ULTRAPRODUCTS AND SMALL BOUND PERTURBATIONS

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It is very well-known that two real Banach spaces are isometric if
and only if they are linearly-isometric or that two uniform algebras
are linearly-isometric if and only if they are isomorphic as algebras.
These and similar classical "isometric" results have been extended by
E. Behrends, M. Cambern, J. Gevirtz, R. Rochberg, the author and
others to "almost isometric" cases. Proofs of the extended results are
usually quite technical. In this note we show that using ultraproducts
of Banach spaces we can in some cases deduce an "almost isometric"
result from the classical one in just a few lines.

0. It is a well-known classical result of Ulam that an isometry T
from a real Banach space X onto a real Banach space Y with $T(0) =
0$ is automatically linear. More recently, in 1982, Gevirtz [5] proved
that this result is stable:

THEOREM. *Let T be a map from a Banach space X onto a Banach
space Y with $T(0) = 0$ such that*

$$(1 - \varepsilon)\|x - y\| \le \|Tx - Ty\| \le (1 + \varepsilon)\|x - y\|, \quad \text{for } x, y \in X,$$

then

$$\|T(x + y) - Tx - Ty\| \le \varepsilon'(\|x\| + \|y\|), \quad \text{for } x, y \in X$$

where $\varepsilon' \to 0$ as $\varepsilon \to 0$.

The proof of the above result repeats, roughly speaking, the basic
idea of Ulam's proof but is much longer and much more technical.
The intent of this note is to draw attention to the method of ultra-
products of Banach spaces. Using this method we can extend in just
a few lines some "isometric" results to "almost isometric" cases. This
includes the theorem of Gevirtz.

1. In this section we give a definition of the ultraproduct of Banach
spaces and list some basic results. We refer to the paper by Heinrich
[6] for a more extended exposition.

We denote by $\mathbb{N}$ the set of all positive integers and by $\mathscr{F}$ a non-
prime ultrafilter of subsets of $\mathbb{N}$. That is, we assume that $\mathscr{F}$ is a

proper subset of $2^{\mathbb{N}}$ which does not contain a one point set and such that

$$A \cap B \in \mathscr{F} \qquad \text{if } A, B \in \mathscr{F},$$
$$B \in \mathscr{F} \qquad \text{if } B \supseteq A \in \mathscr{F},$$
$$A \in \mathscr{F} \text{ or } B \in \mathscr{F} \quad \text{if } A \cup B \in \mathscr{F}.$$

Throughout this paper we assume $\mathscr{F}$ is fixed.

DEFINITION. Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence of complex numbers. We write

$$\lim_{\mathscr{F}} a_n = g \quad \text{if } \forall \varepsilon > 0 \; \exists A \in \mathscr{F} \; \forall n \in A \quad |a_n - g| \leq \varepsilon.$$

It is easy to observe that $\lim_{\mathscr{F}} a_n$ exists for any bounded sequence of complex numbers. To get a useful alternative definition let $p \in \beta \mathbb{N} \backslash \mathbb{N}$, where $\beta \mathbb{N}$ is the maximal compactification of $\mathbb{N}$. Since $a = (a_n)_{n=1}^{\infty}$ is a continuous bounded function on $\mathbb{N}$ it can be uniquely extended to a continuous function $\tilde{a}$ on $\beta \mathbb{N}$. We have $\tilde{a}(p) = \lim_{\mathscr{F}} a_n$ where $\mathscr{F}$ is the set of all neighborhoods of p, restricted to $\mathbb{N}$.

DEFINITION. Let $(X_n)_{n=1}^{\infty}$ be a sequence of normed spaces and let $m(X_n)$ be the space of all norm bounded sequences $(x_n)_{n=1}^{\infty}$ with $x_n \in X_n$. We introduce a seminorm $\|\cdot\|_{\mathscr{F}}$ on $m(X_n)$ by $\|(x_n)_{n=1}^{\infty}\|_{\mathscr{F}} = \lim_{\mathscr{F}} \|x_n\|$. The ultraproduct $\prod_{\mathscr{F}} X_n$ of $(X_n)_{n=1}^{\infty}$ is the quotient space of the space $m(X_n) \mod \ker \|\cdot\|_{\mathscr{F}}$.

