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In the present paper we examine rank-2 stable bundles over Q_3 with $c_1 = 0$ and $c_2 = 2$ or 4.

This paper is a continuation of [7] where rank-2 Fano bundles over $\mathbb{P}^3$ and Q_3 were studied. Let us recall that a bundle $\mathscr{E}$ is called Fano if its projectivization $\mathbb{P}(\mathscr{E})$ is a Fano manifold, i.e. a manifold with ample first Chern class $c_1(\mathbb{P}(\mathscr{E}))$. In the present paper we examine rank-2 stable bundles over Q_3 with $c_1 = 0$ and $c_2 = 2$ or 4. These are the cases whose knowledge was necessary to complete the classification of rank-2 Fano bundles over Q_3. They are very different: if $\mathscr{E}$ is stable with $c_1 = 0$, $c_2 = 2$ then its first twist $\mathscr{E}(1)$ is spanned by global sections (see Proposition 1), whereas if $c_2 = 4$ then for a general $\mathscr{E}$ from a component in the moduli $\mathscr{E}(1)$ has no section at all (Proposition 3). We complete the classification of rank-2 Fano bundles over Q_3. The results of §3 from [7] and of the present paper can be summarized in the following

THEOREM. *Let $\mathscr{E}$ be a rank-2 Fano bundle over Q_3. If $c_1\mathscr{E} = -1$ then $\mathscr{E}$ is either $\mathscr{O} \oplus \mathscr{O}(-1)$ or the spinor bundle $\underline{E}$. If $c_1\mathscr{E} = 0$ then $\mathscr{E}$ is either $\mathscr{O} \oplus \mathscr{O}$, or $\mathscr{O}(-1) \oplus \mathscr{O}(1)$, or any stable bundle with $c_2 = 2$ (see a corollary in* §1 *for a complete description of such bundles).*

Let us recall that the spinor bundle $\underline{E}$ on an odd-dimensional quadric $Q_{2\nu+1}$ is the restriction of the universal 2^ν-bundle on the Grassmannian $\mathrm{Gr}(2^\nu, 2^{\nu+1})$. Then $\underline{E}^* = \underline{E}(1)$. On an even-dimensional quadric $Q_{2\nu}$, $\nu \geq 2$, there are two spinor bundles, corresponding to the two reguli of ν-planes. The following characterization of the bundles with no intermediate cohomology was proved in [1]:

THEOREM. *For a vector bundle F on Q_n, $n \geq 2$, it is $H^i(F(l)) = 0$ for all $0 < i < n$, $l \in \mathscr{Z}$, if and only if F is a direct sum of line bundles $\mathscr{O}(l)$ and of their tensor product with spinor bundles.*

1. Bundles with $c_1 = 0$, $c_2 = 2$. In this section we prove the following

PROPOSITION 1. *Let* $\mathscr{E}$ *be a stable bundle on* Q_3 *with* $c_1 = 0$, $c_2 = 2$. *Then* $\mathscr{E}(1)$ *is globally generated* (*and therefore is Fano*).

Then, in view of the Proposition (3.2) from [7] we have:

COROLLARY. *Any stable rank-*2 *bundle on* Q_3 *with* $c_1 = 0$, $c_2 = 2$ *is the pullback of a null correlation bundle on* $\mathbb{P}^3$ *via some double covering* $Q_3 \to \mathbb{P}^3$ (*see* [5] *for a definition of the null correlation bundle*).

To prove the proposition we apply a technique of "killing H^1", developed by Horrocks, see the final acknowledgments in [2]. Namely, starting from a bundle $\mathscr{F}$ with, say, $H^1(\mathscr{F}(-1)) \neq 0$, we take a nontrivial extension of $\mathscr{F}(-1)$ by $\mathcal{O}$ which corresponds to this element of the cohomology. Then the middle bundle of the exact sequence that forms the extension has "simpler" cohomology than the initial one. Eventually, we obtain a bundle with no intermediate cohomology and we use classification theorems of such bundles, see [1]. The proof will be divided into several steps.

