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**THE CLASSIFICATION OF FLAT COMPACT COMPLETE
SPACE-FORMS WITH METRIC OF SIGNATURE (2, 2)**

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Those flat compact complete space-forms with metric of signature $(2,2)$ are classified up to finite covers. The simply transitive subgroups of $R^4 \rtimes \text{SO}(2, 2)$ are classified up to conjugation.

1. Introduction.

(1.1) If $\Gamma \subseteq R^4 \rtimes \text{SO}(2, 2)$ and Γ acts on R^{p+q} freely and properly discontinuously with compact quotient, then $X = R^{p+q}/\Gamma$ is a flat compact complete space-form with metric of signature (p, q) . Recently D. Fried [3] has classified those flat compact complete space-forms with metric of signature $(1,3)$ upto finite covers. Ravi S. Kulkarni pointed out that Fried's method can be applied to the case $(p, q) = (2, 2)$. The basic idea of Fried's method is in the following theorem:

(1.2) **THEOREM.** *Suppose X is a flat compact complete space-form with fundamental group $\Gamma \subseteq R^4 \rtimes \text{SO}(2, 2)$. Then there is a uniquely determined subgroup H of $R^4 \rtimes \text{SO}(2, 2)$ that acts simply transitively on R^4 and $H \cap \Gamma = \pi$ has finite index in Γ .*

(1.3) In §2 we classify those subgroups of $R^4 \rtimes \text{SO}(2, 2)$ that act on R^4 simply transitively, up to the conjugacy of $R^4 \rtimes \text{O}(2, 2)$. Every such subgroup, as a Lie group, is isomorphic to one of the following:

$$R^4, \quad R \times \text{Nil}^3, \quad \text{Nil}^4, \quad R \times \left\{ R^2 \rtimes \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}; t \in R \right\}, \\ R \times \{R^2 \rtimes \text{SO}(2)\}.$$

All of them, except the last one, correspond to Γ 's. Their uniform lattices are known, cf. [3] and [7].

(1.4) To prove Theorem (1.2), we first prove in §3 that Γ is virtually solvable. This result confirms a conjecture by Milnor in a special case. In [6], it is conjectured that the fundamental group of a complete affinely flat manifold is virtually polycyclic. Our result, combined with

Fried's result, shows that this conjecture is true for compact pseudo-Riemannian 4-manifolds.

(1.5) In §4 we complete the proof of Theorem 1.2, using the theory of crystallographic hull developed by Fried and Goldman, cf. [4]. In §5, we give our classification. By comparing our list with Fried's, we obtain an interesting fact: as differential manifolds, they are the same coset spaces of the form H/Γ , where H is a Lie group isomorphic to R^4 , $R \times \text{Nil}^3$, Nil^4 or $R \times \{R^2 \rtimes \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}; t \in R\}$ and Γ is a uniform lattice of H . These Lie groups have simply transitive representations as affine motions and when the signature is (2,2) (resp. (3,1)), the images of the representations are $R^4 \rtimes \text{SO}(2, 2)$ (resp. $R^4 \rtimes \text{SO}(3, 1)$).

(1.6) *Notations and some properties of $\text{SO}(2, 2)$ and $\text{so}(2, 2)$.* Throughout this paper we will call $\{e_i\}$, $1 \leq i \leq 4$, a standard basis s.t. the metric Q , w.r.t. this basis, has the form

$$Q(v, v) = v_1v_3 + v_2v_4,$$

where $v = \sum_{i=1}^4 v_i e_i$. The full group of orientation-preserving isometries is $R^4 \rtimes \text{SO}(2, 2)$ and

$$\text{SO}(2, 2) = \left\{ g \in \text{SL}_4(R); {}^t g \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right\},$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The infinitesimal isometries are $R^4 \rtimes \text{so}(2, 2)$ and

$$(1.6.1) \quad \text{so}(2, 2) = \left\{ X \in \mathfrak{gl}_4(R); {}^t X \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} X = 0 \right\} \\ = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & d \\ a_{21} & a_{22} & -d & 0 \\ 0 & c & -a_{11} & -a_{21} \\ -c & 0 & -a_{12} & -a_{22} \end{pmatrix}; a_{ij}, d, c \in R \right\}.$$

(1.6.2) $\text{so}(2, 2) = L_1 \oplus L_2$, where $L_i \simeq \mathfrak{sl}_2(R)$, $i = 1, 2$; and

$$L_1 = \left\{ \begin{pmatrix} a & b & & \\ c & -a & & \\ & & -a & -c \\ & & -b & a \end{pmatrix}; a, b, c \in R \right\}, \\ L_2 = \left\{ \begin{pmatrix} a' & 0 & 0 & d' \\ 0 & a' & -d' & 0 \\ 0 & c' & -a' & 0 \\ -c' & 0 & 0 & -a' \end{pmatrix}; a', d', c' \in R \right\}.$$

L_1, L_2 are permuted by an element of $\text{O}(2, 2)$.

(1.6.3) It is easy to show that any Cartan subalgebra of $\mathfrak{so}(2, 2)$ is conjugate under $O(2, 2)$ to one of the following:

$$(1) \quad \left\{ \begin{pmatrix} a & & & \\ & b & & \\ & & -a & \\ & & & -b \end{pmatrix} ; a, b, \in R \right\},$$

$$(2) \quad \left\{ \begin{pmatrix} 0 & a & 0 & b \\ -a & 0 & -b & 0 \\ 0 & b & 0 & a \\ -b & 0 & -a & 0 \end{pmatrix} ; a, b, \in R \right\},$$

$$(3) \quad \left\{ \begin{pmatrix} a & b & & \\ -b & a & & \\ & & -a & b \\ & & -b & -a \end{pmatrix} ; a, b, \in R \right\}.$$

