_i z_i$ with $\sum_{i=1}^m \alpha_i |x_F^*(x - z_i)| > \varepsilon$ for a suitable choice of $z_i \in D$ and $\alpha_i > 0$ with $\sum_{i=1}^m \alpha_i = 1$.

We shall use property (∗) to construct an increasing sequence $\{\pi_n\}_{n \geq 0}$ of finite partitions of $[0, 1)$, a martingale $\{f_n, \sigma(\pi_n)\}_{n \geq 0}$, and a sequence $\{x_n^*\}_{n \geq 1}$ in $S(\mathfrak{X}^*)$ such that for each nonnegative integer n:

(1) f_n has the form $f_n = \sum_{E \in \pi_n} x_E \chi_E$ where x_E is in D,
(2) $\int_\Omega |x_{n+1}^*(f_{n+1} - f_n)| \, d\mu \geq \varepsilon$,
(3) if E is in π_n, then E has the form $[a, b)$ and $\mu(E) < 1/2^n$ and
(4) $\pi_0 = \{\Omega\}$.

Condition (3) guarantees that $\bigvee \sigma(\pi_n) = \Sigma$ while condition (2) guarantees that $\{f_n\}$ is not Cauchy in the Pettis norm.

Towards the construction, pick an arbitrary x in D. Set $\pi_0 = \{\Omega\}$ and $f_0 = x\chi_\Omega$. Fix $n \geq 0$. Suppose that a partition π_n of Ω consisting of intervals of length at most $1/2^n$ and a function $f_n = \sum_{E \in \pi_n} x_E \chi_E$ with $x_E \in D$ have been constructed. We now construct f_{n+1}, π_{n+1} and x_{n+1}^* satisfying conditions (1), (2), and (3).

Apply $(*)$ to $F = \{x_E : E \in \pi_n\}$ and find the associated $x_F^* = x_{n+1}^*$ in $S(\mathfrak{X}^*)$. Fix an element $E = [a, b)$ of π_n. We first define $f_{n+1}\chi_E$. Property $(*)$ gives that x_E has the form

$$x_E = \sum_{i=1}^{m} \alpha_i x_i \quad \text{with} \sum_{i=1}^{m} \alpha_i |x_{n+1}^*(x - x_i)| > \varepsilon$$

for a suitable choice of $x_i \in D$ and positive real numbers $\alpha_1, \dots, \alpha_m$ whose sum is one. Using repetition, we arrange to have $\alpha_i < 1/2^{n+1}$ for each i. It follows that there are real numbers $d_0, d_1, \dots, d_m$ such that

$$a = d_0 < d_1 < \cdots < d_{m-1} < d_m = b$$

and

$$d_i - d_{i-1} = \alpha_i(b - \alpha) \quad \text{for } i = 1, \dots, m.$$

Set

$$f_{n+1}\chi_E = \sum_{i=1}^{m} x_i \chi_{[d_{i-1}, d_i)}.$$

Define f_{n+1} on all of Ω similarly. Let π_{n+1} be the partition consisting of all the intervals $[d_{i-1}, d_i)$ obtained from letting E range over π_n.

Clearly, f_{n+1} and π_{n+1} satisfy conditions (1) and (3). Condition (2) is also satisfied since for each $E = [a, b)$ in π_n we have, using the above notation,

$$\begin{aligned}\int_E |x_{n+1}^*(f_{n+1} - f_n)|\, d\mu &= \sum_{i=1}^{m} \int_{d_{i-1}}^{d_i} |x_{n+1}^*(x_i - x_E)|\, d\mu \\ &= (b - a) \sum_{i=1}^{m} \alpha_i |x_{n+1}^*(x_i - x_E)| \geq \mu(E)\varepsilon.\end{aligned}$$

To insure that $\{f_n\}$ is indeed a martingale, we need to compute $E_n(f_{n+1})$. Fix $E = [a, b)$ in π_n. Using the above notation, we have for almost all t in E,

$$\begin{aligned}E_n(f_{n+1})(t) &= \frac{1}{b - a} \int_a^b f_{n+1}\, d\mu \\ &= \frac{1}{b - a} \sum_{i=1}^{m} \int_{d_{i-1}}^{d_i} f_{n+1}\, d\mu = \sum_{i=1}^{m} \frac{d_i - d_{i-1}}{b - a} x_i \\ &= \sum_{i=1}^{m} \alpha_i x_i = x_E = f_n(t).\end{aligned}$$

Thus $E_n(f_{n+1}) = f_n$ a.e., as needed.

This completes the necessary constructions. □

We need the following lemma which we will prove after the proof of Theorem 2.10.

