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**THE LOCAL STRUCTURE OF SOME MEASURE-ALGEBRA
HOMOMORPHISMS**

RUSSELL DAVID LYONS

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Extending classical theorems, we obtain representations for bounded linear transformations from L -spaces to Banach spaces with a separable predual. In the case of homomorphisms from a convolution measure algebra to a Banach algebra, we obtain a generalization of Šreider's representation of the Gelfand spectrum via generalized characters. The homomorphisms from the measure algebra on a LCA group, G , to that on the circle are analyzed in detail. If the torsion subgroup of G is denumerable, one consequence is the following necessary and sufficient condition that a positive finite Borel measure on G be continuous: $\exists \gamma_\alpha \rightarrow \infty$ in \widehat{G} such that $\forall n \neq 0 \hat{\mu}(\gamma_\alpha^n) \rightarrow 0$.

1. Introduction. Given a measurable space X and a (bounded) complex measure μ on X , the Banach space dual of $L^1(\mu)$ is commonly represented as $L^\infty(\mu)$. We shall call M an L -space on X if M is a Banach space of complex measures on X (under the measure norm) such that $\nu \ll \mu \in M \Rightarrow \nu \in M$ [Sc]. Šreider [Šr] gave a representation of the dual M^* of M as a space of so-called generalized functions, i.e., families of functions $f_\mu \in L^\infty(\mu)$ satisfying

$$(1.1) \quad \nu \ll \mu \Rightarrow f_\nu = f_\mu \quad \nu\text{-a.e.},$$

$$(1.2) \quad \sup_{\mu \in M} \|f_\mu\|_{L^\infty(\mu)} < \infty.$$

The representation of M^* , like that of $L^1(\mu)^*$, is by integration:

$$\mu \mapsto \int f_\mu d\mu.$$

Now, given two Banach spaces, B_1 and B_2 , we denote by $L(B_1, B_2)$ the Banach space of bounded linear transformations from B_1 to B_2 . Since $M^* = L(M, \mathbf{C})$, we may ask, in generalizing the above, for a representation of $L(M, B)$, where B is an arbitrary Banach space. Again, the case where $M = L^1(\mu)$ is classical [DS]; here, the hypothesis that B has a separable predual is made. In §2, we extend this theorem to general L -spaces M in a manner similar to Šreider's representation above. In essence, functions are replaced by

B -valued functions. Our treatment will be entirely self contained, thus giving an apparently new proof of [DS, Theorem VI.8.6]. However, another point of view could be adopted. Namely, if we use the Radon-Nikodym theorem to identify $L(\mu) = \{\nu \ll \mu : \nu \text{ bounded}\}$ with $L^1(\mu)$, then we may regard an L -space M as the direct limit $\lim_{\mu \in M} L^1(\mu)$, where M is directed by \ll and for $\nu \ll \mu$, $L^1(\nu)$ is included in $L^1(\mu)$. Now $L(\cdot, B)$ is a functor from the category of Banach spaces to its opposite category and, furthermore, is easily checked to be a left adjoint. Since left adjoints preserve direct limits and inverse limits are dual to direct limits, it follows that $L(M, B)$ is the inverse limit $\lim_{\mu \in M} L(L^1(\mu), B)$, where, for $\nu \ll \mu$, $L(L^1(\mu), B)$ is mapped by restriction to $L(L^1(\nu), B)$. Hence, given a representation of $L(L^1(\mu), B)$ (as in [DS]) and a construction of inverse limits, we may obtain a representation of $L(M, B)$. This amounts to the same as our Theorem 2.1.

Now Šreider was actually interested in representing ΔM , the multiplicative linear functionals on M , when M was a convolution measure algebra on a locally compact abelian group. He showed that in addition to (1.1) and (1.2), the following property was necessary and sufficient for f_μ to define an element of ΔM :

$$(1.3) \quad \forall \mu, \nu \geq 0 \quad f_{\mu * \nu}(xy) = f_\mu(x)f_\nu(y) \quad \mu \times \nu\text{-a.e. } [(x, y)].$$

We, too, are mainly interested in the subset of homomorphisms $\text{Hom}(M, B) \subseteq L(M, B)$ when B is a Banach algebra. A similar condition to (1.3) is found in Theorem 3.2. In particular, when $M = M(G)$, the complex Borel measures on a locally compact abelian group, G , and $B = M(\mathbb{T})$, \mathbb{T} the circle, $\text{Hom}(M(G), M(\mathbb{T}))$ contains in a natural way $\text{Hom}(G, \mathbb{T}) = \widehat{G}$. The closure of \widehat{G} in a certain weak topology is related to the behavior of Fourier transforms at infinity and contains much information about a measure μ when regarded locally, i.e., when restricted to $L(\mu)$, or, what is the same, when viewed via the Šreider representation. For example, this analysis will lead to the following surprising result: if the torsion subgroup of G is denumerable, then a positive measure $\mu \in M(G)$ is continuous iff there is a net $\{\gamma_\alpha\} \subseteq \widehat{G}$ tending to infinity such that for all $n \neq 0$, $\lim_\alpha \hat{\mu}(\gamma_\alpha^n) = 0$. Characterizations of certain other classes of measures are found in §4; these have proved useful in [KL] and [L4]. Other analyses of the local structure of the closure of \widehat{G} for certain μ can be found in [L3], [L4], and [L5]. The local structure of \widehat{G} is also related to asymptotic distribution; this relationship, described here, has been used in [KL] and [L4].

The Šreider representation, Theorem 3.2, has been given before in [IgK] for the case $\text{Hom}(M, M(\mathbb{T}))$, M being an L -subalgebra of $M(\mathbb{T})$, though in slightly different notation. An alternative representation for $\text{Hom}(M, M(G))$, where M is a semisimple commutative convolution measure algebra in the sense of Taylor and G is a compact abelian group, analogous to Taylor's representation of ΔM via a structure semigroup, has been given in [InK].

2. The Šreider representation of linear transformations. Suppose that M is an L -space on a measurable space X and that B is a Banach space with a separable predual, B_* . Let $\mathcal{B}(X, B)$ denote the set of maps $f: X \rightarrow B$ which are bounded in B -norm and measurable when B is given the weak* topology from B_* . If $f \in \mathcal{B}(X, B)$ and $\mu \in M$, there is a unique element $\int f d\mu \in B$ defined by the relation

$$\forall b_* \in B_* \left\langle b_*, \int f d\mu \right\rangle = \int_X \langle b_*, f(x) \rangle d\mu(x).$$

If D is a countable dense set in the unit ball of B_* , then the equation

$$\|f(x)\|_B = \sup_{b_* \in D} |\langle b_*, f(x) \rangle|$$

shows that $\|f(\cdot)\|_B$ is measurable. It is clear that

$$\left\| \int f d\mu \right\|_B \leq \int \|f\|_B d|\mu|.$$

The set of equivalence classes of $\mathcal{B}(X, B)$ under equality μ -a.e. will be denoted $\mathcal{B}(X, B)_\mu$, although this distinction will often be ignored.

The following theorem, which we shall term the *Šreider representation*, associates to each element of $L(M, B)$ a certain family of maps in $\mathcal{B}(X, B)$. We denote the image of $\mu \in M$ under $\Sigma \in L(M, B)$ by Σ_μ .