An immediate corollary is

(1.6.4) *If X is in a Cartan subalgebra of $\mathfrak{so}(2, 2)$ and $\det X = 0$, then X must conjugate under $O(2, 2)$ to*

$$(4) \quad \left\{ \begin{pmatrix} a & & & \\ & 0 & & \\ & & -a & \\ & & & 0 \end{pmatrix} \right\},$$

or

$$(5) \quad \left\{ \begin{pmatrix} 0 & a & 0 & a \\ -a & 0 & -a & 0 \\ 0 & a & 0 & a \\ -a & 0 & -a & 0 \end{pmatrix} \right\}.$$

(1.7) We identify $\text{Aff}(n)$, resp. $\mathfrak{aff}(n)$, with

$$\left\{ \begin{pmatrix} A & v \\ 0 & a \end{pmatrix} ; A \in \text{GL}_4(R), v \in R^4 \right\},$$

resp.

$$\left\{ \begin{pmatrix} X & v \\ 0 & 0 \end{pmatrix} ; X \in \mathfrak{gl}_4(R), v \in R^4 \right\},$$

w.r.t. a given basis. Let P_l be the natural homomorphism taking an affine transformation (or an infinitesimal affine transformation) to its linear part. Let $L(G)$ be the Lie algebra of a Lie group G and $A(G)$ be the algebraic hull of G . We will need the following well-known lemma.

(1.7.1) **LEMMA.** *If $G \subseteq \text{Aff}(n)$ s.t. G acts freely on R^n , then every $A \in P_l(G)$ has 1 as an eigenvalue.*

(1.7.2) **LEMMA** (*Kostant and Sullivan, cf. [5]*). *If G is as in (1.7.1), then every $A \in P_l(A(G))$ has 1 as an eigenvalue.*

(1.7.3) **COROLLARY.** *If G is as in (1.7.1), then every $X \in P_l(L(A(G)))$ or $X \in L(A(P_l(G)))$ has 0 as an eigenvalue.*

2. Simply transitive subgroups. We will classify subgroups of $R^4 \rtimes \text{SO}(2, 2)$ that act simply transitively on R^4 . Our classification is up to the conjugation under $R^4 \rtimes \text{O}(2, 2)$. It is well known that a simply transitive group of affine motions must be solvable, connected, simply connected and of dimension 4, cf. [1]. We will start from a special case when the groups are unipotent. The following lemma from Auslander and Scheuneman plays the key role in this section.

(2.1) **LEMMA.** *Let U be a nilpotent Lie group which has a faithful representation $\rho: U \rightarrow \text{Aff}(n)$, let ρ_* be the induced monomorphism of Lie algebras*

$$\rho_*L(U) \rightarrow \left\{ \begin{pmatrix} X & v \\ 0 & 0 \end{pmatrix}; X \in \mathfrak{gl}_n(\mathbb{R}), v \in \mathbb{R}^n \right\} = \text{aff}(n),$$

and let P_l be as in (1.7), let P_t be the projection from an element in $\text{aff}(n)$ to its translation part. Then $\rho(U)$ acts on R^n simply transitively if and only if

- (1) $P_l \circ \rho_*(L(U))$ is nilpotent, and
- (2) $P_t \circ \rho_*(L(U))$ is a linear isomorphism of $L(U)$ onto R^n .

For a proof, cf. [1]. So unipotent simply transitive subgroups are exactly the following U 's s.t.

$$(2.2) \quad L(U) = \left\{ \begin{pmatrix} X(v) & v \\ 0 & 0 \end{pmatrix}; v \in \mathbb{R}^n \right\},$$

where $X(v)$ is a linear function of v and $P_l(L(U)) = \{X(v); v \in \mathbb{R}^n\}$ is nilpotent.

(2.3) **LEMMA.** *There is a vector $v_0 \in R^4$ such that*

- (i) $P_l(L(U))(v_0) = 0$,
- (ii) $Q(v_0, v_0) = 0$.

Proof. If $V = \{v \in R^4; P_l(L(U))v = 0\}$, then V^\perp is invariant. By Engel's Theorem on V^\perp , V^\perp meets V . \square

Let $\{e_i\}$ be our standard basis. Then we choose $v_0 = e_1$ since $O(2, 2)$ is transitive on $\{v; Q(v, v) = 0\}/v \sim tv$, where $t \in R - \{0\}$.

(2.4) COROLLARY. *W.r.t. the above standard basis, $X \in P_l(L(U))$ has the form*

$$X(v) = \begin{pmatrix} 0 & a & 0 & b \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 \end{pmatrix},$$

where $a = a(v)$ and $b = b(v)$ are linear functions of v .

To find $a(v)$ and $b(v)$, we compute the commutator of $L(U)$.

$$(2.5) \quad \left[\begin{pmatrix} X(v) & v \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} X(v') & v' \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} X(v'') & v'' \\ 0 & 0 \end{pmatrix},$$

where $v'' = X(v)v' - X(v')v$, $X(v'') = X(v)X(v') - X(v')X(v) = 0$.
So

$$a(v'') = b(v'') \equiv 0.$$

Write

$$(2.6) \quad a(v) = \sum_{i=1}^4 a_i v_i, \quad b(v) = \sum_{i=1}^4 b_i v_i.$$

Then we have

$$(2.7) \quad 0 = \sum_{i=1}^4 a_i v_i'', \quad 0 = \sum_{i=1}^4 b_i v_i'',$$

where v_i'' 's are linear functions of a_i , b_i and $v_i v_j'$, $1 \leq i, j \leq 4$, and all coefficients of $v_i v_j'$ must be zero. We obtain

(2.8) LEMMA.

- (i) $a_1 = b_1 = 0$,
- (ii) $a_2 b_4 + a_4^2 = 0$,
- (iii) $a_2 b_2 + a_4 a_2 = 0$,
- (iv) $b_2 b_4 + b_4 a_4 = 0$,
- (v) $b_2^2 + b_4 a_2 = 0$.

(2.9) COROLLARY.

- (i) $b_4(b_2 + a_4) = 0$,
- (ii) $a_2(b_2 + a_4) = 0$,
- (iii) $(b_2 - a_4)(b_2 + a_4) = 0$.