LEMMA 2.9. *If A is in Σ^+ and f in $L_\infty(\mu)$ is not constant a.e. on A, then there is an increasing sequence $\{\pi_n\}$ of positive finite measurable partitions of A such that $\bigvee \sigma(\pi_n) = \Sigma \cap A$ and for each n*

$$\mu\left(\bigcup\left\{E : E \in \pi_n,\ \frac{\int_E f\,d\mu}{\mu(E)} \geq \frac{\int_A f\,d\mu}{\mu(A)}\right\}\right) = \frac{\mu(A)}{2},$$

and so

$$\mu\left(\bigcup\left\{E : E \in \pi_n,\ \frac{\int_E f\,d\mu}{\mu(E)} < \frac{\int_A f\,d\mu}{\mu(A)}\right\}\right) = \frac{\mu(A)}{2}.$$

THEOREM 2.10. *If all bounded subsets of $\mathfrak{X}$ are midpoint Bocce dentable, then $\mathfrak{X}$ has the complete continuity property.*

Proof. Let all bounded subsets of $\mathfrak{X}$ be midpoint Bocce dentable. Fix a bounded linear operator T from L_1 into $\mathfrak{X}$. We shall show that the subset $T^*(B(\mathfrak{X}^*))$ of L_1 satisfies the Bocce criterion. Then an appeal to Fact 1.1.7 shows that $\mathfrak{X}$ has the complete continuity property.

To this end, fix $\varepsilon > 0$ and B in Σ^+. Let F denote the vector measure from Σ into $\mathfrak{X}$ given by $F(E) = T(\chi_E)$. Since the subset $\{\frac{F(E)}{\mu(E)} : E \subset B \text{ and } E \in \Sigma^+\}$ of $\mathfrak{X}$ is bounded, it is midpoint Bocce dentable. Accordingly, there is a finite collection $\mathscr{F}$ of subsets of B each in Σ^+ such that for each x^* in the unit ball of $\mathfrak{X}^*$ there is a set A in $\mathscr{F}$ such that if

$$\frac{F(A)}{\mu(A)} = \frac{1}{2}\frac{F(E_1)}{\mu(E_1)} + \frac{1}{2}\frac{F(E_2)}{\mu(E_2)}$$

for some subsets E_i of B with $E_i \in \Sigma^+$, then

$$\frac{1}{2}\left|\frac{x^*F(E_1)}{\mu(E_1)} - \frac{x^*F(A)}{\mu(A)}\right| + \frac{1}{2}\left|\frac{x^*F(E_2)}{\mu(E_2)} - \frac{x^*F(A)}{\mu(A)}\right| < \varepsilon. \tag{1}$$

Fix x^* in the unit ball of $\mathfrak{X}^*$ and find the associated A in $\mathscr{F}$. By definition, the set $T^*(B(\mathfrak{X}^*))$ will satisfy the Bocce criterion provided that Bocce-osc $(T^*x^*)|_A \leq \varepsilon$.

If $T^*x^* \in L_1$ is constant a.e. on A, then the Bocce-osc $(T^*x^*)|_A$ is zero and we are finished. So assume T^*x^* is not constant a.e. on A.

For a finite positive measurable partition π of A, denote

$$f_\pi = \sum_{E\in\pi} \frac{F(E)}{\mu(E)} \chi_E$$

and

$$E_\pi^+ = \bigcup \left\{ E \in \pi : \frac{x^* F(E)}{\mu(E)} \geq \frac{x^* F(A)}{\mu(A)} \right\}$$

and

$$E_\pi^- = \bigcup \left\{ E \in \pi : \frac{x^* F(E)}{\mu(E)} < \frac{x^* F(A)}{\mu(A)} \right\} .$$

Note that for E in Σ

$$x^* F(E) = \int_E (x^* T^*)\, d\mu .$$

Compute

$$\begin{aligned}
(2) \qquad & \int_A \left| x^* f_\pi - \frac{x^* F(A)}{\mu(A)} \right| d\mu \\
&= \sum_{E\in\pi} \int_E \left| \frac{x^* F(E)}{\mu(E)} - \frac{x^* F(A)}{\mu(A)} \right| d\mu \\
&= \mu(A) \sum_{E\in\pi} \frac{\mu(E)}{\mu(A)} \left| \frac{x^* F(E)}{\mu(E)} - \frac{x^* F(A)}{\mu(A)} \right| \\
&= \mu(A) \left[\frac{\mu(E_\pi^+)}{\mu(A)} \left| \frac{x^* F(E_\pi^+)}{\mu(E_\pi^+)} - \frac{x^* F(A)}{\mu(A)} \right| \right. \\
&\qquad \left. + \frac{\mu(E_\pi^-)}{\mu(A)} \left| \frac{x^* F(E_\pi^-)}{\mu(E_\pi^-)} - \frac{x^* F(A)}{\mu(A)} \right| \right] .
\end{aligned}$$

Since the L_1-function $T^* x^*$ is bounded, for now we may view $T^* x^*$ as an element in L_∞. Lemma 2.9 allows us to apply property (1) to equation (2). For applying Lemma 2.9 to A with $f \equiv T^* x^*$ produces an increasing sequence $\{\pi_n\}$ of positive measurable partitions of A satisfying

$$\bigvee \sigma(\pi_n) = \Sigma \cap A \quad \text{and} \quad \mu(E_{\pi_n}^+) = \frac{\mu(A)}{2} = \mu(E_{\pi_n}^-) .$$