Step 1. Using the information on the spectrum of stable bundles, [3], we calculate the cohomology of $\mathscr{E}(1)$:

i

0	0	0	0
1	1	0	0
0	0	1	1
0	0	0	0

$h^i(\mathscr{E}(j))$

$j = -2$ $j = 0$ j

Step 2. Let us take a nontrivial extension

$$0 \to \mathscr{E}(-1) \to B \to \mathcal{O} \to 0 \tag{1}$$

which corresponds to a non-zero element of $\operatorname{Ext}^1(\mathcal{O}, \mathscr{E}(-1)) = H^1(\mathscr{E}(-1))$. The extension is non-trivial; hence the connecting homomorphism $\delta\colon H^0(\mathcal{O}) \to H^1(\mathscr{E}(-1))$ is a non-zero map. Then we may fill out the cohomology diagram for $B(j)$ as follows:

i

0	0	0	0
1	1	0	0
0	0	0	a
0	0	0	

$h^i(B(j))$

$j = -2$ $j = 0$ j

with $a \leq 1$.

Step 3. Let us take $B' = B^*(-1)$. The Chern classes of B' are the following: $c_1 = -1$, $c_2 = 2$, $c_3 = -2$. The cohomology of B' can be easily derived from that of B and the result is

i

	0	0	0
a	0	0	0
0	0	1	1
0	0	0	0

$h^i(B'(j))$

$j = -2$ $j = 0$ j

We see that $\dim H^1(B'(-1)) = \dim \mathrm{Ext}^1(\mathcal{O}, B'(-1)) = 1$, so that we consider an extension

$$0 \longrightarrow B'(-1) \longrightarrow C(-1) \longrightarrow \mathcal{O} \longrightarrow 0 \tag{2}$$

corresponding to a non-zero element of $H^1(B'(-1))$.

Step 4. We then calculate that C is a rank-4 vector bundle with all Chern classes zero and the cohomology

i

	0	0	0
a	0	0	0
0	0	0	b
0	0	0	

$h^i(C(j))$

$j = -2$ $j = 0$ j

with $a \leq 1$, $b \leq 1$.

Step 5. Let

$$0 \longrightarrow C \longrightarrow D \longrightarrow \mathcal{O} \longrightarrow 0 \tag{3}$$

be a non-trivial extension (if $b = 1$) or the splitting one (if $b = 0$). In both cases all Chern classes of D vanish and the cohomology of D

is

i

$5-a$	0	0	0
a	0	0	0
0	0	0	0
0	0	0	5
	$j=-2$		$j=0$

$h^i(D(j))$

j

Step 6. In a similar way we get rid of a (possibly) positive a. Let us take

$$0 \longrightarrow D^* \longrightarrow F \longrightarrow \mathcal{O} \longrightarrow 0, \tag{4}$$

corresponding to a generator of $H^1(D^*) = H^2(D(-3))$. The bundle F is uniquely determined up to proportionality in $\operatorname{Ext}^1(\mathcal{O}, D^*) = H^1(D^*)$. It is a 6-bundle on Q_3 with no intermediate cohomology, with $H^0(F(-1))$ vanishing and all Chern classes equal zero.

Claim. F is either $\mathcal{O}^6$ or $\mathcal{O}^2 \oplus \underline{E} \oplus \underline{E}^*$, where $\underline{E}$ is the spinor bundle on Q_3.

Proof. It follows easily from the characterization of bundles with no intermediate cohomology.

Step 7. If F is $\mathcal{O}^6$, then D and C in (4) and (3) must be trivial. Dualizing (2) then gives the sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}^4 \longrightarrow B(1) \longrightarrow 0 \tag{5}$$

whose second exterior power is

$$0 \longrightarrow B \longrightarrow \mathcal{O}^6 \longrightarrow B^* \longrightarrow 0 \tag{6}$$

—notice that $B^* = \bigwedge^2[B(1)]$ because B is of rank 3 and $c_1(B) = -2$. Therefore $\mathcal{E}(1)$ is globally generated, because it is an image of B^* (see (1)).