(2.10) Now we can get some necessary conditions for the nontranslation unipotent simply transitive subgroups. If $b_2 + a_4 \neq 0$, then $b_4 = a_2 = 0$. By (2.8) (ii) and (v), $b_2^2 = a_4^2 = 0$ and we get a contradiction. So $b_2 + a_4 = 0$, and we have three subcases:

(2.10.1) $b_2 = a_4 = b_4 = a_2 = 0$, but $(a_3, b_3) \neq (0, 0)$, i.e.,

$$\begin{cases} a(v) = a_3v_3 \\ b(v) = b_3v_3. \end{cases}$$

(2.10.2) $b_2 + a_4 = 0$ but $b_2 \neq 0$, $a_4 \neq 0$. Then by (2.8) $b_4 \neq 0$, $a_2 \neq 0$, i.e.

$$\begin{cases} a(v) = a_2v_2 + a_3v_3 + a_4v_4 \\ b(v) = b_2v_2 + b_3v_3 + b_4v_4. \end{cases}$$

(2.10.3) $b_2 = 0$, $a_4 = 0$, $(a_2, b_4) \neq (0, 0)$. By (2.8), $b_4a_2 = 0$, so

$$\begin{cases} a(v) = a_2v_2 + a_3v_3 \\ b(v) = b_3v_3, \end{cases}$$

or

$$\begin{cases} a(v) = a_3v_3 \\ b(v) = b_3v_3 + b_4v_4. \end{cases}$$

(2.11) THEOREM. *Up to conjugacy under $R^4 \rtimes O(2, 2)$, the nontranslation unipotent simply transitive groups U of $R^4 \rtimes SO(2, 2)$, have the following Lie algebras:*

$$L(U) = \left\{ \begin{pmatrix} X(v) & v \\ 0 & 0 \end{pmatrix}; v \in R^4 \right\},$$

where

$$X(v) = \begin{pmatrix} 0 & a(v) & 0 & b(v) \\ 0 & 0 & -b(v) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a(v) & 0 \end{pmatrix},$$

$a(v)$ and $b(v)$ are listed in the following table:

Type of $L(U)$	$a(v)$	$b(v)$	isomorphism type as an abstract Lie algebra
I-1	v_3	v_3	$N_3 \oplus R$
I-2	v_3	$-v_3$	$N_3 \oplus R$
I-3	v_3	0	$N_3 \oplus R$
II-1	$v_2 + v_4 + tv_3, (t \geq 0)$	$-v_2 - v_4$	N_4
II-2	$-v_2 + v_4 + tv_3, (t \geq 0)$	$-v_2 + v_4$	N_4
II-3	v_2	v_3	N_4

The equivalence classes are uniquely determined by the type of $L(U)$ and the parameter t (in Type II).

Proof. The discussion of the conjugacy under $R^4 \rtimes O(2, 2)$ is long and tedious. We will only write down a brief one for subcase (2.10.2). We give the following lemma without proof.

(2.11.1) LEMMA. If $a(v) \neq 0, b(v) \neq 0, a'(v') \neq 0, b'(v') \neq 0,$ and if there is a matrix $A = (a_{ij}) \in O(2, 2)$ such that

$$A^{-1} \begin{pmatrix} 0 & a(v) & 0 & b(v) \\ 0 & 0 & -b(v) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a(v) & 0 \end{pmatrix} A = \begin{pmatrix} 0 & a'(v') & 0 & b'(v') \\ 0 & 0 & -b'(v') & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a'(v') & 0 \end{pmatrix},$$

then either

$$(1) \quad \begin{cases} a'(v') = \frac{a_2 a_{22}^2}{a_{11}} v'_2 + \left\{ \frac{a_{22} a_{23}}{a_{11}} a_2 + \frac{a_{22}}{a_{11}^2} a_3 + \frac{a_{22} a_{43}}{a_{11}} a_4 \right\} v'_3 \\ \quad + \frac{a_4}{a_{11}} v'_4 \\ b'(v') = \frac{b_2}{a_{11}} v'_2 + \left\{ \frac{a_{23}}{a_{11} a_{22}} b_2 + \frac{1}{a_{11}^2 a_{22}} b_3 + \frac{a_{43}}{a_{11} a_{22}} b_4 \right\} v'_3 \\ \quad + \frac{b_4}{a_{11} a_{22}^2} v'_4 \end{cases}$$

where $a_{11}a_{22} \neq 0$; or

$$(2) \quad \begin{cases} a'(v') = \frac{b_4 a_{42}^2}{a_{11}} v'_2 + \left\{ \frac{a_{42} a_{23}}{a_{11}} b_2 + \frac{a_{42}}{a_{11}^2} b_3 + \frac{a_{42} a_{43}}{a_{11}} b_4 \right\} v'_3 \\ \quad + \frac{b_2}{a_{11}} v'_4 \\ b'(v') = \frac{a_4}{a_{11}} v'_2 + \left\{ \frac{a_{23}}{a_{11} a_{42}} a_2 + \frac{1}{a_{11}^2 a_{42}} a_3 + \frac{a_{43}}{a_{11} a_{42}} a_4 \right\} v'_3 \\ \quad + \frac{a_4}{a_{11} a_{41}^2} v'_4 \end{cases}$$

where $a_{11}a_{42} \neq 0$.