For $\pi = \pi_n$, condition (2) becomes

$$\begin{aligned}
(3) \qquad & \int_A \left| x^* f_{\pi_n} - \frac{x^* F(A)}{\mu(A)} \right| d\mu \\
&= \mu(A) \left[\frac{1}{2} \left| \frac{x^* F(E_{\pi_n}^+)}{\mu(E_{\pi_n}^+)} - \frac{x^* F(A)}{\mu(A)} \right| \right. \\
&\qquad \left. + \frac{1}{2} \left| \frac{x^* F(E_{\pi_n}^-)}{\mu(E_{\pi_n}^-)} - \frac{x^* F(A)}{\mu(A)} \right| \right] .
\end{aligned}$$

Since $F(A)/\mu(A)$ has the form

$$\begin{aligned}\frac{F(A)}{\mu(A)} &= \frac{\mu(E_{\pi_n}^+)}{\mu(A)}\frac{F(E_{\pi_n}^+)}{\mu(E_{\pi_n}^+)} + \frac{\mu(E_{\pi_n}^-)}{\mu(A)}\frac{F(E_{\pi_n}^-)}{\mu(E_{\pi_n}^-)} \\ &= \frac{1}{2}\frac{F(E_{\pi_n}^+)}{\mu(E_{\pi_n}^+)} + \frac{1}{2}\frac{F(E_{\pi_n}^-)}{\mu(E_{\pi_n}^-)},\end{aligned}$$

applying property (1) to equation (3) yields that for each π_n

$$\int_A \left| x^* f_{\pi_n} - \frac{x^* F(A)}{\mu(A)} \right| d\mu < \mu(A)\varepsilon .$$

Since $\bigvee \sigma(\pi_n) = \Sigma \cap A$ and

$$\begin{aligned}(x^* f_{\pi_n})|_A &= \sum_{E\in\pi_n} \frac{x^* F(E)}{\mu(E)}\chi_E \\ &= \sum_{E\in\pi_n} \frac{\int_E (T^* x^*)\, d\mu}{\mu(E)}\chi_E = E_{\pi_n}(T^* x^*)|_A ,\end{aligned}$$

we have that $(x^* f_{\pi_n})|_A$ converges to $(T^* x^*)|_A$ in L_1-norm. Hence,

$$\text{Bocce-osc}(T^* x^*)|_A \equiv \frac{\int_A |(T^* x^*) - [\int_A (T^* x^*)\, d\mu/\mu(A)]|\, d\mu}{\mu(A)} \leq \varepsilon .$$

Thus $T^*(B(\mathfrak{X}^*))$ satisfies the Bocce criterion, and so as needed, $\mathfrak{X}$ has the complete continuity property. □

We now verify Lemma 2.9.

Proof of Lemma 2.9. Fix A in Σ^+ and f in $L_\infty(\mu)$. Without loss of generality, f is not constant a.e. on A and $\int_A f\, d\mu = 0$. Find P and N in Σ satisfying

$$A = P \cup N, \quad \mu(P) = \frac{\mu(A)}{2} = \mu(N), \quad P \cap N = \varnothing$$

and

$$\int_P f\, d\mu \equiv 2M > 0, \qquad \int_N f\, d\mu \equiv -2M < 0 .$$

Approximate f by a simple function $\tilde{f}(\cdot) = \sum \alpha_i \chi_{A_i}(\cdot)$ satisfying

(1) $\|f - \tilde{f}\|_{L_\infty} < M$,
(2) $\bigcup A_i = A$ and the A_i are disjoint,
(3) $A_i \subset P$ if $i \leq m$ and $A_i \subset N$ if $i > m$ for some positive integer m.

Note that

$$P = \bigcup_{i \le m} A_i \quad \text{and} \quad N = \bigcup_{i > m} A_i .$$

To find the sequence $\{\pi_n\}$, we shall first find an increasing sequence $\{\pi_n^P\}$ of partitions of P and an increasing sequence $\{\pi_n^N\}$ of partitions of N. Then π_n will be the union of π_n^P and π_n^N. To this end, for each A_i obtain an increasing sequence of partitions of A_i:

$$\begin{tikzcd}[column sep=small] & & & A_i \equiv E_1^{i0} \arrow[dll] \arrow[drr] & & & \\ & E_1^{i1} \arrow[dl] \arrow[dr] & & & & E_2^{i1} \arrow[dl] \arrow[dr] & \\ E_1^{i2} & & E_2^{i2} & & E_3^{i2} & & E_4^{i2} \end{tikzcd}$$

$$\cdots$$

such that for $n = 0, 1, 2, \ldots$ and $k = 1, \ldots, 2^n$

$$E_{2k-1}^{i\,n+1} \cup E_{2k}^{i\,n+1} = E_k^{in}, \quad E_{2k-1}^{i\,n+1} \cap E_{2k}^{i\,n+1} = \varnothing, \quad \mu(E_k^{in}) = \frac{\mu(A_i)}{2^n}.$$

For each positive integer n, let π_n^P be the partition of P given by

$$\pi_n^P = \{P_k^n : k = 1, \ldots, 2^n\} \quad \text{where } P_k^n = \bigcup_{i \le m} E_k^{in},$$

π_n^N be the partition of N given by

$$\pi_n^N = \{N_k^n : k = 1, \ldots, 2^n\} \quad \text{where } N_k^n = \bigcup_{i > m} E_k^{in},$$

and π_n be the partition of A given by

$$\pi_n = \pi_n^P \cup \pi_n^N .$$

The sequence $\{\pi_n\}$ has the desired properties. Since

$$\mu(P_k^n) = \sum_{i \le m} \frac{\mu(A_i)}{2^n} = \frac{\mu(P)}{2^n} = \frac{\mu(A)}{2^{n+1}}$$

and

$$\mu(N_k^n) = \sum_{i < m} \frac{\mu(A_i)}{2^n} = \frac{\mu(N)}{2^n} = \frac{\mu(A)}{2^{n+1}},$$

any element in π_n has measure $\mu(A)/2^{n+1}$. Thus $\bigvee \sigma(\pi_n) = \Sigma \cap A$.