Step 8. We now want to exclude the case $F = \mathcal{O}^2 \oplus \underline{E} \oplus \underline{E}^*$. Assume this is the case. Let us look at the epimorphism $F \longrightarrow \mathcal{O}$ in (4). Its dual is an embedding $\mathcal{O} \subset \mathcal{O} \oplus \mathcal{O} \oplus \underline{E} \oplus \underline{E}^*$. Because $H^0(\underline{E}) = 0$ and $\underline{E}^*$ has no non-vanishing sections, see [1], then the embedding map sends $\mathcal{O}$ into $\mathcal{O} \oplus \mathcal{O}$. Hence the bundle D^* in (4) is equal to $\mathcal{O} \oplus \underline{E} \oplus \underline{E}^*$. In the same way we conclude that $C = \underline{E} \oplus \underline{E}^*$, so instead of (5) we get

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \underline{E} \oplus \underline{E}^* \longrightarrow B(1) \longrightarrow 0. \tag{7}$$

Raising this sequence to the second symmetric power, making use of the identity $B^* = \bigwedge^2[B(1)]$ again and recalling that

$$\begin{aligned}\overset{2}{\bigwedge}(\underline{E} \oplus \underline{E}^*) &= \overset{2}{\bigwedge}(\underline{E}) \oplus (\underline{E} \otimes \underline{E}^*) \oplus \overset{2}{\bigwedge}(\underline{E}^*)\\ &= \mathscr{O}(-1) \oplus \mathscr{E}nd(\underline{E}) \oplus \mathscr{O}(1),\end{aligned}$$

we obtain an analogue of (6):

$$(8) \qquad 0 \longrightarrow B \longrightarrow \mathscr{O}(1) \oplus \mathscr{E}nd(\underline{E}) \oplus \mathscr{O}(-1) \longrightarrow B^* \longrightarrow 0,$$

whose twist by -1 is

$$(9) \quad 0 \longrightarrow B(-1) \longrightarrow \mathscr{O} \oplus [\mathscr{E}nd(\underline{E})](-1) \oplus \mathscr{O}(-2) \longrightarrow B^*(-1) \longrightarrow 0,$$

which contradicts the cohomology tables from Step 2 and Step 3—namely that $B(-1)$ and $B^*(-1)$ have no sections.

2. Bundles with $c_1 = 0$, $c_2 = 4$. In view of the results of [7] the following completes the proof of the theorem stated at the beginning of the paper.

PROPOSITION 2. *A vector bundle* $\mathscr{E}$ *on* Q_3 *which has* $c_1 = 0$, $c_2 = 4$ *cannot be Fano.*

Proof. First let us note that an unstable $\mathscr{E}$ with $c_1 = 0$, $c_2 = 4$ cannot be Fano—this is proved at the beginning of §3 in [7]. So let us assume that $\mathscr{E}$ is stable. Using the spectrum technique [3], we calculate the cohomology of $\mathscr{E}(j)$ to be

i

0	0	0	0
4	2	0	0
0	0	2	4
0	0	0	0

$h^i(\mathscr{E}(j))$

$j = -2$ $\quad j = 0$ $\quad j$

Consider the natural bilinear map

$$\vartheta \colon H^1(\mathscr{E}(-1)) \times H^0(\mathscr{O}(1)) \longrightarrow H^1(\mathscr{E}).$$

We see that $\dim H^1(\mathscr{E}(-1)) = 2$, $\dim H^0(\mathscr{O}(1)) = 5$ and moreover $\dim H^1(\mathscr{E}) = 4$. The bilinear lemma [4] gives the existence of s and h such that $(s, h) = 0$. Hence there is a section of $\mathscr{E}|Q_2$ over a (not necessarily smooth) hyperplane section of the quadric. The section vanishes at four points. These points are not necessarily distinct, but they are not collinear since otherwise the splitting type of $\mathcal{E}$ on this line would be $(-c, c)$ with $c \geq 2$, contradicting the ampleness of $\mathcal{E}(2)$. Let us take a conic C that passes through at least three of these points, counted with multiplicities. Then $\mathcal{E}|C = \mathcal{O}_C(-d) \oplus \mathcal{O}_C(d)$ with $d \geq 3$, because the section has at least triple zero. But this implies that there exists an effective 1-cycle C' associated to the section of $\mathcal{E}(-3)|C$; the cycle C' is numerically equivalent to $\xi_{\mathcal{E}(-3)} \cdot p^{-1}(C)$, where $\xi_{\mathcal{E}(-3)}$ is the relative hyperplane divisor on $\mathbb{P}(\mathcal{E})$ associated to $\mathcal{E}(-3)$ i.e. a class whose restriction to a fiber of the projection p; $\mathbb{P}(\mathcal{E}) \to Q_3$ is a hyperplane and $p_*\mathcal{O}(\xi_{\mathcal{E}(-3)}) = \mathcal{E}(-3)$. Then $H \cdot C' = 2$, $\xi_{\mathcal{E}'} = -d$, where H is the pullback of the hyperplane divisor from Q_3 and $\xi_{\mathcal{E}}$ is equivalent to $\xi_{\mathcal{E}(-3)} + 3H$. Because the anticanonical divisor of $\mathbb{P}(\mathcal{E})$ is equivalent to $2\xi_{\mathcal{E}} + 3H$, we have