THEOREM 2.1. *Let M be an L -space and B a Banach space with a separable predual. There is a bijection between $L(M, B)$ and the set of elements $\{b_{\cdot, \mu}\}_{\mu \in M} \in \prod_{\mu \in M} \mathcal{B}(X, B)_\mu$ which satisfy*

$$(i) \quad \sup_{\mu \in M} \| \|b_{x, \mu}\|_B \|_{L^\infty(\mu)} < \infty$$

and

$$(ii) \quad \forall \nu \ll \mu \in M \quad b_{x, \nu} = b_{x, \mu} \quad \nu\text{-a.e. } [x]$$

such that if Σ corresponds to $\{b_{\cdot, \mu}\}_{\mu \in M}$ (written $\Sigma \sim b_{\cdot, \cdot}$), then

$$(iii) \quad \forall \mu \in M \quad \Sigma_\mu = \int b_{x, \mu} d\mu(x)$$

and

$$(iv) \quad \|\Sigma\|_{L(M, B)} = \sup_{\mu \in M} \| \|b_{x, \mu}\|_B \|_{L^\infty(\mu)}.$$

Proof. Given $\{b_{\cdot, \mu}\}$ satisfying (i) and (ii), define Σ by (iii). If $\mu, \nu \in M$, then by (ii), we have $b_{x, \mu} = b_{x, |\mu|+|\nu|}$ μ -a.e., whence $\Sigma_\mu = \int b_{x, |\mu|+|\nu|} d\mu(x)$. In conjunction with similar equations for Σ_ν and $\Sigma_{\mu+\nu}$, this equation shows that $\Sigma_\mu + \Sigma_\nu = \Sigma_{\mu+\nu}$. Similarly, for $\alpha \in \mathbf{C}$, $\Sigma_{\alpha\mu} = \alpha\Sigma_\mu$, whence Σ is linear. Let K denote the quantity in (i). Then

$$\begin{aligned} \|\Sigma\| &= \sup_{\|\mu\| \leq 1} \|\Sigma_\mu\| = \sup_{\|\mu\| \leq 1} \left\| \int b_{x, \mu} d\mu(x) \right\| \\ &\leq \sup_{\|\mu\| \leq 1} \int \|b_{x, \mu}\| d|\mu|(x) \leq K. \end{aligned}$$

To show that $\|\Sigma\| = K$, choose any nonzero $\mu \in M$ and $\varepsilon > 0$. Let $0 \neq \nu \in L(\mu)$ be such that $\| \|b_{\cdot, \mu}\|_B - \| \|b_{\cdot, \mu}\|_B \|_{L^\infty(\mu)} \|_{L^\infty(\nu)} < \varepsilon$. Let S be the unit sphere of B . Since the unit ball of B is weak* compact, there exists a finite number of elements, b_*^1, \dots, b_*^n , of the unit ball of B_* such that

$$S = \bigcup_{i=1}^n \{b \in S : |\langle b_*^i, b \rangle - 1| < \varepsilon\}.$$

Therefore $\exists 0 < \omega \in L(\nu) \exists i \| \langle b_*^i, b_{x, \mu} / \|b_{x, \mu}\|_B \rangle - 1 \|_{L^\infty(\omega)} < \varepsilon$. We have

$$\begin{aligned} \|\Sigma\| &\geq \frac{\|\Sigma_\omega\|}{\|\omega\|} \geq \frac{1}{\|\omega\|} |\langle b_*^i, \Sigma_\omega \rangle| = \frac{1}{\|\omega\|} \left| \int \langle b_*^i, b_{x, \mu} \rangle d\omega(x) \right| \\ &\geq \frac{1}{\|\omega\|} \int \|b_{x, \mu}\|_B d\omega(x) - \varepsilon K \geq \| \|b_{\cdot, \mu}\|_B \|_{L^\infty(\mu)} - \varepsilon(K+1). \end{aligned}$$

Thus $\|\Sigma\| = K$.

Conversely, let $\Sigma \in L(M, B)$. Fix $\mu \in M$. For $b_* \in B_*$, we denote by $b_* \circ \Sigma$ the map $\nu \mapsto \langle b_*, \Sigma_\nu \rangle$. Restricted to $L(\mu)$, this map is a bounded linear functional and hence can be represented by a function $g_{b_*} \in L^\infty(\mu)$. Choose a countable linearly independent set D whose

linear span over \mathbf{Q} , D' , is dense in B_* . If $b_* = \sum_{i=1}^n \alpha_i d_*^i$, $d_*^i \in D$, $\alpha_i \in \mathbf{Q}$, define

$$h_{b_*} = \sum_{i=1}^n \alpha_i g_{d_*^i}.$$

Then $b_* \mapsto h_{b_*}(x)$ is rational-linear on D' for every $x \in X$. Furthermore, $h_{b_*} = g_{b_*}$ μ -a.e., whence by countability of D' ,

$$(2.1) \quad \forall b_* \in D' \quad |h_{b_*}(x)| \leq \|b_* \circ \Sigma\| \leq \|b_*\| \cdot \|\Sigma\|$$

for μ -a.e. x . Now for every x such that (2.1) holds, $b_* \mapsto h_{b_*}(x)$ extends from D' to all of B_* as a bounded linear functional, hence element of B , call it $f(x)$. This defines $f(x)$ μ -a.e. and shows that, given any $b_* \in B_*$, if $b_* = \lim_{n \rightarrow \infty} b_*^n$ ($b_*^n \in D'$), then

$$(2.2) \quad \langle b_*, f(x) \rangle = \lim_{n \rightarrow \infty} \langle b_*^n, f(x) \rangle = \lim_{n \rightarrow \infty} h_{b_*^n}(x)$$

for every x where f is defined. Write $b_{*,\mu}$ for the equivalence class of f . From Equation (2.1), we see that $\|f(x)\| \leq \|\Sigma\|$ for every x where f is defined. Together with (2.2), this shows that $b_{*,\mu} \in \mathcal{B}(X, B)_\mu$ and gives (i). Now for $\nu \in L(\mu)$ and $b_* \in D'$, we have

$$\begin{aligned} \left\langle b_*, \int f d\nu \right\rangle &= \int \langle b_*, f(x) \rangle d\nu(x) = \int h_{b_*}(x) d\nu(x) \\ &= \int g_{b_*}(x) d\nu(x) = \langle b_*, \Sigma_\nu \rangle. \end{aligned}$$

Since D' is dense, (iii) follows. We claim that $b_{*,\mu}$ is uniquely determined by the property just established:

$$\forall \nu \in L(\mu) \quad \Sigma_\nu = \int b_{x,\mu} d\nu(x).$$

Indeed, if we also have that $\forall \nu \in L(\mu) \quad \Sigma_\nu = \int b'_{x,\mu} d\nu(x)$ for some $b'_{x,\mu} \in \mathcal{B}(X, B)_\mu$, then

$$\forall b_* \in D' \quad \forall \nu \in L(\mu) \quad \int \langle b_*, b_{x,\mu} \rangle d\nu(x) = \int \langle b_*, b'_{x,\mu} \rangle d\nu(x),$$

whence for μ -a.e. $x \quad \forall b_* \in D' \quad \langle b_*, b_{x,\mu} \rangle = \langle b_*, b'_{x,\mu} \rangle$, i.e., $b_{x,\mu} = b'_{x,\mu}$ μ -a.e. Thus (ii) follows. The same argument shows that if $\Sigma \sim b_{*,\cdot}$ and $\Sigma \sim b'_{*,\cdot}$, then $b_{*,\cdot} = b'_{*,\cdot}$. \square

We define the *weak* operator topology* (W^*OT) on $L(M, B)$ as the weakest topology such that $\forall \mu \in M \quad \forall b_* \in B_* \quad \Sigma \mapsto \langle b_*, \Sigma_\mu \rangle$ is continuous. It is an elementary exercise to show that the unit ball of $L(M, B)$ is W^*OT compact.