As for the other properties, since $\tilde{f}$ takes the value α_i on $E_k^{in} \subset A_i$ we have

$$\begin{aligned}\int_{P_k^n} \tilde{f}\,d\mu &= \sum_{i\le m} \int_{E_k^{in}} \tilde{f}\,d\mu = \sum_{i\le m} \alpha_i \mu(E_k^{in}) \\ &= \frac{1}{2^n} \sum_{i\le m} \alpha_i \mu(A_i) = \frac{1}{2^n} \int_P \tilde{f}\,d\mu > 0\end{aligned}$$

and likewise

$$\int_{N_k^n} \tilde{f}\,d\mu = \frac{1}{2^n} \int_N \tilde{f}\,d\mu < 0.$$

We chose $\tilde{f}$ close enough to f so that the above inequalities still hold when we replace $\tilde{f}$ by f,

$$\begin{aligned}\int_{P_k^n} f\,d\mu &\ge \int_{P_k^n} (\tilde{f} - M)\,d\mu \\ &= \frac{1}{2^n} \int_P \tilde{f}\,d\mu - M\mu(P_k^n) \\ &\ge \frac{1}{2^n} \int_P (f - M)\,d\mu - \frac{M\mu(A)}{2^{n+1}} \\ &= \frac{1}{2^n} \int_P f\,d\mu - \frac{M\mu(A)}{2^{n+1}} - \frac{M\mu(A)}{2^{n+1}} \\ &> \frac{M}{2^n} - \frac{M\mu(A)}{2^n} = \frac{M[1-\mu(A)]}{2^n} \\ &\ge 0\end{aligned}$$

and likewise

$$\int_{N_k^n} f\,d\mu < \frac{M[\mu(A)-1]}{2^n} \le 0.$$

Thus the other properties of the lemma are satisfied since for each n,

$$\begin{aligned}\mu\left(\bigcup\left\{E : E \in \pi_n, \int_E f\,d\mu \ge 0\right\}\right) &= \mu\left(\bigcup\left\{E : E \in \pi_n^P\right\}\right) \\ &= \mu(P) = \frac{\mu(A)}{2}\end{aligned}$$

and so

$$\mu\left(\bigcup\left\{E : E \in \pi_n, \int_E f\,d\mu < 0\right\}\right) = \frac{\mu(A)}{2}.$$

Note that the partitions $\{\pi_n\}$ are nested by construction. □

3. Bushes and trees. In this section, we examine which Banach spaces allow certain types of bushes and trees to grow in them. First let us review some known implications.

A Banach space $\mathfrak{X}$ fails the RNP precisely when a bounded δ-bush grows in $\mathfrak{X}$. Thus if a bounded δ-tree grows in $\mathfrak{X}$ then $\mathfrak{X}$ fails the RNP. The converse is false; the Bourgain-Rosenthal space [**BR**] fails the RNP yet has no bounded δ-trees. However, if $\mathfrak{X}$ is a dual space then the converse does hold.

Bourgain [**B2**] showed that if $\mathfrak{X}$ fails the CCP then a bounded δ-tree grows in $\mathfrak{X}$. The converse is false; the dual of the James Tree space has a bounded δ-tree and the CCP. It is well-known that if a bounded δ-Rademacher tree grows in $\mathfrak{X}$ then $\mathfrak{X}$ fails the CCP. Riddle and Uhl [**RU**] showed that the converse holds in a dual space. This section's main theorem, Theorem 3.1 below, makes precise exactly which types of bushes and trees grow in a Banach space failing the CCP.

THEOREM 3.1. *The following statements are equivalent.*

(1) $\mathfrak{X}$ *fails the CCP.*
(2) *A bounded separated δ-tree grows in $\mathfrak{X}$.*
(3) *A bounded separated δ-bush grows in $\mathfrak{X}$.*
(4) *A bounded δ-Rademacher tree grows in $\mathfrak{X}$.*

The remainder of this section is devoted to proving Theorem 3.1. That (1) implies (2) will follow from Theorem 3.2 below. All the other implications are straightforward and will be verified shortly. As usual, we start with some definitions.