DEFINITION. Let $X_n, Y_n, n \in \mathbb{N}$, be sequences of normed spaces and let $T_n \colon X_n \to Y_n$ be a sequence of maps such that

$$(1) \qquad \|T_n(x_n)\| \leq K \|x_n\| \quad \text{for } n \in \mathbb{N}, \; x_n \in X_n.$$

(We do not assume that T_n are linear.) Let $\prod_{\mathscr{F}} T_n$ denote the map from $\prod_{\mathscr{F}} X_n$ into $\prod_{\mathscr{F}} Y_n$ defined by $\prod_{\mathscr{F}} T_n([x_n]_{\mathscr{F}}) = [T_n(x_n)]_{\mathscr{F}}$.

For $(x_n)_{n=1}^{\infty} \in m(X_n)$ we denote by $[x_n]_{\mathscr{F}}$ the corresponding element of $\prod_{\mathscr{F}} X_n$. If X_n are equal to a fixed normed space X then $\prod_{\mathscr{F}} X = \prod_{\mathscr{F}} X_n$ is called an ultrapower of X.

From (1) it follows that $\prod_{\mathscr{F}} T_n$ is well-defined and that

$$(2) \qquad \left\| \prod_{\mathscr{F}} T_n([x_n]_{\mathscr{F}}) \right\|_{\mathscr{F}} \leq K \|[x_n]_{\mathscr{F}}\|_{\mathscr{F}}, \qquad [x_n]_{\mathscr{F}} \in \prod_{\mathscr{F}} X_n.$$

Note that if X_n is not only a Banach space but also a Banach algebra then we can carry this multiplicative structure to $\prod_{\mathscr{F}} X_n$ by defining

$$[x_n]_{\mathscr{F}} \cdot [y_n]_{\mathscr{F}} = [x_n \cdot y_n]_{\mathscr{F}}, \quad \text{for } [x_n]_{\mathscr{F}}, [y_n]_{\mathscr{F}} \in \prod_{\mathscr{F}} X_n.$$

Here is a list of some basic properties of ultraproducts:

$1°$. $\prod_{\mathscr{F}} X_n$ is a Banach space, that is $\prod_{\mathscr{F}} X_n$ is complete even if the X_n are not.

$2°$. A map from X into $\prod_{\mathscr{F}} X$ defined by $x \mapsto [x]_{\mathscr{F}}$ (mapping x onto the sequence constantly equal to x) is an isometric embedding of X into $\prod_{\mathscr{F}} X$. This map is surjective if and only if X is finite dimensional.

$3°$. If $T_n \colon X_n \to Y_n$ are all linear then $\prod_{\mathscr{F}} T_n$ is a linear map with $\| \prod_{\mathscr{F}} T_n \| = \lim_{\mathscr{F}} \| T_n \|$.

$4°$. If $T_n \colon X_n \to Y_n$ is a sequence of invertible maps with

$$\sup \left\{ \frac{\| T_n x_n \|}{\| x_n \|}, \frac{\| x_n \|}{\| T_n x_n \|} : x_n \in X_n, \ x_n \neq 0 \right\} < \infty$$

then $\prod_{\mathscr{F}} T_n$ is invertible and $(\prod_{\mathscr{F}} T_n)^{-1} = \prod_{\mathscr{F}} (T_n^{-1})$.

$5°$. If $X_n = C(K_n)$ then $\prod_{\mathscr{F}} X_n = C(K)$, where K is compact.

$6°$. If X_n are closed subalgebras of $C(K_n)$, then $\prod_{\mathscr{F}} X_n$ is a closed subalgebra of $C(K)$.