For $\mu \in M$, let $L(M, B)_\mu$ denote the set of Šreider representations $b_{\cdot, \mu}$ of elements of $L(M, B)$. We give $L(M, B)_\mu$ the weak topology generated by the maps $b_{\cdot, \mu} \mapsto \int \langle b_*, b_{x, \nu} \rangle d\nu(x)$ ($b_* \in B_*$, $\nu \in L(\mu)$). Thus, the W^* OT is the inverse limit of these topologies, i.e., it is the weak topology generated by the maps $\Sigma \mapsto b_{\cdot, \mu}$ ($\mu \in M$) from $L(M, B) \rightarrow L(M, B)_\mu$, where $\Sigma \sim b_{\cdot, \cdot}$.

Every decomposition $M = I \oplus J$ of M as a direct sum of closed subspaces yields an addition on $L(M, B)$ as follows: if $\Pi^1, \Pi^2 \in L(M, B)$, then we may define

$$(2.3) \quad \Sigma_\mu = \Pi_{\mu_I}^1 + \Pi_{\mu_J}^2,$$

where $\mu = \mu_I + \mu_J$, $\mu_I \in I$, $\mu_J \in J$. If $\Sigma \sim b_{\cdot, \cdot}$, $\Pi^i \sim b^i_{\cdot, \cdot}$, and $I \perp J$, then $b_{x, \mu} = b^1_{x, \mu_I} + b^2_{x, \mu_J}$ μ -a.e.

The case where $B = M(Y)$, the space of complex regular Borel measures on a locally compact metric space, Y , is of interest. A predual of B is the separable space $C_0(Y)$ of continuous functions vanishing at infinity. We shall denote the Šreider representation of Σ by $\sigma_{x, \mu}$ in this case; thus, if $f \in C_0(Y)$ and $\mu \in M$,

$$(2.4) \quad \int_Y f d\Sigma_\mu = \int_X \left(\int_Y f d\sigma_{x, \mu} \right) d\mu(x).$$

(If Y is separable and a countable union of complete subspaces, then (2.4) holds for $f \in \mathcal{B}(Y, \mathbb{C})$ since it is preserved under bounded pointwise limits. In particular, for Borel sets $E \subseteq Y$,

$$\Sigma_\mu(E) = \int_X \sigma_{x, \mu}(E) d\mu(x).$$

Let M^+ denote the nonnegative elements of M and likewise for $M^+(Y)$. We say that $\Sigma \in L(M, M(Y))$ is *positive* if it carries M^+ into $M^+(Y)$. It is easy to see from (2.4) applied to $|\mu|$ that $\Sigma \geq 0$ iff $\forall \mu \in M \quad \forall^e x[\mu] \quad \sigma_{x, \mu} \geq 0$ (“ $\forall^e x[\mu]$ ” means “for μ -a.e. x ”—see [L1]). It is also easy to show that if $\Sigma \geq 0$, then $\nu \ll \mu \Rightarrow \Sigma_\nu \ll \Sigma_{|\mu|}$ and $|\Sigma_\nu| \leq \Sigma_{|\mu|}$.

3. The Šreider representation of homomorphisms. Let G be a locally compact semigroup with separately continuous multiplication. Then $M(G)$ is a Banach algebra under convolution [W]. Let M be an L -subalgebra of $M(G)$, i.e., a subalgebra which is also an L -subspace, and let B be a Banach algebra with a separable predual such that

multiplication is separately weak* measurable and

$$(3.1) \quad \forall f \in \mathcal{B}(G, B) \quad \forall b \in B \quad \forall \mu \in M$$

$$\int f(x) \cdot b \, d\mu(x) = \left(\int f \, d\mu \right) \cdot b$$

$$\& \int b \cdot f(x) \, d\mu(x) = b \cdot \int f \, d\mu.$$

In order to state some sufficient conditions that (3.1) be true, we define the following multiplication on $B^* \times B$. If $b \in B$ and $b^* \in B^*$, then $b' \mapsto \langle b' \cdot b, b^* \rangle$ is a bounded linear functional on B ; we denote it by $b^* \cdot b$. Let $\overline{B_*}^{sw^*}$ be the smallest subspace of B^* containing (canonically) B_* which is closed under weak* sequential limits. Let ΔB be the subset of B^* consisting of the multiplicative linear functionals.

PROPOSITION 3.1. *Let B be a Banach algebra with a separable predual. Right multiplication on B is weak* measurable and the first equation of (3.1) holds if any of the following conditions is satisfied:*

- (i) $B_* \cdot B \subseteq \overline{B_*}^{sw^*}$.
- (ii) Right multiplication is weak* continuous.
- (iii) Right multiplication is weak* measurable and $\overline{B_*}^{sw^*} \cap \Delta B$ separates points in B .

Proof. The class of $b^* \in B^*$ such that $b \mapsto \langle b, b^* \rangle$ is weak* measurable contains B_* and is closed under weak* sequential limits. Thus, all elements of $\overline{B_*}^{sw^*}$ are weak* measurable. Now right multiplication is weak* measurable iff $\forall b \in B \quad \forall b_* \in B_* \quad b' \mapsto \langle b_*, b' \cdot b \rangle$ is weak* measurable. But $\langle b_*, b' \cdot b \rangle = \langle b', b_* \cdot b \rangle$, whence this condition is equivalent to all elements of $B_* \cdot B$ being weak* measurable. The sufficiency of (i) for measurability is now obvious. Also, the class of weak* measurable $b^* \in B^*$ such that

$$\left\langle \int f \, d\mu, b^* \right\rangle = \int \langle f, b^* \rangle \, d\mu$$

is closed under weak* sequential limits by the bounded convergence theorem, hence contains $\overline{B_*}^{sw^*}$. Thus, if (i) holds, then $\forall b_* \in B_* \quad \forall b \in B$

$$\left\langle b_*, \int f \cdot b \, d\mu \right\rangle = \int \langle b_*, f \cdot b \rangle \, d\mu = \int \langle f, b_* \cdot b \rangle \, d\mu$$

$$= \left\langle \int f \, d\mu, b_* \cdot b \right\rangle = \left\langle b_*, \left(\int f \, d\mu \right) \cdot b \right\rangle,$$

whence the first equation of (3.1).

Now (ii) is equivalent to $B_* \cdot B \subseteq B_*$ since B_* is the set of weak* continuous linear functionals on B . Thus, sufficiency follows from that of (i). Finally, if (iii) holds, then for $f \in \mathcal{B}(G, B)$, $b \in B$, $\mu \in M$, and $b^* \in \overline{B_*}^{sw*} \cap \Delta B$, we have

$$\begin{aligned} \left\langle \int f \cdot b \, d\mu, b^* \right\rangle &= \int \langle f \cdot b, b^* \rangle \, d\mu = \int \langle f, b^* \rangle \langle b, b^* \rangle \, d\mu \\ &= \int \langle f, b^* \rangle \, d\mu \cdot \langle b, b^* \rangle = \left\langle \int f \, d\mu, b^* \right\rangle \cdot \langle b, b^* \rangle \\ &= \left\langle \left(\int f \, d\mu \right) \cdot b, b^* \right\rangle, \end{aligned}$$

from which the first equation of (3.1) follows. \square

Let $\mathcal{B}_0(G, B)$ denote the Baire-measurable functions from G to B , where B is given the weak* topology. For $\mu, \nu \in M(G)$, let $\mu \times \nu$ denote, besides the usual product measure, also its unique extension to a regular Borel measure in $M(G \times G)$. If $f \in \mathcal{B}_0(G, B)$ and $\mu, \nu \in M(G)$, then

$$\begin{aligned} \int f \, d\mu * \nu &= \int f(xy) \, d\mu \times \nu(x, y) \\ &= \iint f(xy) \, d\mu(x) \, d\nu(y), \end{aligned}$$

as can be seen by applying any $b_* \in B_*$ [W].

