RICCI CURVATURE AND VOLUME GROWTH

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We give an example of a complete manifold $M^m$ of nonnegative Ricci curvature for which the volume of distance tubes around a totally geodesic submanifold $L^l$ divided by the corresponding volume in $L \times \mathbb{R}^{m-l}$ goes to infinity. Recall that in the case of nonnegative sectional curvature, this quotient is nonincreasing and bounded by $1$.

1. Introduction. One of the fundamental tools in the study of Ricci curvature is the Bishop-Gromov volume inequality, which states that in a complete manifold $M^m$ of Ricci curvature $\geq (m-1)\kappa$, the map

$$r \mapsto \frac{\text{vol} B_r(p)}{\text{vol} (D_r, \hat{g}_\kappa)}$$

is monotonically nonincreasing. Here, $B_r(p)$ is the ball of radius $r$ around $p \in M$, and $(D_r, \hat{g}_\kappa)$ is a ball of same radius in the simply connected space of constant sectional curvature $\kappa$. Under somewhat different assumptions, this inequality still holds when $p$ is replaced by a compact, totally geodesic submanifold $L^l$ of $M$: The comparison space now becomes $(L \times D_r, g_\kappa)$, where for $x = (x_0, x_1)$ in the tangent space of $L \times D_r$ at $(p, u)$, $g_\kappa(x, x) = c_\kappa^2(|u|) \hat{g}(x_0, x_0) + \hat{g}_\kappa(x_1, x_1)$. (Here $\hat{g}$ is the metric on $L$ induced by the imbedding $L \hookrightarrow M$, and $c_\kappa$ is the solution of the equation $c''_\kappa + \kappa c_\kappa = 0$, with $c_\kappa(0) = 1$, $c'_\kappa(0) = 0$.) The volume inequality now reads (cf. [4], [3], [6]):

(*) If the radial sectional curvatures of $M$ are $\geq \kappa$, then

$$q_L(r) \equiv \frac{\text{vol} B_r(L)}{\text{vol} (L \times D_r, g_\kappa)}$$

is a nonincreasing function of $r$, with $q_L(0) = 1$. (A 2-plane $\sigma \subset M_q$ is said to be radial if it contains the tangent vector of some minimal geodesic from $q$ to $L$.)

(**) If all sectional curvatures of $M$ are $\geq \kappa$, then $q_L(r') = q_L(r)$ for some $0 < r' < r$ only if the normal bundle of $L \hookrightarrow M$ is flat with respect to the induced connection, and $B_r(L)$ is (locally) isometric to $(L \times D_r, g_\kappa)$.
In this note, we show that (*) no longer holds in general if one only assumes $\text{Ric}_{\mathcal{M}} > (m - 1)\kappa$ (see also [1] for a related result): In fact, the quotient $q_L(r)$ may go to infinity as $r \to \infty$. Moreover, even if the radial sectional curvatures are $\geq \kappa$—so that (*) must hold—(**) is no longer true if one replaces $K_{\mathcal{M}} \geq \kappa$ by $\text{Ric}_{\mathcal{M}} \geq (m - 1)\kappa$. More precisely, we have:

1.1. Theorem. Let $L = \mathbb{C}P^1$, and $M = \mathbb{C}P^2$. Then

(a) The normal bundle $E$ of $L \hookrightarrow M$ admits a complete metric of nonnegative Ricci curvature such that

$$q_L(r) \overset{\text{def}}{=} \frac{\text{vol} B_r(L)}{\text{vol}(L \times D_r, g_0)}$$

goes monotonically to infinity as $r \to \infty$.

(b) There is a complete metric on $M$ with the following properties:

1. $L$ is totally geodesically imbedded in $M$.
2. $\text{Ric}_{\mathcal{M}} \geq 3$, and the radial sectional curvatures are $\geq 1$.
3. $q_L(r) \overset{\text{def}}{=} \frac{\text{vol} B_r(L)}{\text{vol}(L \times D_r, g_0)} \equiv 1$ for $r \leq \varepsilon$, provided $\varepsilon$ is sufficiently small.