Perhaps it is easiest to define a bush via martingales. If $\{\pi_n\}_{n\geq 0}$ is an increasing sequence of finite positive interval partitions of $[0, 1)$ with $\bigvee \sigma(\pi_n) = \Sigma$ and $\pi_0 = \{\Omega\}$ and if $\{f_n, \sigma(\pi_n)\}_{n\geq 0}$ is an $\mathfrak{X}$-valued martingale, then each f_n has the form

$$f_n = \sum_{E\in\pi_n} x_E^n \chi_E$$

and the system

$$\{x_E^n : n = 0, 1, 2, \ldots \text{ and } E \in \pi_n\}$$

is a bush in $\mathfrak{X}$. Moreover, every bush is realized this way. A bush is a *δ-bush* if the corresponding martingale satisfies for each positive

While typing this paper, I learned that H. P. Rosenthal has also recently obtained the result that if $\mathfrak{X}$ fails the CCP then a bounded δ-Rademacher tree grows in $\mathfrak{X}$.

integer n

(i) $$\|f_n(t) - f_{n-1}(t)\| > \delta .$$

A bush is a *separated δ-bush* if there exists a sequence $\{x_n^*\}_{n\geq 1}$ in $S(\mathfrak{X}^*)$ such that the corresponding martingale satisfies for each positive integer n

(ii) $$|x_n^*(f_n(t) - f_{n-1}(t))| > \delta .$$

In this case we say that the bush is separated by $\{x_n^*\}$. Clearly a separated δ-bush is also a δ-bush.

Observation that (3) *implies* (1) *in Theorem* 3.1. If a bounded separated δ-bush grows in a subset D of $\mathfrak{X}$, then condition (ii) guarantees that the corresponding D-valued martingale $\{f_n, \sigma(\pi_n)\}$ is not Pettis-Cauchy since

$$\|f_n - f_{n-1}\|_{\text{Pettis}} \geq \int_\Omega |x_n^*(f_n(t) - f_{n-1}(t))| \, d\mu > \delta .$$

Thus, if a bounded separated δ-bush grows in $\mathfrak{X}$ then $\mathfrak{X}$ fails the CCP (Fact 1.5). □

If each π_n is the nth dyadic partition then we call the bush a (dyadic) tree. Let us rephrase the above definitions for this case, without the help of martingales. A *tree* in $\mathfrak{X}$ is a system of the form $\{x_k^n : n = 0, 1, \dots ; k = 1, \dots, 2^n\}$ satisfying for $n = 1, 2, \dots$ and $k = 1, \dots, 2^{n-1}$

(iii) $$x_k^{n-1} = \frac{x_{2k-1}^n + x_{2k}^n}{2} .$$

Condition (iii) guarantees that $\{f_n\}$ is indeed a martingale. It is often helpful to think of a tree diagrammatically:

$$\begin{tikzcd}[column sep=tiny]
 & & & & & & & x_1^0 \arrow[dllll] \arrow[drrrr] & & & & & & & \\
 & & & x_1^1 \arrow[dll] \arrow[drr] & & & & & & & & x_2^1 \arrow[dll] \arrow[drr] & & & \\
 & x_1^2 \arrow[dl] \arrow[dr] & & & & x_2^2 \arrow[dl] \arrow[dr] & & & & x_3^2 \arrow[dl] \arrow[dr] & & & & x_4^2 \arrow[dl] \arrow[dr] & \\
x_1^3 & & x_2^3 & & x_3^3 & & x_4^3 & & x_5^3 & & x_6^3 & & x_7^3 & & x_8^3
\end{tikzcd}$$

$$\dots$$

It is easy to see that (iii) is equivalent to

$$\text{(iii}'\text{)} \qquad x^n_{2k-1} - x^n_{2k} = 2(x^n_{2k-1} - x^{n-1}_k) = 2(x^{n-1}_k - x^n_{2k}) .$$

A tree $\{x^n_k\}$ is a *δ-tree* if for $n = 1, 2, \ldots$ and $k = 1, \ldots, 2^{n-1}$

$$\text{(iv)} \qquad \|x^n_{2k-1} - x^{n-1}_k\| \equiv \|x^n_{2k} - x^{n-1}_k\| > \delta .$$

An appeal to (iii$'$) shows that (iv) is equivalent to

$$\text{(iv}'\text{)} \qquad \|x^n_{2k-1} - x^n_{2k}\| > 2\delta .$$

A tree $\{x^n_k\}$ is a *separated δ-tree* if there exists a sequence $\{x^*_n\}_{n\geq 1}$ in $S(\mathfrak{X}^*)$ such that for $n = 1, 2, \ldots$ and $k = 1, \ldots, 2^{n-1}$

$$\text{(v)} \qquad |x^*_n(x^n_{2k-1} - x^{n-1}_k)| \equiv |x^*_n(x^n_{2k} - x^{n-1}_k)| > \delta .$$

Another appeal to (iii$'$) shows that (v) is equivalent to

$$\text{(v}'\text{)} \qquad |x^*_n(x^n_{2k-1} - x^n_{2k})| > 2\delta .$$

Furthermore, by switching x^n_{2k-1} and x^n_{2k} when necessary, we may assume that (v$'$) is equivalent to

$$\text{(v}''\text{)} \qquad x^*_n(x^n_{2k-1} - x^n_{2k}) > 2\delta .$$

Since a separated δ-tree is also a separated δ-bush, (2) implies (3) in Theorem 3.1.

