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Let X and Y be two rearrangement invariant spaces on a measure space (Ω, Σ, μ) with a finite, nonatomic measure μ. We show that if there exists a non-zero order continuous disjointness preserving operator $T: X \to Y$, then $X \subseteq Y$. This result has many consequences. For example, if $T: L_p(\Omega, \Sigma, \mu) \to L_q(\Omega, \Sigma, \mu)$ $(0 < p < q \le \infty)$ preserves disjointness, then $T \equiv 0$.

1. Notation and preliminary facts. Recall that a (linear) operator $T: X \to Y$ between vector lattices is said to be a *disjointness preserving operator* if $|x_1| \wedge |x_2| = 0$ in X implies $|Tx_1| \wedge |Tx_2| = 0$ in Y. All vector lattices are assumed to be Archimedean, and all operators on normed or linear metric spaces are assumed to be continuous.

Let (Ω, Σ, μ) be a measure space with a finite σ-additive non-atomic measure and $S(\Omega, \Sigma, \mu)$ be the space of all (equivalence classes of) measurable real valued functions. Throughout the work we will use the representation of the space S as the space $C_\infty(Q)$ of all continuous extended functions on the Stone space Q of S. (See **[10]** for details.) We retain the same notation μ for the corresponding measure on Q, which is defined on the σ-algebra Σ_Q consisting of all subsets of the form $(E\backslash N) \cup (N\backslash E)$, where E is a *clopen* (closed and open) subset of Q and N is a first category subset of Q. It is well known that $\mu(D) = 0$ if and only if D is a nowhere dense subset of Q. (Any extremally disconnected space Q with such a measure is sometimes called a hyperstonian space.) A subspace X of $S(\Omega, \Sigma, \mu)$ is called a rearrangement invariant (r.i.) ideal if

(i) X is an order ideal in S, and

(ii) If $x \in X$, $y \in S$, and x and y are equimeasurable, in symbols $x \sim y$, then $y \in X$.

If, in addition, X is equipped with a Banach norm $\|\cdot\|$ such that

(iii) $x_1, x_2 \in X$ and $|x_1| \le |x_2| \Rightarrow \|x_1\| \le \|x_2\|$, and

(iv) $x_1, x_2 \in X$ and $x_1 \sim x_2 \Rightarrow \|x_1\| = \|x_2\|$,

then X is called a r.i. Banach function space. We refer to [7] for the basic facts concerning r.i. ideals and Banach spaces. (Let us mention incidentally that up to an equivalent renorming (i), (ii), and (iii) imply (iv). See [**1**] or [**7**, p. 115].) All necessary information about Banach and vector lattices can be found in [**4**, **10**].

2. The following theorem is the main result of this article.

THEOREM 1. *If X and Y are r.i. ideals and $X \not\subseteq Y$, then every order continuous disjointness preserving operator $T: X \to Y$ is identically equal to zero, i.e., $T \equiv 0$.*

We precede the proof of this theorem with several immediate corollaries.

COROLLARY 2. *Let X and Y be two r.i. Banach function spaces and X have order continuous norm. If $T: X \to Y$ is a nonzero disjointness preserving operator, then $X \subseteq Y$.*

An alternative proof of this corollary can be obtained using Lemma 5.2 in [**6**].

COROLLARY 3. *There is no nontrivial disjointness preserving operator from $L_p(\Omega, \Sigma, \mu)$ into $L_q(\Omega, \Sigma, \mu)$ for $0 < p < q \leq \infty$.*

REMARK. In a special case of L_p-spaces $(1 \leq p \leq \infty)$, when Ω is an open subset of R^n and μ is Lebesgue measure, this result was earlier obtained by a quite different method by M. Drahklin [**5**].

COROLLARY 4 (*L. Potepun* [**9**]). *Order isomorphic r.i. ideals coincide. That is, if X and Y are order isomorphic r.i. ideals, then $X = Y$.*

Proof. Let T be an order isomorphism of X onto Y. Obviously, T and T^{-1} are order continuous and, hence, by Theorem 1, $X \subseteq Y$ and $Y \subseteq X$, i.e., $X = Y$. The original proof in [**9**] was much more difficult. □