The Šreider representation of $\text{Hom}(M, B)$, the continuous homomorphisms from M to B , satisfies one property additional to those in Theorem 2.1.

THEOREM 3.2. *Let G be a locally compact semigroup with separately continuous multiplication and M an L -subalgebra of $M(G)$. Let B be a Banach algebra with a separable predual and separately weak* measurable multiplication satisfying (3.1). Let $\Sigma \in L(M, B)$ and choose $b_{\cdot, \mu} \in \mathcal{B}_0(G, B)$ ($\mu \in M$) so that $\Sigma \sim b_{\cdot, \cdot}$. Then $\Sigma \in \text{Hom}(M, B)$ iff*

$$(3.2) \quad \forall \mu, \nu \in M^+ \quad b_{xy, \mu * \nu} = b_{x, \mu} \cdot b_{y, \nu} \quad \text{for } \mu \times \nu\text{-a.e. } (x, y).$$

Proof. Suppose first that (3.2) is satisfied. Then for $\mu, \nu \in M$,

$$\begin{aligned}\Sigma_{\mu*\nu} &= \int b_{t,|\mu|*|\nu|} d\mu * \nu(t) = \iint b_{xy,|\mu|*|\nu|} d\mu(x) d\nu(y) \\ &= \iint b_{x,|\mu|} \cdot b_{y,|\nu|} d\mu(x) d\nu(y) \\ &= \int \left(\int b_{x,|\mu|} d\mu(x) \right) \cdot b_{y,|\nu|} d\nu(y) \\ &= \int b_{x,|\mu|} d\mu(x) \cdot \int b_{y,|\nu|} d\nu(y) = \Sigma_\mu \cdot \Sigma_\nu.\end{aligned}$$

Conversely, if $\Sigma \in \text{Hom}(M, B)$, then given $\mu, \nu \in M^+$, we have for all $\mu' \in L(\mu)$ and $\nu' \in L(\nu)$,

$$\begin{aligned}\int b_{xy, \mu*\nu} d\mu' \times \nu'(x, y) &= \int b_{t, \mu*\nu} d\mu' * \nu'(t) = \Sigma_{\mu'*\nu'} \\ &= \Sigma_{\mu'} \cdot \Sigma_{\nu'} = \int b_{x, \mu} d\mu'(x) \cdot \int b_{y, \nu} d\nu'(y) \\ &= \iint b_{x, \mu} \cdot b_{y, \nu} d\mu'(x) d\nu'(y) \\ &= \int b_{x, \mu} \cdot b_{y, \nu} d\mu' \times \nu'(x, y).\end{aligned}$$

Since the span of $L(\mu) \times L(\nu)$ is dense in $L(\mu \times \nu)$, (3.2) follows. \square

If multiplication in B is jointly weak* continuous (for example, if $B_* \cap \Delta B$ separates points in B), then the unit ball in $\text{Hom}(M, B)$ is easily shown to be W^* OT compact. An example where compactness fails is $\text{Hom}(M(\mathbf{R}), M(\mathbf{R}))$: define T_n ($n \geq 1$) in the unit ball by

$$\int_{\mathbf{R}} f(x) d(T_n)_\mu(x) = \int_{\mathbf{R}} f(nx) d\mu(x) \quad (f \in C_0(\mathbf{R}))$$

and let $\Sigma: \mu \mapsto \mu(\{0\})\delta(0)$, where $\delta(0)$ is the Dirac measure at 0. Then $T_n \rightarrow \Sigma$ in W^* OT, but

$$\Sigma \in L(M(\mathbf{R}), M(\mathbf{R})) \setminus \text{Hom}(M(\mathbf{R}), M(\mathbf{R})).$$

We define the following multiplication on $L(M, B)$: if $\Sigma \sim b_{\cdot, \cdot}$ and $\Pi \sim b_{\cdot, \cdot}$, then $\Sigma \cdot \Pi$ is defined by its Šreider representation $b_{x, \mu} \cdot b'_{x, \mu}$. When B is commutative, $\text{Hom}(M, B)$ is closed under multiplication. It is easily verified that if multiplication in B is separately weak* continuous, then multiplication in $L(M, B)$ is separately W^* OT continuous.

Suppose that $M = I \oplus J$, where I is a closed ideal and J is a closed subalgebra. If $\Pi^1, \Pi^2 \in \text{Hom}(M, B)$ satisfy

$$(3.3) \quad \forall \mu \in I \quad \forall \nu \in J \quad \Pi_{\mu*\nu}^1 = \Pi_\mu^1 \cdot \Pi_\nu^2 \quad \& \quad \Pi_{\nu*\mu}^1 = \Pi_\mu^2 \cdot \Pi_\nu^1,$$

then the “sum” Σ of Π^1 and Π^2 defined in (2.3) is a homomorphism.

4. Limit points of group homomorphisms. If H is a locally compact group, then convolution is separately weak* continuous in $M(H)$. Indeed, if $\mu_\alpha, \mu, \nu \in M(H)$ with $\mu_\alpha \xrightarrow{w^*} \mu$, then for $f \in C_0(H)$, the map $x \mapsto \int f(xy) d\nu(y)$ lies in $C_0(H)$, whence

$$\begin{aligned} \int f d\mu_\alpha * \nu &= \iint f(xy) d\nu(y) d\mu_\alpha(x) \\ &\rightarrow \iint f(xy) d\nu(y) d\mu(x) = \int f d\mu * \nu, \end{aligned}$$

which is to say that $\mu_\alpha * \nu \xrightarrow{w^*} \mu * \nu$. A similar argument applies to $\nu * \mu_\alpha$. Thus, if G is a locally compact semigroup with separately continuous multiplication and H is a locally compact metrizable group, then the preceding section applied to $\text{Hom}(M, M(H))$ for any L -subalgebra M of $M(G)$. Every continuous homomorphism $\varphi: G \rightarrow H$ yields an element of $\text{Hom}(M, M(H))$, which we also denote by φ , defined by $\langle f, \varphi_\mu \rangle = \langle f \circ \varphi, \mu \rangle$ for $f \in C_0(H)$. The Šreider representation of such a φ is particularly simple: $\varphi \sim \delta(\varphi(x))$ (independent of μ), where $\delta(t)$ denotes the Dirac measure at t .