A tree $\{x^n_k : n = 0, 1, \ldots ; k = 1, \ldots, 2^n\}$ is called a *δ-Rademacher tree* **[RU]** if for each positive integer n

$$\left\| \sum_{k=1}^{2^{n-1}} (x^n_{2k-1} - x^n_{2k}) \right\| > 2^n \delta .$$

Perhaps a short word on the connection between Rademacher trees and the Rademacher functions $\{r_n\}$ is in order. In light of our discussion in §1, there is a one-to-one correspondence between all bounded trees in $\mathfrak{X}$ and all bounded linear operators from L_1 into $\mathfrak{X}$. If $\{x^n_k\}$ is a bounded tree in $\mathfrak{X}$ with associated operator T, then it is easy to verify that $\{x^n_k\}$ is a δ-Rademacher tree precisely when $\|T(r_n)\| > \delta$ for all positive integers n.

Fact that (4) *implies* (1) *in Theorem* 3.1 **[RU]**. Let $\{f_n\}$ be the (dyadic) martingale associated with a δ-Rademacher tree $\{x^n_k\}$. If x^*

is in $\mathfrak{X}^*$ and I_k^n is the dyadic interval $[(k-1)/2^n, k/2^n)$ then

$$\begin{aligned}
\int_\Omega |x^*(f_n - f_{n-1})|\,d\mu &= \sum_{k=1}^{2^{n-1}} \int_{I_k^{n-1}} |x^*(f_n - f_{n-1})|\,d\mu \\
&= \sum_{k=1}^{2^{n-1}} \left[\int_{I_{2k-1}^n} |x^*(x_{2k-1}^n - x_k^{n-1})|\,d\mu \right. \\
&\qquad\qquad \left. + \int_{I_{2k}^n} |x^*(x_{2k}^n - x_k^{n-1})|\,d\mu \right] \\
&= \frac{1}{2^n} \sum_{k=1}^{2^{n-1}} [|x^*(x_{2k-1}^n - x_k^{n-1})| + |x^*(x_{2k}^n - x_k^{n-1})|] \\
&= \frac{1}{2^n} \sum_{k=1}^{2^{n-1}} |x^*(x_{2k-1}^n - x_{2k}^n)| \quad \text{by (iii}'\text{)} \\
&\geq \frac{1}{2^n} \left| x^* \left(\sum_{k=1}^{2^{n-1}} (x_{2k-1}^n - x_{2k}^n) \right) \right| .
\end{aligned}$$

From this we see that $\{f_n\}$ is not Cauchy in the Pettis norm since

$$\begin{aligned}
\|f_n - f_{n-1}\|_{\text{Pettis}} &= \sup_{x^* \in B(\mathfrak{X}^*)} \int_\Omega |x^*(f_n - f_{n-1})|\,d\mu \\
&\geq \sup_{x^* \in B(\mathfrak{X}^*)} \frac{1}{2^n} \left| x^* \left(\sum_{k=1}^{2^{n-1}} (x_{2k-1}^n - x_{2k}^n) \right) \right| \\
&= \frac{1}{2^n} \left\| \sum_{k=1}^{2^{n-1}} (x_{2k-1}^n - x_{2k}^n) \right\| \\
&> \frac{1}{2^n} 2^n \delta = \delta .
\end{aligned}$$

Thus if a bounded δ-Rademacher tree grows in a subset D of $\mathfrak{X}$, then there is a bounded D-valued martingale that is not Pettis-Cauchy and so $\mathfrak{X}$ fails the CCP (Fact 1.5). □

Observation that (2) *implies* (4) *in Theorem* 3.1. A separated δ-tree can easily be reshuffled so that it is a δ-Rademacher tree. For if $\{x_k^n\}$ is a separated δ-tree then we may assume, by switching x_{2k-1}^n and x_{2k}^n when necessary, that there is a sequence $\{x_n^*\}$ in $S(\mathfrak{X}^*)$ satisfying

$$x_n^*(x_{2k-1}^n - x_{2k}^n) > 2\delta .$$

With this modification $\{x_k^n\}$ is a δ-Rademacher tree since

$$\left\|\sum_{k=1}^{2^{n-1}}(x_{2k-1}^n - x_{2k}^n)\right\| \geq \left|\sum_{k=1}^{2^{n-1}} x_n^*(x_{2k-1}^n - x_{2k}^n)\right|$$
$$= \sum_{k=1}^{2^{n-1}} x_n^*(x_{2k-1}^n - x_{2k}^n) > \sum_{k=1}^{2^{n-1}} 2\delta = 2^n\delta . \qquad \square$$

It should be noted that a bounded $\tilde{\delta}$-Rademacher tree need neither be a δ-tree nor a separated δ-tree. For example, consider the c_0-valued dyadic martingale $\{f_n\}$ given by

$$f_n = (s_0, \dots, s_n, 0, 0, \dots),$$

where the function s_n from $[0, 1]$ into $[-1, 1]$ is defined by

$$s_n = \begin{cases} (-1)^k 2^{-n} & \text{if } \omega \in I_k^n \text{ with } k \leq 2; \\ (-1)^k & \text{if } \omega \in I_k^n \text{ with } k > 2. \end{cases}$$

The tree associated with $\{f_n\}$ is a $\frac{1}{4}$-Rademacher tree but is neither a δ-tree nor a separated δ-tree for any positive δ. Thus, since a δ-tree grows in a space failing the CCP, the notion of a separated δ-tree is more desirable than that of a δ-Rademacher tree for characterizing the CCP.