Write $a'(v') = \sum_{i=2}^4 a'_i v'_i$ and $b'(v') = \sum_{i=2}^4 b'_i v'_i$, then from (2.11.1)

$$a'_2 b'_4 = a'_4 b'_2 = \frac{a_4 b_2}{a_{11}^2} = -\frac{a_4^2}{a_{11}^2} < 0,$$

since $a_4 = -b_2 \neq 0$. So we can choose a_{11} such that $a'_2 b'_4 = a'_4 b'_2 = -1$, i.e. $a_4/a_{11} = \pm 1$. Next we use (1) (resp. (2)) if $a_4/a_{11} = 1$ (resp. -1), and choose a_{22} (resp. a_{42}) to reduce

$$\begin{pmatrix} a'_2 & a'_4 \\ b'_2 & b'_4 \end{pmatrix}$$

to

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{if } a_2 a_4 > 0,$$

or

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{if } a_2 a_4 < 0.$$

Now a'_3, b'_3 have the form

$$\begin{cases} a'_3 = z_1 + \frac{a_{22}}{a_{11}^2} a_3 \\ b'_3 = \pm z_1 + \frac{1}{a_{11}^2 a_{22}} b_3, \end{cases} \quad \text{or} \quad \begin{cases} a'_3 = z_2 + \frac{a_{42}}{a_{11}^2} b_3 \\ b'_3 = \pm z_2 + \frac{1}{a_{11}^2 a_{42}} a_3, \end{cases}$$

where z_1 (resp. z_2) depends on a_{23}, a_{43} (resp. a_{23}, a_{43}) and $z_i, i = 1, 2$ can assume any real number. We can choose z_i so that $b'_3 = 0$ and we can choose the sign of a_{22} (resp. a_{42}) so that $a'_3 \geq 0$. So we can find an $A \in O(2, 2)$ such that

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X & v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

is of Type II-1 or Type II-2. We can replace $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ by $\begin{pmatrix} A & w \\ 0 & 1 \end{pmatrix}$ and show that the translation part doesn't contribute to the classification.

We omit the rest of the proof. □

(2.12) To handle the general case, namely when the simply transitive group of affine motion is non-unipotent solvable, we need the following lemma from Auslander, cf. [1].

(2.12.1) LEMMA. *Let H be an n -dimensional, connected, simply connected, solvable Lie group acting simply transitively as affine motions on R^n . Let $A(H)$ be the algebraic hull of H and let U be the unipotent radical of $A(H)$. Then U operates simply transitively as affine motions on R^n .*

Now all such nontranslation U 's are known from (2.11), and we'll study them first.

(2.12.2) LEMMA. *Let H, U be as in (2.12.1) and assume that U is not the translation group T . Then $H = U$.*

Proof. W.r.t. the standard basis $\{e_i\}$, $1 \leq i \leq 4$, we know

$$L(P_l(U)) = \left\{ \begin{pmatrix} 0 & a(v) & 0 & b(v) \\ 0 & 0 & -b(v) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a(v) & 0 \end{pmatrix}; v \in R^4 \right\}.$$

Notice that $A(H)$ is contained in the normalizer of U , we have

$$[L(A(H)), L(U)] \subseteq L(U), \quad [L(P_l(A(H))), L(P_l(U))] \subseteq L(P_l(U)).$$

Since for

$$Y = \begin{pmatrix} a_{11} & a_{12} & 0 & d \\ a_{21} & a_{22} & -d & 0 \\ 0 & c & -a_{11} & -a_{21} \\ -c & 0 & -a_{12} & -a_{22} \end{pmatrix}, \quad X = \begin{pmatrix} 0 & a(v) & 0 & b(v) \\ 0 & 0 & -b(v) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a(v) & 0 \end{pmatrix},$$

we have

$$[Y, X] = \begin{pmatrix} -a(v)a_{21} + b(v)c & a(v)(a_{11} - a_{22}) & 0 & b(v)(a_{11} + a_{22}) \\ 0 & a(v)a_{21} + b(v)c & -b(v)(a_{11} + a_{22}) & 0 \\ 0 & 0 & a(v)a_{21} - b(v)c & 0 \\ 0 & 0 & -a(v)(a_{11} - a_{22}) & -a(v)a_{21} - b(v)c \end{pmatrix}.$$

So

$$\begin{cases} -a(v)a_{21} + b(v)c = 0, \\ a(v)a_{21} + b(v)c = 0, \end{cases}$$

i.e.

$$\begin{cases} a(v)a_{21} = 0 \\ b(v)c = 0 \end{cases}$$

for any a_{21} , c , $a(v)$ and $b(v)$, $v \in R^4$.

By (2.11), we can always find a v so that $a(v) \neq 0$, so we must have $a_{21} = 0$. Similarly $c = 0$, unless $b(v) \equiv 0$. So we have two cases.

Case 1. Type of $L(U)$ is I-1, I-2 or II.

$L(P_l(A(H)))$ is contained in

$$\left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & d \\ 0 & a_{22} & -d & 0 \\ 0 & 0 & -a_{11} & 0 \\ 0 & 0 & -a_{12} & -a_{22} \end{pmatrix} ; a_{11}, a_{12}, a_{22}, d \in R \right\}.$$

Case 2. Type of $L(U)$ is I-3.

$L(P_l(A(H)))$ is contained in

$$\left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & c & -a_{11} & 0 \\ -c & 0 & -a_{12} & -a_{22} \end{pmatrix} ; a_{11}, a_{12}, a_{22}, c \in R \right\}.$$

It's easy to show that matrices in Case 1 and Case 2 are conjugate under $O(2, 2)$. We will only write down a proof for Case 1; a proof for Case 2 can be obtained similarly.

Again let $Y \in L(P_l(A(H)))$. Then by (1.7.3) $\det Y = 0$, so $a_{11}a_{22} = 0$, i.e. $a_{11} = 0$ or $a_{22} = 0$.

If $a_{22} = 0$, then an element in $L(A(H))$ has the form

$$\begin{pmatrix} Y & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 & d(v) & v_1 \\ 0 & 0 & -d & 0 & v_2 \\ 0 & 0 & -a_{11} & 0 & v_3 \\ 0 & 0 & -a_{12} & 0 & v_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{for some } v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}.$$

By subtracting an element $\begin{pmatrix} X(v) & v \\ 0 & 0 \end{pmatrix} \in L(U)$, we have

$$\begin{pmatrix} Y - X(v) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} - a(v) & 0 & d - b(v) & 0 \\ 0 & 0 & -d + b(v) & 0 & 0 \\ 0 & 0 & -a_{11} & 0 & 0 \\ 0 & 0 & -a_{12} + a(v) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in L(A(H)).$$