We identify $\text{Hom}(G, H)$ with a subset of $\text{Hom}(M(G), M(H))$ in the above manner. Our aim is to study the set

$$\Lambda = \overline{\text{Hom}(G, H)} \setminus \text{Hom}(G, H)$$

and its local structure

$$\Lambda(\mu) = \{\Sigma_\mu : \Sigma \in \Lambda\}, \quad \check{\Lambda}(\mu) = \{\check{\sigma}_\mu : \check{\sigma}_\cdot \in \check{\Lambda}\},$$

where $\check{\Lambda}$ consists of the Šreider representations of elements of Λ . Since all elements of $\text{Hom}(G, H)$ are positive and lie in the unit ball, the same holds for Λ . (In fact, every positive homomorphism lies in the unit ball: if $0 \leq \Sigma \in \text{Hom}(M(G), M(H))$, then for $\mu \in M(G)$ and $n \geq 1$, we have

$$\|\Sigma_\mu\|^n \leq \|\Sigma_{|\mu|}\|^n = \|\Sigma_{|\mu|}^n\| = \|\Sigma_{|\mu|^n}\| \leq \|\Sigma\| \cdot \|\mu\|^n = \|\Sigma\| \cdot \|\mu\|^n,$$

whence $\|\Sigma\| \leq 1$.)

We are particularly interested in the case where G is a locally compact abelian group and H is a circle group, \mathbf{T} . In this case,

$\text{Hom}(G, \mathbf{T}) = \widehat{G}$, the dual of G , and the identification of \widehat{G} as a subset of $\text{Hom}(M(G), M(\mathbf{T}))$ preserves the usual topology of \widehat{G} (of uniform convergence on compact subsets). Furthermore, as \widehat{G} lies in the unit ball of $\text{Hom}(M(G), M(\mathbf{T}))$, it follows that $\widetilde{\widehat{G}} = \widehat{G} \cup \Lambda$ is a compactification of \widehat{G} .

Recall that a sequence $\{x_k\}_{k=1}^\infty \subseteq G$ is said to have an *asymptotic distribution* σ , written $\{x_k\} \sim \sigma$, if

$$\frac{1}{K} \sum_{k=1}^K \delta(x_k) \xrightarrow{w^*} \sigma \quad \text{as } K \rightarrow \infty.$$

For $n \in \mathbf{Z}$ and $\Sigma \in \text{Hom}(M(G), M(\mathbf{T}))$, define $\widehat{\Sigma}(n) \in \Delta M(G)$ by $\langle \mu, \widehat{\Sigma}(n) \rangle = \widehat{\Sigma}_\mu(n)$. We write the Šreider representation of $\chi \in \Delta M(G)$ as $\chi_\mu(x)$. Thus, if $\Sigma \sim \sigma$, and $\chi = \widehat{\Sigma}(n)$, then

$$\chi_\mu(x) = \widehat{\sigma}_{x, \mu}(n).$$

Note that for all n , the map $\Sigma \mapsto \widehat{\Sigma}(n)$ from $(\text{Hom}(M(G), M(\mathbf{T})), W^* \text{OT})$ to $\Delta M(G)$ (with its usual Gelfand topology) is continuous. We regard the Fourier transform as a restriction of the Gelfand transform; thus, in accordance with the Šreider representation, we have $\widehat{\mu}(\gamma) = \int \gamma d\mu$ for $\gamma \in \widehat{G}$.

PROPOSITION 4.1. *Let G be a locally compact abelian group and $\Lambda = \widetilde{\widehat{G}} \setminus \widehat{G}$ in $\text{Hom}(M(G), M(\mathbf{T}))$. Then*

(i) Λ is closed topologically and under multiplication by elements of $\widetilde{\widehat{G}}$;

(ii) if $\sigma_x, \tau_x \in \widetilde{\Lambda}(\mu)$, then $\sigma_x * \tau_x \in \widetilde{\Lambda}(\mu)$;

(iii) $\Lambda(\mu) = \{\nu \in M(\mathbf{T}) : \exists \text{ net } \{\gamma_\alpha\} \subseteq \widehat{G} \ (\gamma_\alpha \rightarrow \infty \ \& \ \forall n \in \mathbf{Z} \ \widehat{\mu}(\gamma_\alpha^n) \rightarrow \widehat{\nu}(n))\}$;

(iv) $\Lambda(\mu) = \{\sigma \in \mathcal{B}(G, M(\mathbf{T}))_\mu : \exists \text{ net } \{\gamma_\alpha\} \subseteq \widehat{G} \ (\gamma_\alpha \rightarrow \infty \ \& \ \forall n \in \mathbf{Z} \ \gamma_\alpha^n \rightarrow \widehat{\sigma}(n) \text{ weak* in } L^\infty(\mu))\}$;

(v) if G is metrizable, then the nets in (iii) and (iv) can be replaced by sequences and $\Lambda(\mu) = \{\sigma \in \mathcal{B}(G, M(\mathbf{T}))_\mu : \exists \gamma_j \in \widehat{G} \ (\gamma_j \rightarrow \infty \ \& \ \text{for every subsequence } \gamma_{j_k}, \forall^e x[\mu] \ \{\gamma_{j_k}(x)\}_{k=1}^\infty \sim \sigma_x)\}$.

Proof. Suppose that $\Sigma \in \Lambda$ is the limit of a net $\{\gamma_\alpha\} \subseteq \widehat{G}$. Then $\widehat{\Sigma}(n) = \lim \gamma_\alpha^n$ in $\Delta M(G)$ for all $n \in \mathbf{Z}$. Now if $\gamma_\alpha \rightarrow \gamma \in \widehat{G}$, then $\gamma_\alpha^n \rightarrow \gamma^n$, whence $\Sigma = \gamma$. But since $\Lambda \cap \widehat{G} = \emptyset$, this is impossible, and so $\gamma_\alpha \rightarrow \infty$ in \widehat{G} . In particular, $\widehat{\Sigma}(1)$ is 0 on $L^1(G)$ [HMP,

p. 136, Proposition 4] and consequently Λ is closed. It is clear that $\Lambda \cdot \widehat{G} \subseteq \Lambda$, from which (i) now follows. Statement (ii) ensues as well. Now if $\nu \in \Lambda(\mu)$, then let $\widehat{G} \ni \gamma_\alpha \rightarrow \Sigma \in \Lambda$ be such that $\nu = \Sigma_\mu$. Then $\gamma_\alpha \rightarrow \infty$ and $(\gamma_\alpha)_\mu \xrightarrow{w^*} \Sigma_\mu = \nu$, which gives the inclusion \subseteq of (iii). On the other hand, if $\gamma_\alpha \rightarrow \infty$ and $\forall n \ \widehat{\mu}(\gamma_\alpha^n) \rightarrow \widehat{\nu}(n)$, then by compactness of \widehat{G} , we can choose a subnet $\{\gamma'_\beta\}$ of $\{\gamma_\alpha\}$ converging to some Σ . Since $\gamma'_\beta \rightarrow \infty$, it follows that $\Sigma \in \Lambda$ and $\nu = \Sigma_\mu \in \Lambda(\mu)$. This completes the proof of (iii). The proof of (iv) is analogous. Finally, if G is metrizable, then $L^1(\mu)$ is separable for $\mu \in M(G)$ and so $L(M(G), M(\mathbf{T}))_\mu$ is metrizable. Thus, if $\mu \in M(G)$ and $\gamma_\alpha \rightarrow \Sigma \sim \sigma, \dots$, pick any non-zero $\rho \in L^1(G)$ and a subsequence $\{\delta(\gamma_{\alpha_j}(\cdot))\}$ converging to $\sigma_{\cdot, |\mu|+|\rho|}$ in $L(M(G), M(\mathbf{T}))_{|\mu|+|\rho|}$. Then $\gamma_{\alpha_j} = \delta(\gamma_{\alpha_j}(\cdot)) \wedge (1) \xrightarrow{w^*} (\widehat{\Sigma}(1))_\rho = 0$ in $L^\infty(\rho)$, whence $\gamma_{\alpha_j} \rightarrow \infty$ in \widehat{G} , and $\gamma_{\alpha_j}^n \xrightarrow{w^*} (\widehat{\Sigma}(n))_\mu = \widehat{\sigma}_{\cdot, \mu}(n)$ in $L^\infty(\mu)$. This shows the sufficiency of sequences for (iii) and (iv). Furthermore, if $\forall n \ \gamma_j^n \rightarrow \widehat{\sigma}_{\cdot}(n)$ weak* in $L^\infty(\mu)$, then by [L2, Lemma 5], there is a subsequence $\{\gamma'_j\}$ of $\{\gamma_j\}$ such that every further subsequence $\{\gamma'_{j_k}\}$ satisfies

$$(4.1) \quad \forall^e x[\mu] \{\gamma'_{j_k}(x)\}_{k=1}^\infty \sim \sigma_x.$$