3. Three auxiliary lemmas. The space Q and measure μ below are as defined above.

LEMMA 5. *Let A be a nonvoid clopen subset of Q and let φ be a continuous open mapping from A into Q. Put $B = \varphi(A)$. Then there exists a nonvoid clopen subset B_1 of B and a constant $K > 0$ such that for any measurable $D \subset B_1$*

$$K^{-1}\mu(D) \leq \mu(\varphi^{-1}(D)) \leq K\mu(D).$$

Proof. The set $B = \varphi(A)$ is evidently a clopen subset of Q. We introduce a new measure γ on the σ-algebra Σ_Q by letting $\gamma(D) := \mu(\varphi^{-1}(D \cap B))$, $(D \in \Sigma_Q)$. Obviously, B is the support set of the measure γ. Let us verify that γ is absolutely continuous with respect to μ. Take an arbitrary measurable set D with $\mu(D) = 0$. Hence D is nowhere dense in Q. Since φ is open the set $\varphi^{-1}(D)$ $(= \varphi^{-1}(D \cap B))$ is also nowhere dense and thus $\mu(\varphi^{-1}(D \cap B)) = 0$. This proves that γ is absolutely continuous with respect to μ and, consequently, by the Radon-Nikodym theorem there exists a nonnegative function $h \in L_1(\Omega, \Sigma, \mu)$ such that $\gamma(D) = \int_D h\, d\mu$ for each measurable set D. Take a nonvoid clopen subset $B_1 \subset B$ and a constant $K > 0$ so that $K^{-1} \leq h(q) \leq K$ for each $q \in B_1$. Clearly B_1 and K satisfy the desired properties. □

LEMMA 6. *Let X and Y be two r.i. ideals on a (finite nonatomic measure) space (Ω, Σ, μ). If $X \not\subseteq Y$, then for each set $D \in \Sigma$ with $\mu(D) > 0$ there is a function $x \in X$ such that its support* $\operatorname{supp}(x) \subset D$ *and $x \notin Y$. Moreover, x can be chosen to be a step function.*

The proof is straightforward and is omitted. We only mention that for infinite measures this lemma is false and it is the only place where the finiteness of the measure μ is essential (see 5.4 below).

LEMMA 7. *Let Y be a r.i. ideal and $\tilde{y} = \sum_{n=1}^{\infty} d_n \chi_{E_n} \in Y$ be a step function, where $\{E_n\}$ $(n = 1, 2, \ldots)$ is a sequence of pairwise disjoint measurable sets. Also, let $\{D_n\}$ be a second sequence of pairwise disjoint measurable sets such that $K^{-1} \leq \mu(D_n)/\mu(E_n) \leq K$ for some $K > 0$. Then the step function $x = \sum_{n=1}^{\infty} d_n \chi_{D_n}$ likewise belongs to Y.*

4. Proof of Theorem 1. Let $T: X \to Y$ be an order continuous disjointess preserving operator from X into Y and let $X \not\subseteq Y$. We must show that $T \equiv 0$. The gist of the proof lies in an application of the multiplicative representation of disjointness preserving operators obtained in [**2**].

By Theorem A in [**2**], the operator T admits a global multiplicative representation, i.e., there exists a clopen set $E \subset Q$, a function $e \in C_\infty(Q)$ and a continuous mapping φ from E into Q, such that for each $x \in X$ and each $q \in Q$

$$(Tx)(q) = e(q)x(\varphi(q)), \quad \text{if } q \in E, \quad \text{and} \quad (Tx)(q) = 0 \text{ otherwise.}$$

The order continuity of T implies that the mapping φ is open (see [**2**, Lemma 4.1] or [**8**, Prop. 8]). Without loss of generality we may assume that $T \geq 0$. If $T \not\equiv 0$, then the set E is nonvoid and $E_0 := \{q \in E : 0 < e(q) < \infty\}$ is a dense open subset of E. (It is possible that $E_0 = E$.) Let us fix some constant $M > 0$ such that the clopen set $A = \mathrm{cl}\{q \in E_0 : M^{-1} < e(q) < M\}$ is nonvoid.