For any $\begin{pmatrix} X(v') & v' \\ 0 & 0 \end{pmatrix} \in L(U)$, we have

$$\left[\begin{pmatrix} Y - X(v) & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} X(v') & v' \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & a_{11}a(v') & 0 & a_{11}b(v') & a_{11}v'_1 + (a_{12} - a(v))v'_2 + (d - b(v))v'_4 \\ 0 & 0 & -a_{11}b(v') & 0 & -(d - b(v))v'_3 \\ 0 & 0 & 0 & 0 & -a_{11}v'_3 \\ 0 & 0 & -a_{11}a(v') & 0 & -(a_{12} - a(v))v'_3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in L(U).$$

But we know that

$$\begin{pmatrix} 0 & a_{11}a(v') & 0 & a_{11}b(v') & a_{11}v'_1 \\ 0 & 0 & -a_{11}b(v') & 0 & a_{11}v'_2 \\ 0 & 0 & 0 & 0 & a_{11}v'_3 \\ 0 & 0 & -a_{11}a(v') & 0 & a_{11}v'_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in L(U).$$

So we have

- (1) $a_{11}v'_1 = a_{11}v'_1 + (a_{12} - a(v))v'_2 + (d - b(v))v'_4$;
- (2) $a_{11}v'_2 = -(d - b(v))v'_3$;
- (3) $a_{11}v'_3 = -a_{11}v'_3$;
- (4) $a_{11}v'_4 = -(a_{12} - a(v))v'_3$.

From (3) we get $a_{11} = 0$. Then (2), resp. (4), implies $d = b(v)$, resp. $a_{12} = a(v)$, i.e. $Y = X(v)$. So $\begin{pmatrix} Y & v \\ 0 & 0 \end{pmatrix} \in L(U)$.

If $a_{11} = 0$, let $\begin{pmatrix} Y & v \\ 0 & 0 \end{pmatrix} \in L(A(H))$. By subtracting an element $\begin{pmatrix} X(v) & v \\ 0 & 0 \end{pmatrix} \in L(U)$, we have

$$\begin{pmatrix} Y - X(v) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a - 12 - a(v) & 0 & d - b(v) & 0 \\ 0 & a_{22} & -(d - b(v)) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(a_{12} - a(v)) & -a_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in L(A(H)).$$

Then for any $\begin{pmatrix} X(v') & v' \\ 0 & 0 \end{pmatrix} \in L(U)$, we have

$$\left[\begin{pmatrix} Y - X(v) & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} X(v') & v' \\ 0 & 0 \end{pmatrix} \right] \\ = \begin{pmatrix} 0 & -a_{22}a(v') & 0 & a_{22}b(v') & (a_{12} - a(v))v'_2 + (d - b(v))v'_4 \\ 0 & 0 & -a_{22}b(v') & 0 & a_{22}v'_2 - (d - b(v))v'_3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{22}a(v') & 0 & -(a_{12} - a(v))v'_3 - a_{22}v'_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in L(U).$$

Let $a(v) = \sum a_i v_i$, $b(v) = \sum b_i v_i$, $2 \leq i \leq 4$ as in (2.6) ($a_1 = b_1 = 0$) and let

$$v'' = \begin{pmatrix} (a_{12} - a(v))v'_2 + (d - b(v))v'_4 \\ a_{22}v'_2 - (d - b(v))v'_3 \\ 0 \\ -(a_{12} - a(v))v'_3 - a_{22}v'_4 \end{pmatrix}.$$

Then

$$\begin{cases} -a_{22}a(v') = a(v'') = a_2(a_{22}v'_2 - (d - b(v))v'_3) \\ \quad + a_4(-(a_{12} - a(v))v'_3 - a_{22}v'_4), \\ a_{22}b(v') = b(v'') = b_2(a_{22}v'_2 - (d - b(v))v'_3) \\ \quad + b_4(-(a_{12} - a(v))v'_3 - a_{22}v'_4), \end{cases}$$

i.e.

$$\begin{cases} -a_{22}(a_2v'_2 + a_3v'_3 + a_4 + v'_4) = a_2(a_{22}v'_2 - (d - b(v))v'_3) \\ \quad + a_4(-(a_{12} - a(v))v'_3 - a_{22}v'_4), \\ a_{22}(b_2v'_2 + b_3v'_3 + b_4v'_4) = b_2(a_{22}v'_2 - (d - b(v))v'_3) \\ \quad + b_4(-(a_{12} - a(v))v'_3 - a_{22}v'_4), \end{cases}$$

i.e.

$$\begin{cases} 2a_{22}a_2v'_2 + (a_3a_{22} - a_4(a_{12} - a(v)) - a_2(d - b(v)))v'_3 = 0, \\ 2a_{22}b_4v'_4 + (b_3a_{22} + b_4(a_{12} - a(v)) + b_2(d - b(v)))v'_3 = 0. \end{cases}$$

Letting v'_i 's vary, we have

- (1) $a_{22}a_2 = 0$;
- (2) $a_{22}b_4 = 0$;
- (3) $a_3a_{22} - a_4(a_{12} - a(v)) - a_2(d - b(v)) = 0$;
- (4) $b_3a_{22} + b_4(a_{12} - a(v)) + b_2(d - b(v)) = 0$.

If $a_{22} \neq 0$, we must have $a_2 = b_4 = 0$ by (1) and (2). According to (2.10), this implies $b_2 = a_4 = b_4 = a_2 = 0$. Then (3) and (4) lead to

$$\begin{cases} a_3 a_{22} = 0, \\ b_3 a_{22} = 0, \end{cases}$$

i.e. $a_3 = b_3 = 0$. So $U = T$, and we have a contradiction. So $a_{22} = 0$. We always have $a_{11} = a_{22} = 0$, i.e. $A(H)$ is unipotent; so H is unipotent. But any unipotent connected Lie group is Zariski closed, so $H = A(H)$. U , as the unipotent radical of H must be H itself. □

(2.12.3) Now consider the case when the unipotent radical $A(H)$ is precisely the group T of translations of R^4 . Suppose $H \neq T$, i.e. H is not unipotent.

(2.12.3.1) LEMMA. $P_l(H)$ is abelian.