Conversely, if $\{\gamma_j\}$ is a sequence, every subsequence of which satisfies (4.1), then we claim $\gamma_j^n \rightarrow \widehat{\sigma}_{\cdot}(n)$ weak* for every n . If not, then for some n there would be a subsequence $\{\gamma'_{j_k}\}$ converging to a different limit χ . Then also

$$\frac{1}{K} \sum_{k=1}^K \gamma'_{j_k} \xrightarrow{w^*} \chi$$

and by (4.1),

$$\frac{1}{K} \sum_{k=1}^K \gamma'_{j_k} \xrightarrow{w^*} \widehat{\sigma}_{\cdot}(n).$$

Therefore $\chi = \widehat{\sigma}_{\cdot}(n)$, a contradiction. Thus (v) follows from (iv). \square

When \widehat{G} is regarded as a subset of $\Delta M(G)$, we shall use the notation Γ rather than \widehat{G} to avoid confusion. Let $T_n \in \text{Hom}(G, G)$ denote the map $x \mapsto x^n$ ($n \in \mathbf{Z}$), as well as the corresponding map induced in $\text{Hom}(M(G), M(G))$. Thus, for $\Sigma \in \text{Hom}(M(G), M(\mathbf{T}))$, we obtain $\Sigma \circ T_n \in \text{Hom}(M(G), M(\mathbf{T}))$; note that if $\Sigma = \gamma \in \widehat{G}$, then $\gamma \circ T_n = \gamma^n$.

PROPOSITION 4.2. *Let G be a LCA group and*

$$\Sigma \in \text{Hom}(M(G), M(\mathbf{T})).$$

Then $\Sigma \in \widehat{G}$ iff $\widehat{\Sigma}(1) \in \overline{\Gamma}$ and $\forall n \in \mathbf{Z}$ $\widehat{\Sigma}(n) = \widehat{\Sigma}(1) \circ T_n$. The map $\Sigma \mapsto \widehat{\Sigma}(1)$ is an isomorphism from \widehat{G} onto $\overline{\Gamma}$ sending \widehat{G} to Γ .

Proof. If $\Sigma \in \widehat{G}$, let $\widehat{G} \ni \gamma_\alpha \xrightarrow{W^*OT} \Sigma$. Since $\hat{\gamma}_\alpha(n) = \gamma_\alpha^n$, we have $\gamma_\alpha^n \rightarrow \widehat{\Sigma}(n)$ for all n . In particular, $\widehat{\Sigma}(1) \in \overline{\Gamma}$. Also, $\widehat{\Sigma}(n) = \lim \gamma_\alpha^n = \lim \gamma_\alpha \circ T_n = (\lim \gamma_\alpha) \circ T_n = \widehat{\Sigma}(1) \circ T_n$. Conversely, if $\widehat{\Sigma}(1) \in \overline{\Gamma}$ and $\forall n$ $\widehat{\Sigma}(n) = \widehat{\Sigma}(1) \circ T_n$, then let $\gamma_\alpha \rightarrow \widehat{\Sigma}(1)$. Choose a convergent subnet $\gamma'_\beta \rightarrow \Pi$ in $\text{Hom}(M(G), M(\mathbf{T}))$. Then from the above, $\widehat{\Pi}(n) = \widehat{\Pi}(1) \circ T_n = \widehat{\Sigma}(1) \circ T_n = \widehat{\Sigma}(n)$ for all n , whence $\Sigma = \Pi \in \widehat{G}$.

It follows from this that the map $\Sigma \mapsto \widehat{\Sigma}(1)$ is injective. Surjectivity onto $\overline{\Gamma}$ is proved by a compactness argument similar to the above. \square

We write $M(G) = M_c(G) \oplus M_d(G)$ for the decomposition of a measure into its continuous and discrete parts. Then $h_d: \mu \mapsto \int_G d\mu_d = \hat{\mu}_d(0)$ is in $\overline{\Gamma} \setminus \Gamma$ [HMP, pp. 136–7, (4.1.4)]. We denote the element of Λ corresponding to h_d by Π^d . If G has at most countably many torsion elements, then we claim that

$$\widehat{\Pi}^d(n) = \begin{cases} 0 & \text{if } n = 0, \\ h_d & \text{if } n \neq 0, \end{cases}$$

whence

$$\Pi_\mu^d = \hat{\mu}_c(0)\lambda + \hat{\mu}_d(0)\delta(0),$$

where λ is Lebesgue measure on \mathbf{T} . To see this, note first that

$$\widehat{\Pi}^d(0): \mu \mapsto (\mu \circ T_0^{-1}) \wedge (0) = \hat{\mu}(0).$$

Second, if $n \neq 0$, then for all $g \in G$, there are, by the supposition, denumerably many $x \in G$ such that $x^n = g$. Therefore

$$(\mu \circ T_n^{-1})(\{g\}) = \sum_{x^n=g} \mu(\{x\}),$$

whence

$$\begin{aligned} \widehat{\Pi}^d(n): \mu &\mapsto \sum_{g \in G} (\mu \circ T_n^{-1})(\{g\}) \\ &= \sum_{g \in G} \sum_{x^n=g} \mu(\{x\}) = \sum_{x \in G} \mu(\{x\}) = \hat{\mu}_d(0). \end{aligned}$$

This proves the claim.

Related elements of Λ are $\Sigma \cdot \Pi^d$ for $\Sigma \in \widetilde{\widehat{G}}$; if, as above, the torsion subgroup of G is denumerable, then

$$(\Sigma \cdot \Pi^d)_\mu = \hat{\mu}_c(0)\lambda + \Sigma_{\mu_d}.$$

Thus, if we set $\Pi: \mu \mapsto \hat{\mu}(0)\lambda$, then $\Sigma \cdot \Pi^d$ is the sum of Π and Σ defined by (2.3) and (3.3) from the decomposition $M = M_c \oplus M_d$. An interesting example is $G = \mathbf{T}$ and $\Sigma: \mu \mapsto \mu$; in this case, $(\Sigma \cdot \Pi^d)_\mu = \hat{\mu}_c(0)\lambda + \mu_d$.