To complete the proof of Theorem 3.1, we need only to show that (1) implies (2). Towards this end, let $\mathfrak{X}$ fail the CCP. An appeal to Theorem 2.7 gives that there is a bounded non-midpoint-Bocce-dentable subset of $\mathfrak{X}$. In such a set, we can construct a separated δ-tree. This construction is made precise in the following theorem.

THEOREM 3.2. *A separated δ-tree grows in a non-midpoint-Bocce-dentable set.*

Proof. Let D be a subset of $\mathfrak{X}$ that is not midpoint Bocce dentable. Accordingly, there is a $\delta > 0$ satisfying:

($*$) for each finite subset F of D there is a norm one linear functional x_F^* such that each x in F has the form $x = (x_1 + x_2)/2$ with $|x_F^*(x_1 - x_2)| > \delta$ for a suitable choice of x_1 and x_2 in D.

We shall use the property ($*$) to construct a tree $\{x_k^n : n = 0, 1, \dots ; k = 1, \dots, 2^n\}$ in D that is separated by a sequence $\{x_n^*\}_{n \geq 1}$ of norm one linear functionals.

Towards this construction, pick an arbitrary x_1^0 in D. Apply $(*)$ with $F = \{x_1^0\}$ and find $x_F^* = x_1^*$. Property $(*)$ provides x_1^1 and x_2^1 in D satisfying

$$x_1^0 = \frac{x_1^1 + x_2^1}{2} \quad \text{and} \quad |x_1^*(x_1^1 - x_2^1)| > \delta .$$

Next apply $(*)$ with $F = \{x_1^1, x_2^1\}$ and find $x_F^* = x_2^*$. For $k = 1$ and 2, property $(*)$ provides x_{2k-1}^2 and x_{2k}^2 in D satisfying

$$x_k^1 = \frac{x_{2k-1}^2 + x_{2k}^2}{2} \quad \text{and} \quad |x_2^*(x_{2k-1}^2 + x_{2k}^2)| > \delta .$$

Instead of giving a formal inductive proof we shall be satisfied by finding $x_1^3, x_2^3, \ldots, x_8^3$ in D along with x_3^*. Apply $(*)$ with $F = \{x_1^2, x_2^2, x_3^2, x_4^2\}$ and find $x_F^* = x_3^*$. For $k = 1, 2, 3$ and 4, property $(*)$ provides x_{2k-1}^3 and x_{2k}^3 in D satisfying

$$x_k^2 = \tfrac{1}{2}(x_{2k-1}^3 + x_{2k}^3) \quad \text{and} \quad |x_3^*(x_{2k-1}^3 - x_{2k}^3)| > \delta .$$

It is now clear that a separated δ-tree grows in such a set D. □

REMARK 3.3. Theorem 2.7 presents several dentability characterizations of the CCP. Our proof that (1) implies (2) in Theorem 3.1 uses part of one of these characterizations; namely, if $\mathfrak{X}$ fails the CCP then there is a bounded non-midpoint-Bocce-dentable subset of $\mathfrak{X}$. If $\mathfrak{X}$ fails the CCP, then there is also a bounded non-weak-norm-one-dentable subset of $\mathfrak{X}$ (Theorem 2.7). In the closed convex hull of such a set we can construct a martingale that is not Pettis-Cauchy [**PU**, Theorem II.7]; furthermore, the bush associated with this martingale is a separated δ-bush. However, it is unclear whether this martingale is a dyadic martingale, thus the separated δ-bush may not be a tree. If $\mathfrak{X}$ fails the CCP, then there is also a bounded non-Bocce-dentable subset of $\mathfrak{X}$ (Theorem 2.7). In such a set we can construct a martingale that is not Pettis-Cauchy (Theorem 2.8), but it is unclear whether the bush associated with this martingale is a separated δ-bush.

REMARK 3.4. The δ-tree that Bourgain [**B2**] constructed in a space failing the CCP is neither a separated δ-tree nor a δ-Rademacher tree since the operator associated with his tree is Dunford-Pettis.

4. Localization. We now localize the results thus far. We define the CCP for bounded subsets of $\mathfrak{X}$ by examining the behavior of certain bounded linear operators from L_1 into $\mathfrak{X}$. Before determining precisely which operators let us set some notation and consider an example.

Let $F(L_1)$ denote the positive face of the unit ball of L_1, i.e.

$$F(L_1) = \{f \in L_1 : f \geq 0 \text{ a.e. and } \|f\| = 1\},$$

and let Δ denote the subset of $F(L_1)$ given by

$$\Delta = \left\{ \frac{\chi_E}{\mu(E)} : E \in \Sigma^+ \right\}.$$

Note that the L_1-norm closed convex hull of Δ is $F(L_1)$.

Some care is needed in localizing the CCP. The example below (due to Stegall) illustrates the trouble one can encounter in localizing the RNP.