Proof. $P_l(H) \simeq H / \text{Ker}(P_l|_H) = H / (H \cap T) \subseteq A(H) / T$, but $A(H) / T$ is abelian (cf. [2], $A(H) / U(H)$ is abelian, since $A(H)$ is solvable and algebraic). □

(2.12.3.2) LEMMA. $\dim P_l(H) = 1$; $P_l(H)$ is diagonalizable in \mathbb{C} .

Proof. $P_l(H)$ is a connected abelian subgroup of $\text{SO}_0(2, 2)$, so $\dim P_l(H) \leq 2$. By (1.7.3) $\det X = 0$ for every $X \in L(P_l(H))$, i.e. 0 is an eigenvalue of X . Since $X \in \mathfrak{so}(2, 2)$, so

$$X = \begin{pmatrix} a_{11} & a_{12} & 0 & d \\ a_{21} & a_{22} & -d & 0 \\ 0 & c & -a_{11} & -a_{21} \\ -c & 0 & -a_{12} & -a_{22} \end{pmatrix},$$

and

$$\begin{aligned} \det(X - \lambda I) &= \lambda^4 + (2dc - 2a_{12}a_{21} - a_{11}^2 - a_{22}^2)\lambda^2 \\ &\quad + (-a_{11}a_{22} + a_{12}a_{21} + dc)^2 \\ &= \lambda^4 + \{-4a_{12}a_{21} - (a_{11} - a_{12})^2\}\lambda^2, \end{aligned}$$

since 0 is an eigenvalue. So the eigenvalues of X are $\{0, 0, 0, 0\}$ or $\{0, 0, \lambda, -\lambda\}$, $\lambda \neq 0$, $\lambda \in R$ or $\sqrt{-1}R$. If $\dim P_l(H) = 2$, then by (1.6.2) $\mathfrak{so}(2, 2) = L_1 \oplus L_2$, $L_i \simeq \mathfrak{sl}_2(R)$. So $L(P_l(H)) = RX_1 + RX_2$ where $X_i \in L_i$, $i = 1, 2$. But by (1.6.2)

$\det(X_1 - \lambda I) = \lambda^4 + 2(a^2 + bc)\lambda^2 + (a^2 + bc)^2$, and

$$\det(X_2 - \lambda I) = \lambda^4 + 2(b'c' - a'^2)\lambda^2 + (b'c' - a'^2)^2.$$

So zero is an eigenvalue of X_i , $i = 1, 2$, if and only if all the eigenvalues of X_i are zero. This means $P_i(H)$ is unipotent and leads to a contradiction. So $\dim P_i(H) = 1$, $L(P_i(H)) = RX$ and X has eigenvalues $\{0, 0, \lambda, -\lambda\}$, $\lambda \neq 0$, $\lambda \in R$ or $\sqrt{-1}R$. Since X is an infinitesimal isometry, it is diagonalizable. \square

(2.12.3.3) COROLLARY. $L(P_i(H))$ is contained in a Cartan subalgebra of $\mathfrak{so}(2, 2)$ and is conjugate under $O(2, 2)$ to

$$(1) \quad \begin{pmatrix} a & & 0 \\ & 0 & \\ & & -a \\ 0 & & & 0 \end{pmatrix}$$

$$(2) \quad \begin{pmatrix} 0 & a & 0 & a \\ -a & 0 & -a & 0 \\ 0 & a & 0 & a \\ -a & 0 & -a & 0 \end{pmatrix}.$$

Proof. By (1.6.4). \square

Since H is simply transitive, the map $P_i: L(H) \rightarrow R^4$ is a linear isomorphism, so in (2.12.3.3) we have $a = \sum_{i=1}^4 a_i v_i$, where

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

is the corresponding translation part. Since T is the unipotent radical of $A(H)$, we have $[L(H), L(H)] \subseteq L(T) = R^4$. By computing the commutator and using the fact that H is simply transitive, we must have $a(v) = a_2 v_2 + a_4 v_4$, $(a_2, a_4) \neq (0, 0)$ in Case (1) and $a(v) = a_1(v_1 - v_3) + a_2(v_2 - v_4)$, $(a_1, a_2) \neq (0, 0)$ in Case (2). Finally, by considering the conjugation under $R^4 \rtimes O(2, 2)$, we get

(2.12.4) THEOREM. If $H \subseteq R^4 \rtimes SO(2, 2)$ acts simply transitively on R^4 and H is not unipotent, then H is conjugate under $R^4 \rtimes O(2, 2)$

to one of the following:

(i) Type III-1:

$$\begin{pmatrix} a(v) & & & 0 \\ & 0 & & \\ & & -a(v) & \\ 0 & & & 0 \end{pmatrix} \begin{pmatrix} v \\ \\ \\ \end{pmatrix},$$

where $a(v) = tv_2 + v_4$, $t > 0$ and $L(H) \simeq R \oplus \{R^2 \rtimes R \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}$.

(ii) Type III-2:

$$\begin{pmatrix} 0 & a(v) & 0 & a(v) \\ -a(v) & 0 & -a(v) & 0 \\ 0 & a(v) & 0 & a(v) \\ -a(v) & 0 & -a(v) & 0 \end{pmatrix} \begin{pmatrix} v \\ \\ \\ \end{pmatrix},$$

where $a(v) = t(v_1 - v_3)$, $t > 0$ and $L(H) = R \oplus \{R^2 \rtimes R \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$. The type and the parameter t determine the equivalence classes uniquely.