Provided still that G has a denumerable torsion subgroup, the Šreider representation $\pi_{x,\mu}^d$ of Π^d is given by

$$(4.2) \quad \pi_{x,\mu}^d = \begin{cases} \lambda & \text{if } \mu(\{x\}) = 0, \\ \delta(0) & \text{if } \mu(\{x\}) \neq 0. \end{cases}$$

Let $\lambda \in \mathcal{B}(G, M(\mathbf{T}))_\mu$ be defined by $\lambda(x) \equiv \lambda$. Then from [HMP, p. 70, Corollaire 2] and Proposition 4.2 (or from (4.2) and the following proposition),

$$(4.3) \quad \mu \in M_c(G) \Leftrightarrow \lambda \in \widetilde{\widehat{\Lambda}}(\mu).$$

This yields other characterizations of $M_c(G)$ when combined with Proposition 4.1 (iv), (v). For example,

$$\begin{aligned} \mu \in M_c(G) &\Leftrightarrow \exists \gamma_\alpha \rightarrow \infty \forall \nu \in L(\mu) \forall n \neq 0 \hat{\nu}(\gamma_\alpha^n) \rightarrow 0 \\ &\Leftrightarrow \exists \gamma_\alpha \rightarrow \infty \forall \gamma \in \widehat{G} \forall n \neq 0 \hat{\mu}(\gamma\gamma_\alpha^n) \rightarrow 0. \end{aligned}$$

Our next proposition describes $\widetilde{\widehat{\Lambda}}(\mu)$ completely when μ is discrete (cf. [HMP, pp. 67–68]).

PROPOSITION 4.3. *Let G be a LCA group. Let $\widetilde{\widehat{G}}$ denote the Šreider representations of $\widetilde{\widehat{G}} \subseteq \text{Hom}(M(G), M(\mathbf{T}))$ and, for $\mu \in M(G)$, $\widetilde{\widehat{G}}(\mu) = \{\sigma_{\cdot,\mu} : \sigma_{\cdot,\cdot} \in \widetilde{\widehat{G}}\}$. Let G_d denote G with the discrete topology and, for $\mu \in M_d(G)$, let G_d^μ denote the discrete subgroup generated by the mass-points of μ .*

(i) $\forall \Sigma \in \widetilde{\widehat{G}} \exists \varphi \in \widehat{G}_d \forall \mu \in M_d(G) \Sigma_\mu = \sum_{x \in G} \mu(\{x\})\delta(\varphi(x))$ and $\sigma_{x,\mu} = \delta(\varphi(x))$, where $\Sigma \sim \sigma_{\cdot,\cdot}$.

(ii) $\forall \mu \in M_d(G) \widetilde{\widehat{G}}(\mu) \simeq \widehat{G}_d^\mu$.

(iii) $\mu \in M_d(G) \Leftrightarrow \widetilde{\widehat{G}}(\mu)$ is a group (under the multiplication in $L(M(G), M(\mathbf{T}))$).

Proof. (i) Let $\widehat{G} \ni \gamma_\alpha \xrightarrow{W^*OT} \Sigma$. Then for $x \in G$,

$$\delta(\gamma_\alpha(\cdot)) \rightarrow \sigma_{\cdot, \delta(x)} \in L(M(G), M(\mathbf{T}))_{\delta(x)},$$

i.e., $\delta(\gamma_\alpha(x)) = \sigma_{x, \delta(x)}$ eventually. Thus, $\gamma_\alpha(x)$ stabilizes at some point $\varphi(x)$ and $\sigma_{x, \delta(x)} = \delta(\varphi(x))$. The assertions now follow from linearity and properties of the Šreider representation.

(ii) The fact that $\widetilde{\widehat{G}}(\mu)$ can be identified as a compact subgroup of \widehat{G}_d^μ follows from (i). If it were not the whole group, then there would be a nonzero $x \in G_d^\mu$ such that $\varphi(x) = 1$ for all $\varphi \in \widetilde{\widehat{G}}(\mu)$. In particular, $\gamma(x) = 1$ for all $\gamma \in \widehat{G}$, whence $x = 0$, a contradiction.

(iii) This follows from [HMP, p. 68, Proposition 10] and (ii). □

We now arrive at the characterization of positive continuous measures mentioned in the introduction.

THEOREM 4.4. *Let G be a LCA group whose torsion subgroup is denumerable and let $\mu \in M^+(G)$ be positive. Then $\mu \in M_c^+(G)$ iff there is a net $\widehat{G} \ni \gamma_\alpha \rightarrow \infty$ such that for all $n \neq 0$, $\hat{\mu}(\gamma_\alpha^n) \rightarrow 0$.*

Proof. By Proposition 4.1 (iii), this is equivalent to $\mu \in M_c^+(G) \Leftrightarrow \hat{\mu}(0)\lambda \in \Lambda(\mu)$. For $\mu \in M_c^+(G)$, this follows from $\lambda \in \Lambda(\mu)$ (see (4.3)). If $\mu \notin M_c^+(G)$ and $\Sigma \in \Lambda$, then $\Sigma_\mu = \Sigma_{\mu_c} + \Sigma_{\mu_d} \geq \Sigma_{\mu_d}$ since $\mu_c \geq 0$ and $\Sigma \geq 0$. However, by Proposition 4.3(i), Σ_{μ_d} is nonzero and discrete; hence Σ_μ cannot equal $\hat{\mu}(0)\lambda$. □

Because of the interest this theorem may present, we provide the following “elementary” proof and strengthening for the case $G = \mathbf{T}$. If $\mu \in M_c(\mathbf{T})$, then by Wiener’s theorem [K, p. 42], there is a sequence $\{m_k^{(1)}\}$ of density one in \mathbf{N} such that $\hat{\mu}(m_k^{(1)}) \rightarrow 0$. Likewise, there is a sequence $\{m_k^{(n)}\}$ of density one such that $\hat{\mu}(nm_k^{(n)}) = (\widehat{T_n})_\mu(m_k^{(n)}) \rightarrow 0$ since $(T_n)_\mu \in M_c$, for $n \neq 0$. By an elementary intersection argument, we obtain a sequence $\{m_k\}$, still of density one, such that for all $n \neq 0$, $\hat{\mu}(nm_k) \rightarrow 0$. (A similar argument produces a sequence $\{m_k\}$ of density one such that for $n \neq 0$ and all r , $\hat{\mu}(r + nm_k) \rightarrow 0$, i.e., $\delta(m_k x) \rightarrow \lambda$ in $L(M(\mathbf{T}), M(\mathbf{T}))_\mu$, thereby strengthening (4.3).) For the converse, we use the following proof due to Jean-François Méla. Let $K_l(x)$ be the Fejér kernel of order l . Then if $\mu \geq 0$ and if for

all $n \neq 0$, $\hat{\mu}(nm_k) \rightarrow 0$, then

$$\mu(\{0\}) \leq \int_{\mathbf{T}} \frac{1}{2l+1} K_l(m_k x) d\mu(x) \rightarrow \frac{1}{2l+1} \hat{\mu}(0) \quad \text{as } k \rightarrow \infty$$

by hypothesis. Since this is true for all l , it follows that $\mu(\{0\}) = 0$. Now apply this result to $\mu * \tilde{\mu}$, where $\tilde{\mu}(E) = \mu(-E)$.

The local structure of Λ can be used to characterize other classes of measures besides M_c and M_d . If \mathcal{E} is a class of subsets of G , let

$$\mathcal{E}^\perp = \{\mu \in M(G) : \forall E \in \mathcal{E} \ |\mu|(E) = 0\}.$$

Thus, if \mathcal{S} is the class of singletons, $\mathcal{S}^\perp = M_c(G)$.