$7°$. With any element $[x_n^*]_{\mathscr{F}}$ of $\prod_{\mathscr{F}} X_n^*$ we can associate a linear functional on $\prod_{\mathscr{F}} X_n$ by putting $[x_n^*]_{\mathscr{F}}([x_n]) = \lim_{\mathscr{F}} x_n^*(x_n)$ for $[x_n]_{\mathscr{F}} \in \prod_{\mathscr{F}} X_n$. This defines a linear isometric embedding of $\prod_{\mathscr{F}} X_n^*$ into $(\prod_{\mathscr{F}} X_n)^*$ which is surjective if the spaces X_n are superreflexive.

Proofs of properties $1°-7°$ are easy exercises, we show here only $3°$ and $4°$ to get some additional information about the structure of the algebra $\prod_{\mathscr{F}} A_n \subseteq \prod_{\mathscr{F}} C(K_n)$. The algebra $m(C(K_n))$ consists of all continuous bounded functions defined on $(\bigcup_{n=1}^{\infty} K_n)$, the disjoint union of K_n. So $m(C(K_n))$ can be identified with the algebra of all continuous funtions on $S = \beta(\bigcup_{n=1}^{\infty} K_n)$. The kernel of the seminorm $\|(f_n)\|_{\mathscr{F}} = \lim_{\mathscr{F}} \| f_n \|$ on $m(C(K_n)) = C(S)$ is a closed ideal. Any closed ideal J in $C(S)$ is of the form $J = J_K = \{ f \in C(S) : f_{|K} \equiv 0 \}$ where $K = \overline{K} \subseteq S$. We also have $C(S)/J_k \cong C(K)$. Hence, $\prod_{\mathscr{F}} C(K_n)$ can be identified with a subalgebra of $C(K)$ where $K \subset \beta(\bigcup K_n) \setminus \bigcup K_n$. Now, since A_n is a subalgebra of $C(K_n)$, $\prod_{\mathscr{F}} A_n$ is a subalgebra of $C(K)$.

2. In this section we give some applications of the method of ultraproducts. We start with the proof of the theorem of Gevirtz. Assume the result is false. Then there are sequences of Banach spaces X_n and

Y_n, a sequence $T_n\colon X_n \to Y_n$ of surjective maps with $T_n 0 = 0$ and

$$(3) \qquad \left(1 - \frac{1}{n}\right)\|x - y\| \le \|T_n x - T_n y\|$$

$$\le \left(1 + \frac{1}{n}\right)\|x - y\|, \qquad x, y \in X_n,$$

and sequences $x_n \in X_n$, $y_n \in Y_n$ with

$$(4) \qquad \|T_n(x_n + y_n) - T_n x_n - T_n y_n\| \ge \varepsilon'(\|x_n\| + \|y_n\|), \qquad n \in \mathbb{N},$$

where $\varepsilon' > 0$ is a fixed number.

Without loss of generality, by putting

$$\tilde{T}_n(x) = \frac{1}{\|x_n\| + \|y_n\|} T_n((\|x_n\| + \|y_n\|)x), \qquad x \in X_n,$$

and

$$\tilde{x}_n = \frac{x_n}{\|x_n\| + \|y_n\|}, \qquad \tilde{y}_n = \frac{y_n}{\|x_n\| + \|y_n\|}$$

in place of T_n, x_n and y_n, respectively, we can assume that $\|x_n\| + \|y_n\| = 1$ for all $n \in \mathbb{N}$.

Put

$$T_\infty = \prod_{\mathcal{F}} T_n \colon \prod_{\mathcal{F}} X_n \to \prod_{\mathcal{F}} Y_n, \qquad x_\infty = [x_n]_{\mathcal{F}}, \qquad y_\infty = [y_n]_{\mathcal{F}}.$$

By (3) and the property 4°, T_∞ is a surjective isometry. By the theorem of Ulam T_∞ is linear, but from (4) we get

$$\|T_\infty(x_\infty + y_\infty) - T_\infty(x_\infty) - T_\infty(y_\infty)\|_{\mathcal{F}}$$
$$= \lim_{\mathcal{F}} \|T_n(x_n + y_n) - T_n(x_n) - T_n(y_n)\| \ge \varepsilon' > 0$$

which is a contradiction.

To formulate the next result we need some definitions.