(2.13) Combining (2.11) with (2.12.4) and denoting $H = T_4$ as Type 0, we complete the classification of simply transitive subgroups of $R^4 \rtimes SO(2, 2)$. We summarize our result in the following table. We denote

$$A(a, b, v) = \left\{ \left(\begin{pmatrix} 0 & a(v) & 0 & b(v) \\ 0 & 0 & -b(v) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a(v) & 0 \end{pmatrix} v \right); v \in R^4 \right\},$$

$$B(a, v) = \left\{ \left(\begin{pmatrix} a(v) & & & 0 \\ & 0 & & \\ & & -a(v) & \\ 0 & & & 0 \end{pmatrix} v \right), v \in R^4 \right\},$$

$$C(a, v) = \left\{ \left(\begin{pmatrix} 0 & a(v) & 0 & a(v) \\ -a(v) & 0 & -a(v) & 0 \\ 0 & a(v) & 0 & a(v) \\ -a(v) & 0 & -a(v) & 0 \end{pmatrix} v \right); v \in R^4 \right\}.$$

Table of equivalence classes of simply transitive subgroups of $R^4 \rtimes \text{SO}(2, 2)$ (given in the form of subalgebras of $\text{aff}(n)$ w.r.t. a standard basis).

type of $L(H)$	affine form of $L(H)$	isomorphism type as abstract Lie algebra
0	$\left\{ \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}; v \in R^4 \right\};$	R^4
I-1	$A(a, b, v), \begin{cases} a(v) = v_3, \\ b(v) = v_3 \end{cases}$	$R \oplus N_3$
I-2	$A(a, b, v), \begin{cases} a(v) = v_3, \\ b(v) = -v_3 \end{cases}$	$R \oplus N_3$
I-3	$A(a, b, v), \begin{cases} a(v) = v_3, \\ b(v) = 0 \end{cases}$	$R \oplus N_3$
II-1	$A(a, b, v), \begin{cases} a(v) = v_2 + v_4 + tv_3, \\ b(v) = -v_2 - v_4, \quad t \geq 0, \end{cases}$	N_4
II-2	$A(a, b, v), \begin{cases} a(v) = -v_2 + v_4 + tv_3, \\ b(v) = -v_2 + v_4, \quad t \geq 0 \end{cases}$	N_4
II-3	$A(a, b, v), \begin{cases} a(v) = v_2, \\ b(v) = v_3 \end{cases}$	N_4
III-1	$B(a, v), a(v) = tv_2 + v_4, t \in R$	$R \oplus \{R^2 \rtimes R \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}$
III-2	$C(a, v), a(v) = t(v_1 - v_3), t > 0$	$R \oplus \{R^2 \rtimes R \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$

The type of $L(H)$ and the parameter t determine the equivalence classes uniquely.

3. Γ is virtually solvable. A group with a solvable subgroup of finite index is called virtually solvable.

(3.1) **THEOREM.** *If $\Gamma \subset R^4 \rtimes \text{SO}(2, 2)$ and Γ acts freely and properly discontinuously on R^4 with compact quotient, then Γ is virtually solvable.*

Proof. Let $\pi = P_l(\Gamma)$ and $A(\pi)$ be the algebraic hull of Γ . The identity component A_0 is of finite index in $A(\pi)$. We will show A_0 is solvable. The following lemma is due to D. Fried.

(3.2) **LEMMA.** *If A_0 fixes a vector $v \in R^4$ s.t. $Q(v, v) \neq 0$, then A_0 is solvable.*

For a proof, cf. [3].

Assume that A_0 is not solvable. As in (1.7.2), for every $g \in A(\pi)$, $\det(g - I) = 0$. This shows $\det = 0$ on $L(A_0)$ and $\dim A_0 < \dim \text{SO}(2, 2)$. So A_0 contains a semisimpleconnected subgroup S

such that $\dim S = 3$ and $L(S) \simeq \mathfrak{sl}_2(R)$. By (1.6.2) $\det \neq 0$ on l_i so $L(S) \neq L_i$, $i = 1, 2$. So $L(S)$ must be a maximal subalgebra of $\mathfrak{so}(2, 2)$, so $A_0 = S$. Let $P_i: L(S) \rightarrow L_i$, $i = 1, 2$ be the projection map, then $P_i(L(S)) = L_i$, $i = 1, 2$.

(3.3) *Claim.* There is a nonzero vector $v \in R^4$ such that

- (i) $Q(v, v) \neq 0$;
- (ii) $A_0(v) = v$.

To prove the claim, let $0 \neq X \in L(A_0)$ such that RX is a split Cartan subalgebra of $L(A_0)$. Then $h = P_1(RX) \oplus P_2(RX)$ is a split Cartan subalgebra of $\mathfrak{so}(2, 2)$. By (1.6.3) h is conjugate under $O(2, 2)$ to $\{\text{diag} \cdot (a, b, -a, -b); a, b \in R\}$. Since $\det X = 0$ we can rescale and permute coordinates so $X = \text{diag} \cdot (1, 0, -1, 0)$. Let $\{X, Y, Z\}$ be the basis of $L(A)$ such that $[X, Y] = 2Y$, $[X, Z] = -2Z$, $[Y, Z] = X$ and $X = \text{diag} \cdot (1, 0, -1, 0)$. Then $\text{ad } X$ has three real eigenvalues on $\mathfrak{so}(2, 2)$: $\{2, 0, -2\}$. Let E_λ be the corresponding eigenspaces, then

$$\begin{aligned}
 E_2 &= \left\{ \begin{pmatrix} 0 & c & 0 & e \\ 0 & 0 & -e & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -c & 0 \end{pmatrix}; c, e \in R \right\}, \\
 E_{-2} &= \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 \\ 0 & f & 0 & -d \\ -f & 0 & 0 & 0 \end{pmatrix}; d, f, \in R \right\}, \text{ and} \\
 [E_2, E_{-2}] &= \left\{ \begin{pmatrix} cd - ef & & & 0 \\ & -cd - ef & & \\ & & -cd + ef & \\ 0 & & & cd + ef \end{pmatrix}; \right. \\
 &\qquad \qquad \qquad \left. c, d, e, f \in R \right\}.
 \end{aligned}$$

So there are $c, e, d, f \in R$ such that

$$Y = \begin{pmatrix} 0 & c & 0 & e \\ 0 & 0 & -e & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -c & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 \\ 0 & f & 0 & -d \\ -f & 0 & 0 & 0 \end{pmatrix}$$

and $[X, Z] = X$ implies

$$\begin{cases} cd - ef = 1, \\ cd + ef = 0, \end{cases}$$

i.e. $cd = -ef = \frac{1}{2}$, $cdef \neq 0$. Let $v = \frac{1}{c}e_2 - \frac{1}{e}e_4$. It's easy to check that $Q(v, v) = \frac{1}{ce} \neq 0$, $A_0(v) = v$.