EXAMPLE 4.1. We would like to define the RNP for sets in such a way that if a subset D has the RNP then the $\overline{\text{co}}\,D$ also has the RNP. For now, let us agree that a subset D of $\mathfrak{X}$ has the RNP if all bounded linear operators from L_1 into $\mathfrak{X}$ with $T(\Delta) \subset D$ are representable. Let $\mathfrak{X}$ be any separable Banach space without the RNP (e.g. L_1). Renorm $\mathfrak{X}$ to be a strictly convex Banach space. Let D be the unit sphere of $\mathfrak{X}$ and $T\colon L_1 \to \mathfrak{X}$ satisfy $T(\Delta) \subset D$. Since $\mathfrak{X}$ is strictly convex, it is easy to verify that $T(\Delta)$ is a singleton in $\mathfrak{X}$. Thus T is representable and so D has the RNP. If this is to imply that $\overline{\text{co}}\,D$ also has the RNP, then the unit ball of $\mathfrak{X}$ would have the RNP. But if the unit ball of $\mathfrak{X}$ has the RNP then $\mathfrak{X}$ has the RNP; but, $\mathfrak{X}$ fails the RNP. The same problem arises if we replace $T(\Delta) \subset D$ by either $T(F(L_1)) \subset D$ or $T(B(L_1)) \subset D$.

Because of such difficulties, we localize propeties to nonconvex sets by considering their closed convex hull. We now make precise the localized definitions.

DEFINITION 4.2. If D is a closed bounded convex subset of $\mathfrak{X}$, then D has the *complete continuity property* if all bounded linear operators T from L_1 into $\mathfrak{X}$ satisfying $T(\Delta) \subset D$ are Dunford-Pettis. If D is an arbitrary bounded subset of $\mathfrak{X}$, then D has the *complete continuity property* if the $\overline{\text{co}}\,D$ has the complete continuity property.

The RNP for subsets is defined similarly. We obtain equivalent formulations of the above definitions by replacing $T(\Delta) \subset D$ with $T(F(L_1)) \subset D$. Because of the definitions we restrict out attention to closed bounded convex subsets of $\mathfrak{X}$.

We can derive a martingale characterization of the CCP for a closed bounded convex subset K of $\mathfrak{X}$. As in §1, fix an increasing sequence $\{\pi_n\}_{n\geq 0}$ of finite positive interval partitions of Ω such that $\bigvee \sigma(\pi_n) = \Sigma$ and $\pi_0 = \{\Omega\}$. Set $\mathscr{F}_n = \sigma(\pi_n)$. It is easy to see that a martingale $\{f_n, \mathscr{F}_n\}$ takes values in K precisely when the corresponding bounded linear operator T satisfies $T(\Delta) \subset K$. In light of Fact 1.5, we now have the following fact.

FACT 4.3. If K is a closed bounded convex subset of $\mathfrak{X}$, then K has the CCP precisely when all K-valued martingales are Cauchy in the Pettis norm.

Theorem 2.7 localizes to provide the following characterization.

THEOREM 4.4. *Let K be a closed bounded convex subset of $\mathfrak{X}$. The following statements are equivalent.*

(1) *K has the CCP.*
(2) *All the subsets of K are weak-norm-one dentable.*
(3) *All the subsets of K are midpoint Bocce dentable.*
(4) *All the subsets of K are Bocce dentable.*

Proof. It is clear from the definitions that (2) implies (3) and that (4) implies (3). Theorem 2.8 and Fact 4.3 show that (1) implies (4) while [**PU**, Theorem II.7] and Fact 4.3 show that (1) implies (2). So we only need to show that (3) implies (1). For this, slight modifications in the proof of Theorem 2.10 suffice.

Let all subsets of K be midpoint Bocce dentable. Fix a bounded linear operator T from L_1 into $\mathfrak{X}$ satisfying $T(\Delta) \subset K$. We shall show that the subset $T^*(B(\mathfrak{X}^*))$ of L_1 satisfies the Bocce criterion. Then an appeal to Fact 1.1.7 gives that K has the complete continuity property. To this end, fix $\varepsilon > 0$ and B in Σ^+. Let F denote the vector measure from Σ into $\mathfrak{X}$ given by $F(E) = T(\chi_E)$. Since $T(\Delta) \subset K$, the set $\{\frac{F(E)}{\mu(E)} : E \subset B \text{ and } E \in \Sigma^+\}$ is a subset of K and thus is midpoint Bocce dentable. The proof now proceeds as the proof of Theorem 2.10. □

Towards a localized tree characterization, let K be a closed bounded convex subset of $\mathfrak{X}$. If K fails the CCP, then there is a subset of K that is not midpoint Bocce dentable (Theorem 4.4) and hence a separated δ-tree grows in K (Theorem 3.2). A separated δ-tree is a separated δ-bush and, with slight modifications, a δ-Rademacher tree. In light of our discussion in §3, if a separated δ-bush or a δ-Rademacher tree grows in K, then the associated K-valued martingale is not Pettis-Cauchy and so K fails the CCP (Fact 4.3). Thus Theorem 3.1 localizes to provide the following characterization.

THEOREM 4.5. *Let K be a closed bounded convex subset of $\mathfrak{X}$. The following statements are equivalent.*

(1) *K fails the CCP.*
(2) *A separated δ-tree grows in K.*
(3) *A separated δ-bush grows in K.*
(4) *A δ-Rademacher tree grows in K.*

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