By a uniform algebra we mean a sup-norm closed subalgebra with unit, of the algebra $C(K)$ of all continuous complex functions defined on a compact set K.

A linear map T from a Banach space X onto a Banach space Y is called ε-isometry if $\|T\| \le 1 + \varepsilon$ and $\|T^{-1}\| \le 1 + \varepsilon$.

A linear map T from a Banach algebra A into a Banach algebra B is called ε-multiplicative if

$$(5) \qquad \|T(fg) - T(f) \cdot T(g)\| \le \varepsilon \|f\| \|g\|, \qquad f, g \in A.$$

It is well-known that, in general, a linear and multiplicative map $T\colon A \to B$ need not be continuous [14]. It is also well-known that, if

B is commutative and semisimple then a linear, multiplicative map $T: A \to B$ is automatically continuous [15]. The same is true for ε-multiplicative maps. In [8, p. 37] it is shown that an ε-multiplicative map T from a Banach algebra A into a uniform algebra is automatically continuous, so by (5) we have $\|T\| \le 1 + \varepsilon$. The general case of a semisimple commutative algebra B follows easily from this by the same arguments (closed graph theorem) as in the multiplicative case.

THEOREM 2. *Let A and B be uniform algebras. If $T: A \to B$ is ε-multiplicative then T is an ε'-isometry. If $T: A \to B$ is an ε-isometry then $\widetilde{T}: A \to B$ defined by $\widetilde{T}(f) = (Tf)(T1)^{-1}$ is ε''-multiplicative. Here ε, ε', ε'' tend to zero simultaneously.*

This theorem was proved in 1979 by R. Rochberg [13] under some additional assumptions about A and B. The general case was proved in 1983 in [7] (see also [8, p. 35]). On the other hand, the isometric case of this theorem, that is the case where $\varepsilon = \varepsilon' = \varepsilon'' = 0$, is a classical result proven in 1959 by Nagasawa [12]. Using ultraproducts we can simply reduce the general case to the isometric one. We show here, by contradiction, the implication in one direction, the second being equally obvious.

Assume $T_n: A_n \to B_n$ is a $\frac{1}{n}$-isometry between function algebras A_n and B_n.

The map $\prod_{\mathscr{F}} T_n: \prod_{\mathscr{F}} A_n \to \prod_{\mathscr{F}} B_n$ is a linear surjective isometry between function algebras so, by the classical result [15] $\prod_{\mathscr{F}} T_n([1]_{\mathscr{F}}) = [T_n(1)]_{\mathscr{F}}$ is an invertible norm one element of $\prod_{\mathscr{F}} B_n$, with the norm of its inverse equal also to one. Let F_n be an element of $\mathfrak{M}(B_n)$, the space of all linear-multiplicative functionals on B_n. Since $\prod_{\mathscr{F}} F_n \in \mathfrak{M}(\prod_{\mathscr{F}} B_n)$, we have

$$1 = \left| \prod_{\mathscr{F}} F_n([T_n(1)]_{\mathscr{F}}) \right| = \left| \lim_{\mathscr{F}} F_n(T_n(1)) \right|,$$

so for all sufficiently large n, $T_n(1)$ is invertible in B_n with

$$\lim_{\mathscr{F}} \|T_n(1)\| = 1 \quad \text{and} \quad \lim_{\mathscr{F}} \|(T_n(1))^{-1}\| = 1.$$

Hence, we can define a map $\widetilde{T}_n: A_n \to B_n$ by $\widetilde{T}_n f = (T_n f)(T_n(1))^{-1}$ and we have $\lim_{\mathscr{F}} \|\widetilde{T}_n\| = 1 = \lim_{\mathscr{F}} \|\widetilde{T}_n^{-1}\|$.