Combining (3.3) with Lemma (3.2), we have a contradiction, so A_0 must be solvable. \square

4. Proof of Theorem (1.2). The principal tool is the following theorem from [4].

(4.1) **THEOREM (Fried and Goldman).** *Let $\Gamma \subseteq \text{Aff}(n)$ be virtually polycyclic and suppose that Γ acts properly discontinuously on R^n . Then there exists at least one subgroup $H \subseteq \text{Aff}(n)$ containing Γ such that:*

- (a) *H has finitely many components and each component meets Γ ;*
- (b) *H/Γ is compact;*
- (c) *H and Γ have the same algebraic hull in $\text{Aff}(n)$;*
- (d) *if Γ has a subgroup Γ_1 of finite index such that every element of $P_l(\Gamma_1)$ has all real eigenvalues, then H is uniquely determined by the above conditions;*
- (e) *the identity component H_0 of H acts simply transitively on R^n and $H_0 \cap \Gamma$ is a discrete cocompact subgroup of H_0 and is of finite index in Γ .*

Such a subgroup H in (4.1) is called a crystallographic hull for Γ . Since a discrete solvable subgroup of Lie with finitely many components is polycyclic and we proved in §3 that Γ in (1.2) is virtually solvable, by (4.1) we need only to check for the uniqueness of H . By (4.1)-(d), we need only to show that $P_l(\Gamma)$ has a subgroup of finite index with real eigenvalues only. Since H_0 must occur in our table of simply transitive motions and all these simply transitive motions, except Type III-2, have linear parts with only real eigenvalues, we need only to check Type III-2. By Bieberbach's theorem (cf. [8]), any discrete subgroup of Type III-2 meets T in a subgroup of finite index. \square

5. Classification of Γ .

(5.1) LEMMA. *Let Γ be a uniform lattice in a simply transitive group $H \subseteq R^4 \rtimes SO(2, 2)$. Then H is the identity component of the crystallographic hull of Γ if and only if H is not of Type III-2.*

Proof. If H is of Type III-2, then Γ has a subgroup of finite index, say Γ_1 , such that $\Gamma_1 \subset T$. So Γ is virtually abelian. By [4], the crystallographic hull of a virtually abelian affine polycyclic group is itself virtually abelian, so H doesn't arise from any Γ .

In the unipotent cases, the algebraic hull of H is H itself. So $A(\Gamma)$, the algebraic hull of Γ , is contained in H . Since H'_0 , the identity component of the crystallographic hull H' of Γ , acts simply transitively on R^4 , the dimension of H'_0 must be four, and then by (4.1)-(C) we have

$$H'_0 \subseteq H' \subseteq A(H') = A(\Gamma) \subseteq H.$$

So $H = H'_0$; then $H' = H$.

The only remaining case is Type III-1. Since Γ is not unipotent, H'_0 , the identity component of the crystallographic hull H' of Γ , must be nonunipotent solvable, i.e. H'_0 is of Type III-1 and $\Gamma \subseteq H \cap H'_0$. Then it's easy to show that $H'_0 = H$. □

(5.2) COROLLARY. *Up to finite covers, every flat compact complete space-form with metric of signature (2,2) is of the form H/Γ , where H is a simply transitive subgroup of $R^4 \times SO(2, 2)$ of Type 0, Type I, Type II or Type III-1 and Γ is a uniform lattice of H .*

(5.3) *Uniform lattices.* The uniform lattices depend only on the structure of H as a Lie group and do not depend on its embedding in $R^4 \rtimes SO(2, 2)$. Since Type 0 $\simeq R^4$, Type I $\simeq R \times Nil^3$, Type II $\simeq Nil^4$ and Type III-1 $\simeq R \times \{R^2 \rtimes \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}; t \in R\}$, as Lie groups, they are exactly the same group as that listed in [8], and D. Fried gave a list of their uniform lattices there. C. T. C. Wall also studied them, cf. [7]. Here we only write them down to complete our classification.

(5.3.1) The uniform lattices of H are semidirect products $Z^3 \rtimes Z_A$, where $A \in SL(Z)$ has a characteristic polynomial

$$\det(t - A) = (t - 1)(t^2 - bt + 1),$$

where $b \geq 2$ is an integer, and A and b satisfy:

- (i) Type 0: $A = I$, $b = 2$;
- (ii) Type I: $(A - I)^2 = 0$, $A \neq I$, $b = 2$;
- (iii) Type II: $(A - I)^2 \neq 0$, $(A - I)^3 = 0$, $b = 2$;
- (iv) Type III-1: $b \geq 3$.

(Cf. [3] and [7] for a proof.)

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REFERENCES

- [1] L. Auslander, *Simply transitive groups of affine motions*, Amer. J. Math., **99** (1977), 809–821.
- [2] A. Borel, *Linear Algebraic Groups*, Benjamin, New York, 1969.
- [3] D. Fried, *Flat spacetimes*, J. Differential Geometry, **26** (1987), 385–396.
- [4] D. Fried and W. Goldman, *Three-dimensional affine crystallographic groups*, Adv. in Math., **47** (1983), 1–49.
- [5] B. Kostant and D. Sullivan, *The Euler characteristic of an affine space form is zero*, Bull. Amer. Math. Soc., **81** (1975), 937–938.
- [6] J. Milnor, *On fundamental groups of complete affinely flat manifolds*, Adv. in Math., **525** (1977), 178–187.
- [7] C. T. C. Wall, *Geometric structures on compact complete analytic surfaces*, Topology, **25** No. 2, (1986), 119–153.
- [8] H. Zassenhaus, *Beweis eines Satzes uber diskrete Gruppen*, Abh. Math. Sem. Hamb. Univ., **12** (1938), 289–312.

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