2. Ricci curvature for connection metrics. Let $L = \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2$ with the standard metric of curvature $1 \leq K \leq 4$. As in [5], we identify a distance tube $B_r(L)$ around $L$ with $[0, r] \times S^3/\sim$, where all the Hopf fibers are collapsed to a point at $\{0\} \times S^3$. Consider the class $d\sigma^2$ of metrics on $S^3$ obtained by multiplying the standard metric by $f^2(r)$ in the Hopf fiber direction, and by $h^2(r)$ on its orthogonal complement. If $f$ is an odd smooth function with $f'(0) = 1$, and $h$ is even and positive, then the metric $dr^2 + d\sigma^2$ on $[0, r] \times S^3$ extends to $B_r(L)$. The standard metric corresponds to $f(r) = (1/2)\sin 2r$ and $h(r) = \cos r$. Using the same vector fields $X_i$, $0 \leq i \leq 3$, as in [5] (where $X_0$ is radial, $X_1$ is tangent to the Hopf fiber, and $X_2$, $X_3$ are orthogonal to it), we obtain for $R_{ij} := \text{Ric}(X_i/|X_i|, X_j/|X_j|)$:

\begin{align*}
(2-1) & \quad R_{00} = -\frac{f''}{f} - 2\frac{h''}{h}, \\
(2-2) & \quad R_{11} = -\frac{f''}{f} - 2\frac{f'h'}{fh} + 2\frac{f^2}{h^4}, \\
(2-3) & \quad R_{22} = R_{33} = -\frac{h''}{h} - \frac{f'h'}{fh} + \frac{4h^2 - 2f^2 - h'^2h}{h^4}, \\
(2-4) & \quad R_{ij} = 0, \quad i \neq j.
\end{align*}
The proof is straightforward and will be omitted.

This class of metrics is actually a special case of the following construction: Let \((L^l, \hat{g})\) be a Riemannian manifold, and \(R^k \to E \xrightarrow{\pi} L\) a vector bundle with inner product \(\langle \cdot, \cdot \rangle\) and Riemannian connection \(\nabla\). Fix \(0 < r_0 \leq \infty\), and consider the disk bundle \(E_{r_0} = \{ u \in E \mid \langle u, u \rangle < r_0 \}\). If \(\mathcal{V}\) denotes the vertical distribution defined by \(\pi\), and \(\mathcal{H}\) the horizontal distribution determined by the connection, define

\[
g(x, x) = h^2(|u|) \hat{g}(\pi_* x, \pi_* x) \quad (x \in \mathcal{H} \cap T_u E),
\]

where \(h\) is an even, smooth, positive function on \((-r_0, r_0)\). The fibers of \(E_{r_0}\) are endowed with a metric given in polar coordinates by

\[	d r^2 + f^2(r) d\sigma^2,
\]

where \(d\sigma^2\) is the standard metric on the sphere, and \(f\) is an odd, smooth function with \(f'(0) = 1\). We then obtain a metric \(g\) on \(E_{r_0}\) by declaring \(\mathcal{H}\) and \(\mathcal{V}\) to be mutually orthogonal. The fibers of the bundle are totally geodesic submanifolds in this metric, and the projection \(\pi\) restricted to a sphere bundle of radius \(r\) becomes a Riemannian submersion with base \((L, h^2(r) \hat{g})\). One can easily compute the Ricci curvatures by using O'Neill's formula for Riemannian submersions and the Gauss equations (cf. also [2]): If \(\partial_r\) denotes the unit radial vector field (dual to \(d r\)), \(v\) a unit vertical vector orthogonal to \(\partial_r\), and \(x\) a unit horizontal vector, then

\[
\text{(2-5)} \quad \text{Ric}(\partial_r, \partial_r) = -l \frac{h''}{h} - (k - 1) \frac{f''}{f},
\]

\[
\text{(2-6)} \quad \text{Ric}(\partial_r, x) = \text{Ric}(\partial_r, v) = 0,
\]

\[
\text{Ric}(v, v) = -\frac{f''}{f} + (k - 2) \frac{1 - f'^2}{f^2} - l \frac{f'h'}{fh}
\]

\[
+ \sum_{i=1}^{l} \langle A_{x_i} v, A_{x_i} v \rangle,
\]

\[
\text{Ric}(x, x) = -\frac{h''}{h} - (l - 1) \frac{h'^2}{h^2} - (k - 1) \frac{h'f'}{hf}
\]

\[
+ \text{Ric}(\pi_* x, \pi_* x) - 2 \sum_{i=1}^{l} \langle A_{x_i} x_i, A_{x_i} x_i \rangle,
\]

\[
\text{(2-9)} \quad \text{Ric}(v, x) = \langle (\tilde{A} x), v \rangle.
\]
Here, \( \{x_i\} \) is an orthonormal basis of \( \mathcal{H} \), \( A \) is the O'Neill tensor of the submersion with divergence \( \delta A = \sum_{i=1}^l D_{x_i}A(x_i, \cdot) \) (\( D \) is the Levi-Civita connection of \( (E^\circ, g) \)), and \( \text{Ric}^\nabla \) is the Ricci tensor of \( (L, h^2(r)\hat{g}) \).