DEFINITION. A set $E \subseteq G$ is called an *H-set* if there is a sequence $\hat{G} \ni \gamma_k \rightarrow \infty$ such that $\{\gamma_k(x) : k \geq 1, x \in E\}$ is not dense in \mathbf{T} . A set $E \subseteq G$ is called a *Dirichlet set* if there is a sequence $\hat{G} \ni \gamma_k \rightarrow \infty$ such that $\lim_{k \rightarrow \infty} \sup_{x \in E} |\gamma_k(x) - 1| = 0$. A measure $\mu \in M(G)$ is called a *Dirichlet measure* if $\overline{\lim}_{\gamma \rightarrow \infty} |\widehat{|\mu|}(\gamma)| = \|\mu\|$.

For background on *H-sets*, see [Z, Chapters IX, XII]; on Dirichlet sets and measures, see [HMP, pp. 34–35, 240–247]. The following proposition is used in [KL].

PROPOSITION 4.5. *Let G be a LCA group.*

(i) *If G is metrizable, then*

$$\begin{aligned} H^\perp &= \{\mu : \forall \sigma. \in \widetilde{\Lambda}(\mu) \ \forall^e x[\mu] \ \text{supp } \sigma_x = \mathbf{T}\} \\ &= \{\mu : \forall \Sigma \in \Lambda \ \forall \nu \in L(\mu) \ \text{supp } \Sigma_\nu = \mathbf{T}\}. \end{aligned}$$

(ii) *μ is a Dirichlet measure iff the constant function $\delta(\mathbf{0}) \in \Lambda(\mu)$.*

(iii) *$D^\perp = \{\mu : \forall \sigma. \in \widetilde{\Lambda}(\mu) \ \forall^e x[\mu] \ \sigma_x \neq \delta(0)\}$*

Proof. Part (i) follows from Proposition 4.1(v) and a straightforward generalization of [L4, Theorem 13]. Part (ii) follows from Proposition 4.2 and the fact that μ is a Dirichlet measure iff the constant function $\mathbf{1} \in (\overline{\Gamma} \setminus \Gamma)(\mu)$ [HMP, p. 34, Lemma 6]. Part (iii) follows from part (ii) and the fact that D^\perp consists of the measures orthogonal to the Dirichlet measures [HMP, p. 243, Proposition 9]. \square

Our final remarks concern the circle group.

DEFINITION. A positive measure $\mu \in M^+(\mathbf{T})$ is called *C-quasi-symmetric* if for all pairs of adjacent arcs, I and J , on \mathbf{T} of equal

length, $\mu I \leq C \cdot \mu J$. We denote the class of C -quasisymmetric measures by $QS(C)$.

Note that quasisymmetric measures are continuous.

PROPOSITION 4.6. *The class $QS(C)$ is weak* closed. If $\mu \in QS(C)$, then $\Lambda(\mu) \subseteq QS(C)$, $\widetilde{\Lambda}(\mu) \subseteq QS(C)$ in the sense that if $\sigma \in \Lambda(\mu)$, then $\forall^\varepsilon x[\mu] \sigma_x \in QS(C)$, and $\Lambda(\nu) \subseteq QS(C)$ for all $0 \leq \nu \in L(\mu)$.*

Proof. Let $QS(C) \ni \mu_\alpha \xrightarrow{w^*} \nu$. Given adjacent arcs I, J of equal length and $\varepsilon > 0$, pick $f, g \in C(\mathbb{T})$ such that $f \leq \mathbf{1}_I, \mathbf{1}_J \leq g$, $\int(\mathbf{1}_I - f) d\nu \leq \varepsilon$, and $\int(g - \mathbf{1}_J) d\nu \leq \varepsilon$. We have

$$\begin{aligned} \nu I &\leq \int f d\nu + \varepsilon = \lim \int f d\mu_\alpha + \varepsilon \leq \overline{\lim} \mu_\alpha I + \varepsilon \\ &\leq C \cdot \overline{\lim} \mu_\alpha J + \varepsilon \leq C \cdot \lim \int g d\mu_\alpha + \varepsilon = C \int g d\nu + \varepsilon \\ &\leq C \cdot \nu J + (C + 1)\varepsilon. \end{aligned}$$

Since ε was arbitrary, we see that $\nu I \leq C \cdot \nu J$, whence $\nu \in QS(C)$.

Choose $\mu \in QS(C)$. Then $\gamma_\mu \in QS(C)$ for any $\gamma \in \widehat{\mathbb{T}}$. Since $\Lambda(\mu)$ is contained in the weak* closure of $\{\gamma_\mu\}_{\gamma \in \widehat{\mathbb{T}}}$, it follows that $\Lambda(\mu) \subseteq QS(C)$. Suppose that $E \subseteq \mathbb{T}$ and $\mu E > 0$. If I and J are adjacent arcs of equal length and $\varepsilon > 0$, then choose U , a finite union of arcs, such that $\mu(U \Delta E) \leq \varepsilon$. By continuity of μ , we have for all large γ ,

$$\begin{aligned} \mu(E \cap \gamma^{-1}[I]) &\leq \mu(U \cap \gamma^{-1}[I]) + \varepsilon \leq C \cdot \mu(U \cap \gamma^{-1}[J]) + 2\varepsilon \\ &\leq C \cdot \mu(E \cap \gamma^{-1}[J]) + (C + 2)\varepsilon. \end{aligned}$$

Since ε was arbitrary, it follows that $\Lambda(\mu|_E) \subseteq QS(C)$. As $QS(C)$ is a positive cone, we deduce that $\Lambda(\nu) \subseteq QS(C)$ for $0 \leq \nu \in L(\mu)$.

Finally, let $\sigma \in \Lambda(\mu)$. Let P be the essential range of σ , i.e., the smallest weak* closed set P such that $\sigma_x \in P$ μ -a.e. Then P is contained in the weak* closure of $\{\int \sigma_x d\nu(x) : 0 \leq \nu \in L(\mu), \|\nu\| = 1\} = \bigcup\{\Lambda(\nu) : 0 \leq \nu \in L(\mu), \|\nu\| = 1\}$, which, by the above, is contained in $QS(C)$. \square

As an example of the pathology possible for $\Lambda(\mu)$, we present the following observation.

PROPOSITION 4.7. *There is a measure $\mu \in M(\mathbb{T})$ such that for any probability measure $\nu \in M(\mathbb{T})$, there exists $\sigma \in \Lambda(\mu)$ such that $\sigma_x = \nu \mu$ -a.e.*

Proof. Let $\{P_k\}_{k \geq 1}$ be a set of trigonometric polynomials such that $\{P_k \cdot \lambda\}$ is weak* dense in the set of probability measures. Let $\{n_k\} \subseteq \mathbb{N}$ satisfy $n_{k+1} \geq 3n_k \cdot \deg P_k$. Form the generalized Riesz product [HMP, Chapitre 5] $\mu = \prod_{k \geq 1} P_k(n_k x)$. Then given a probability ν , let $P_{k_l} \lambda \xrightarrow{w^*} \nu$. For any $r, m \in \mathbb{Z}$, it is easy to see that $\hat{\mu}(r + mn_{k_l}) \rightarrow \hat{\mu}(r)\hat{\nu}(m)$, i.e., $\delta(n_{k_l} x) \rightarrow \nu$ in $L(M(\mathbb{T}), M(\mathbb{T}))_\mu$. \square

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STANFORD UNIVERSITY
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