$$-K_{\mathbb{P}(\mathcal{E})} \cdot C' = (2\xi_{\mathcal{E}} + 3H) \cdot C' \leq 0,$$

so that $-K_{\mathbb{P}(\mathcal{E})}$ cannot be ample. □

Remark. Although ruled out from our Fano list, the investigation of rank-2 vector bundles $\mathcal{E}$ with $c_1 = 0$, $c_2 = 4$ on Q_3 seems to be an interesting open problem. In particular:

does a general $\mathcal{E}(1)$ have a section?

We believe that the answer is no. So far we can only show

Proposition 3. *In the moduli space of stable bundles with* $c_1 = 0$, $c_2 = 4$ *there is a component containing bundles with* $H^0(\mathcal{E}(1)) = 0$.

Proof. Assume Z is the zero set of a section of such an $\mathcal{E}(1)$. Because of stability, Z is not a surface while the indecomposability of $\mathcal{E}$ shows that Z is not empty. Hence Z must be a curve. By the adjunction formula we have

$$K_Z = \mathcal{O}_Q(-1)|Z; \tag{10}$$

hence no connected component of Z may be a single line.

Since $c_2(\mathcal{E}(1)) = 6$, we conclude that Z has at most three connected components. Let us consider the bundles given as extensions

$$0 \to \mathcal{O} \to \mathcal{E}(1) \to J_C(2) \to 0 \tag{11}$$

where C is the sum of three conics. Let us count how many bundles can be obtained in this way. The conics in Q_3 are in 1-1 correspondence with 2-planes in $\mathbb{P}^4$, hence the dimension of the family of triples of conics is equal to $3 \cdot \dim(\mathrm{Grass}(2, 4)) = 18$. The number of non-isomorphic extensions of the form (11) is equal to the dimension of

$$\mathrm{Ext}^1(J_C(2), \mathscr{O})) = H^0(\mathscr{E}\!\mathit{xt}(\mathscr{O}_C(2), \mathscr{O}) = H^0(\mathscr{O}_C),$$

i.e., to 3, see **[5]**, Ch. I, §5.1. Because proportional extensions give rise to isomorphic bundles, altogether we have a bundle family of dimension $18 + 3 - 1 = 20$. On the other hand, using the obvious relation $\mathscr{E}\!\mathit{nd}(\mathscr{E}) = \mathscr{E} \otimes \mathscr{E}^*$ we calculate using (11) that

$$\dim(H^0(\mathscr{E}\!\mathit{nd}(\mathscr{E}))) = 1, \quad \dim(H^1(\mathscr{E}\!\mathit{nd}(\mathscr{E}))) = 21,$$
$$\dim(H^2(\mathscr{E}\!\mathit{nd}(E))) = 0, \quad \dim(H^3(\mathscr{E}\!\mathit{nd}(\mathscr{E}))) = 0.$$

Therefore a local deformation of a bundle given by (11) need not be such. The bundles that do not arise from deformations of those given by (11) must then come from curves C's having at least four components, which is not possible by (10). Hence $\mathscr{E}(1)$ has no section. Because of the semicontinuity, the same holds for a generic bundle in the same component. □

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