Moreover, if \( \nabla \) is a Yang-Mills connection, then (cf. [2], p. 243):

\[
(2-9') \quad \text{Ric}(v, x) = 0.
\]

In the special case when \( E \) is the normal bundle of \( CP^1 \rightarrow CP^2 \), let \( \nabla \) denote the connection on \( E \) induced by the Levi-Civita connection of the symmetric space \( CP^2 \). Then \( \nabla \) is Yang-Mills since the curvature tensor \( R^\nabla \) is parallel. In particular, \( (2-9') \) holds, and it is straightforward to check that \( (2-5)-(2-9) \) reduce to \( (2-1)-(2-4) \). Notice that the \( A \)-tensor can be expressed in terms of \( R^\nabla \), cf. [6].

### 3. Proof.

**Proof of 1.1(a).** The volume of a distance tube \( B_r(L) \) with respect to the class of metrics described in §2 is given by:

\[
\text{vol } B_r(L) = \int_0^r \text{vol } S_t(L) \, dt = C \cdot \text{vol } (L) \cdot h^{-1}(0) \cdot \int_0^r h^l(t) f^{k-1}(t) \, dt,
\]

where \( S_t(L) \) is a distance sphere around \( L \), \( \text{vol } (L) := \text{vol } (L, h^2(0)\hat{g}) \), and \( C \) is the volume of the standard sphere \( S^{k-1} \subset \mathbb{R}^k \). It thus suffices to find functions \( f \) and \( h \) such that \( (2-1)-(2-3) \) yield \( \text{Ric} \geq 0 \), and \( h^l(r) f^{k-1}(r)/r^{k-1} = h^2(r) f(r)/r \rightarrow \infty \) as \( r \rightarrow \infty \). Let \( f(r) := r/(1 + r^2)^{1/2} \), and \( h(r) := (r/f(r))^\alpha \), where \( \alpha \) is any constant in the interval \([1/2, 1]\). Notice that \( q_L(r) \rightarrow \infty \) as \( r \rightarrow \infty \) if \( \alpha > 1/2 \), and \( q_L(r) \equiv 1 \) for \( \alpha = 1/2 \).

A straightforward calculation shows that \( (2-1)-(2-3) \) become:

\[
(3-1) \quad R_{0,0} = \frac{-3(2\alpha - 1)}{(1 + r^2)^2} + \frac{2\alpha}{1 + r^2} \left( 2 - (\alpha + 1) \frac{r^2}{1 + r^2} \right)
= \frac{\alpha}{1 + r^2} (4 - \varphi_\alpha(r)),
\]

where \( \varphi_\alpha(r) = (3(2\alpha - 1) + 2\alpha(\alpha + 1)r^2)/\alpha(1 + r^2) \). Since \( \varphi_\alpha \) is an increasing function on \([0, \infty)\) with \( \lim_{r \rightarrow \infty} \varphi_\alpha(r) = 2(\alpha + 1) \leq 4 \), we conclude that \( R_{0,0} \geq 0 \).

\[
(3-2) \quad R_{1,1} = \frac{3 - 2\alpha}{(1 + r^2)^2} + 2 \frac{f^2}{h^4} \geq 0.
\]
\( (3-3) R_{2,2} = R_{3,3} = \frac{-3\alpha}{(1 + r^2)^2} + \frac{\alpha}{1 + r^2} \left( 1 - \frac{\alpha r^2}{1 + r^2} \right) + 4 \left( \frac{f(r)}{r} \right)^2 - 2r^2 \left( \frac{f(r)}{r} \right)^2 - \frac{\alpha^2 r^2}{(1 + r^2)^2} \geq (1 + r^2)^{-\alpha}(4 - (\psi_\alpha(r) + \theta_\alpha(r))) \),

where \( \psi_\alpha(r) := 2r^2/(1 + r^2)^{1+\alpha} \), and \( \theta_\alpha(r) := (3\alpha + \alpha^2 r^2)/(1 + r^2)^{2-\alpha} \).

One easily checks that the maximum of \( \psi_\alpha \) equals

\[
\eta(\alpha) = \frac{2}{\alpha(1 + 1/\alpha)^{1+\alpha}} \leq \eta(1/2) = 4/3\sqrt{3},
\]

for \( \alpha \geq 1/2 \). Moreover, \( \theta_\alpha \) is a decreasing function if \( \alpha \leq 1 \), with \( \theta_\alpha(0) = 3\alpha \). Thus:

\[
R_{2,2} = R_{3,3} \geq (1 + r^2)^{-\alpha}(4 - (3 + 4/3\sqrt{3})) > 0,
\]

thereby completing the proof of 1.1(a).