Assume there are $\varepsilon_0 > 0$ and f_n, $g_n \in A_n$, $\|f_n\| = 1 = \|g_n\|$ such that

$$\|\widetilde{T}_n(f_n \cdot g_n) - \widetilde{T}_n(f_n)\widetilde{T}_n(g_n)\| \ge \varepsilon_0.$$

Then

$$(6) \qquad \left\| \prod_{\mathcal{F}} \tilde{T}_n([f_n]_{\mathcal{F}}[g_n]_{\mathcal{F}}) - \prod_{\mathcal{F}} \tilde{T}_n([f_n]_{\mathcal{F}}) \prod_{\mathcal{F}} \tilde{T}_n([g_n]_{\mathcal{F}}) \right\| \geq \varepsilon_0 > 0.$$

On the other hand $\prod_{\mathcal{F}} \tilde{T}_n$ is a linear isometry from $\prod_{\mathcal{F}} A_n$ onto $\prod_{\mathcal{F}} B_n$ which maps the unit onto the unit, so again by the Nagasawa theorem it is multiplicative, which contradicts (6).

A linear projection $P : X \to X$ is called ε-L^p-projection, $1 \leq p \leq \infty$, if

$$(1 - \varepsilon)\|x\| \leq (\|Px\|^p + \|x - Px\|^p)^{1/p} \leq (1 + \varepsilon)\|x\|, \qquad x \in X,$$

with the obvious modification for $p = \infty$. L^p-projections and ε-L^p-projections play important roles in studying structure, isometries and small-bound isomorphisms of various Banach spaces. The main result here is due to E. Behrends [2]. He proved that if $\dim X > 2$ and $p \neq 2$ then X admits a non-trivial L^p-projection for at most one p and any two such projections commute. In [4] this result was extended to ε-L^p-projection as follows.

THEOREM 3. *Let X be a Banach space with $\dim X > 2$. Let $1 \leq p, q \leq \infty$, $p \neq 2$, let $P, Q : X \to X$ be ε-L^p and ε-L^q projections, respectively. Then*

$$|p - q| \leq \varepsilon'(p) \quad and \quad \|PQ - QP\| \leq \varepsilon'(p), \quad where \; \varepsilon' \to 0 \; as \; \varepsilon \to 0.$$

Using the method of ultraproducts we can deduce the above theorem from the result of Behrends in what is now an obvious way. It is enough to notice that $\prod_{\mathcal{F}} P_n$ is an L^p-projection if P_n is an $\frac{1}{n}$-L^{p_n}-projection and $p_n \to p$, as $n \to \infty$.

There are a number of open questions related to the problems discussed here. We conjecture just two of them.

Conjecture 1. Let A be a uniform algebra. Let F be a linear functional on A such that

$$|F(f \cdot g) - F(f)F(g)| \leq \varepsilon\|f\|\|g\|, \qquad f, g \in A.$$

Then there is a linear and multiplicative functional G defined on A such that

$$\|G - F\| \leq \varepsilon', \quad where \; \varepsilon' \to 0 \; as \; \varepsilon \to 0.$$

REMARK. The question whether an almost multiplicative functional is close to a multiplicative one was raised in [8], in connection with the

theory of perturbations of Banach algebras. It was noticed there that any such functional is automatically continuous [8]. B. E. Johnson [10] gave an example of a non-uniform, commutative Banach algebra which does not have the property described in the above conjecture. He proved [11] also that $C(K)$ algebras and the disc algebra $A(D)$ have this property. The problem is open, for uniform algebras in general, e.g. for $H^\infty(D)$—the algebra of all bounded analytic functions defined on the unit disc.

Conjecture 2. Let X, Y be real Banach spaces such that there is a surjective map $T: X \to Y$ with

$$(1 - \varepsilon)\|x - y\| \le \|Tx - Ty\| \le (1 + \varepsilon)\|x - y\|, \quad \text{for } x, y \in X,$$

where $0 < \varepsilon \le \varepsilon_0$ and ε_0 is an absolute constant. Then X and Y are linearly isomorphic.

REMARK. The above statement is known to be true for certain special classes of Banach spaces like uniform algebras [9]. It is also known that this is false, even for $C(K)$ spaces, if we do not assume that ε is close to zero. By the theorem of Gevirtz to prove the conjecture it is enough to show that an almost linear map is close to a linear one.

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