Proof of 1.1(b). When \( h = \cos \), (2-1)-(2-3) become:

(i) \( R_{0,0} = 2 - \frac{f''}{f} \),

(ii) \( R_{1,1} = -\frac{f''}{f} + 2\frac{f'}{f} \sin \frac{f^2}{\cos^4} \),

(iii) \( R_{2,2} = R_{3,3} = 1 + \frac{f'}{f} \sin \frac{4 \cos^2 - 2f^2 - \sin^2 \cos^2}{\cos^4} \).

We will choose \( f \) so that \( f(r) = \sin r \) for \( r \leq \varepsilon \), \( f(r) = \sin r \cos r \) for \( r \geq \pi/4 \), and \( R_{i,i} \geq 3 \). Define \( \kappa := f/\sin \); (i) and (ii) transform into:

(i') \( R_{0,0} = 3 - \frac{k''}{k} - 2\frac{k'}{k} \cos \frac{k'}{k} \sin \),

(ii') \( R_{1,1} = 3 - \frac{k''}{k} - 2\frac{k'}{k} \left( \frac{\cos}{\sin} - \frac{\sin}{\cos} \right) + 2k^2 \frac{\sin^2}{\cos^4} \).

If \( \varepsilon > 0 \) is sufficiently small, there exists a function \( k \) such that \( k \equiv 1 \) on \( [0, \varepsilon] \), \( k \equiv \cos \) on \( [\pi/4, \pi/2] \), and \( k'' \leq 0 \). Then \( R_{0,0}, R_{1,1} \geq 3 \). To show that \( R_{2,2} \geq 3 \), observe that, since \( f \leq \sin \),

\[
F \overset{\text{def}}{=} (4 \cos^2 - 2f^2 - \sin^2 \cos^2)/\cos^4 \geq (4 \cos^2 - 2 \sin^2 - \sin^2 \cos^2)/\cos^4 \overset{\text{def}}{=} G.
\]
Now, the minimum value of $G = (5/\cos^2) - (2/\cos^4) + 1$ on the interval $[0, \pi/4]$ is $G(\pi/4) = 3$. Since $R_{2,2} - F = 2 + (k'/\sin)(k\cos) \geq 1$, the result follows.

We now proceed to show that the radial sectional curvatures are $\geq 1$: Let $x \in T_p L$, and consider a unit-speed geodesic $\gamma$ originating at $p$ and orthogonal to $L$. If $E$ denotes the parallel field along $\gamma$ with $E(0) = x$, then $J := hE$ is a Jacobi field along $\gamma$, cf. [3]. Therefore, $R(E, \dot{\gamma})\dot{\gamma} = -(h''/h)E$, so that $\langle R(E, \dot{\gamma})\dot{\gamma}, E \rangle = 1$. On the other hand, if $v$ is orthogonal to both $\dot{\gamma}(0)$ and $T_p L$, and if $F$ denotes the parallel field along $\gamma$ with $F(0) = v$, then $R(F, \dot{\gamma})\dot{\gamma} = -(f''/f)F$, and

$$\langle R(F, \dot{\gamma})\dot{\gamma}, F \rangle = -f''/f = 1 - (k''/k) - 2(k'/k)(\cos / \sin).$$

This last expression is $\geq 1$ and identically 1 on $[0, \varepsilon]$. The same is therefore true for all radial curvatures.

Finally, observe that the comparison space in [4] or [3] has the same volume growth as $(L \times D_r, g_K)$. It follows that $q_L(r) \equiv 1$ for our choices of $f$ and $h$ when $r \leq \varepsilon$.

4. **Remarks.**

4.1. In 1.1(a), the maximal growth rate for the volume of $B_r(L)$ obtained by our method is of order $r^3$.

4.2. The maximal distance from $L$ with respect to the metric $g$ from 1.1(b) is $\pi/(2\sqrt{\kappa}) = \pi/2$, where $\kappa$ is the infimum of the radial sectional curvatures and the Ricci curvature. Nevertheless, $(M, g)$ is not symmetric, cf. the remark on p. 322 in [3].

4.3. As the general formulas of §2 show, one can produce similar examples on other vector bundles. It is, however, essential to have some information about the divergence of the $A$-tensor, cf. (2-9), (2-9').

**References**


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