If we restrict the mapping φ to A and let $B = \varphi(A)$, then the continuous open mapping $\varphi\colon A \to B$ satisfies the conditions of Lemma 5. Therefore there exists a nonvoid clopen set $B_1 \subset B$ and a constant $K > 0$ such that $K^{-1} \leq \mu(D)/\mu(\varphi^{-1}(D) \cap A) \leq K$ for each measurable $D \subset B_1$. The condition $X \not\subseteq Y$ implies by Lemma 6 that there exists a step function $x = \sum_{n=1}^{\infty} d_n \chi_{D_n}$ such that $x \in X$, $x \notin Y$, $D_n \subset B_1$, and $D_n \cap D_m = 0$ $(n \neq m)$. Since $x \in X$, the function $y = Tx \in Y$. Now let us express y in terms of the multiplicative representation of T. We have

$$y = Tx = e(x \circ \varphi) = e(\cdot)x(\varphi(\cdot)) = e(\cdot)\left(\sum_{n=1}^{\infty} d_n \chi_{D_n}\right)(\varphi(\cdot))$$

$$= e(\cdot)\sum_{n=1}^{\infty} d_n \chi_{D_n}(\varphi(\cdot)) = e(\cdot)\sum_{n=1}^{\infty} d_n \chi_{\varphi^{-1}(D_n)}(\cdot).$$

Since $y \in Y$, we see that $y\chi_A \in Y$ and hence

$$y\chi_A = e\sum_{n=1}^{\infty} d_n \chi_{\varphi^{-1}(D_n)\cap A}.$$

As we know $e(q) \in [M^{-1}, M]$ for each $q \in A$ and therefore the function $\tilde{y} = \sum_{n=1}^{\infty} d_n \chi_{\varphi^{-1}(D_n)\cap A}$ belongs to Y if and only if $y\chi_A \in Y$. Letting $E_n = \varphi^{-1}(D_n) \cap A$, we see that $\tilde{y} = \sum_{n=1}^{\infty} d_n \chi_{E_n} \in Y$ and $K^{-1} \leq \mu(D_n)/\mu(E_n) \leq K$. By Lemma 7 this implies that $x \in Y$, a contradiction, and the proof is finished. □

5. Examples and comments. First, we show that the hypotheses of Theorem 1 cannot be weakened.

5.1. The condition $X \not\subseteq Y$ is essential, since if $X \subseteq Y$, then the identity imbedding $\mathrm{id}\colon X \to Y$ is a nonzero order continuous disjointness preserving operator.

5.2. Here we show that the assumption of order continuity of $T\colon X \to Y$ cannot be dropped. Indeed, let a r.i. space X have a nonzero discrete functional f. Then for each Y we can easily construct a nonzero disjointness preserving operator $T: X \to Y$. To this end take an arbitrary $y \in Y$, $y \neq 0$ and define $Tx = f(x)y$. It is evident that $T \not\equiv 0$ and T preserves disjointness. (A similar argument explains why we do not consider the case of atomic measure spaces. This case is of no interest since each discrete r.i. space always has a nonzero order continuous discrete functional.)

5.3. Recall that a norm $\|\cdot\|$ on a normed lattice Z is said to be *strictly monotone* if $0 \leq z_1 < z_2$ implies $\|z_1\| < \|z_2\|$.

PROPOSITION 8. *If X and Y are r.i. Banach function spaces with strictly monotone norms and T is a positive isometry from X into Y, then $X \subseteq Y$ (and $X = Y$ if T is also onto).*

Proof. It is easy to see (and this observation is due to A. S. Veksler) that each positive isometry preserves disjointness provided the norm in Y is strictly monotone. Thus, Theorem 1 is applicable and hence $X \subseteq Y$. If T is also onto, then, as is shown in [**3**, Thm. 1], T is necessarily an order isomorphism, and now Corollary 4 yields the desired equality $X = Y$. □

5.4. *The case of infinite measure.* Let us assume that $\mu(\Omega) = \infty$. It is a little bit surprising that Theorem 1 does not hold in this case. A simple example is as follows. Take $X = L^2(\mathbf{R})$ and $Y = L^2(\mathbf{R}) \cap L^1(\mathbf{R})$. Clearly X and Y are r.i. Banach function spaces with order continuous norms, $X \not\subseteq Y$ but, nevertheless, there exist nonzero order continuous disjointness preserving operators from X into Y. For example, $T_1 x := x\chi_{[a,b]}$ (where $a < b$ are arbitrary real numbers), or $T_2 x(t) := x(t)/(t^2+1)$ are such operators. Nevertheless, the following version of Theorem 1 still holds.

COROLLARY 9. *Let $\mu(\Omega) = \infty$. If there exists a nonzero order continuous disjointness preserving operator $T: X \to Y$, where X and Y are r.i. ideals, then for each set D of finite measure the subspace $X_D = \{x \in X : \operatorname{supp}(x) \subset D\}$ belongs to Y.*

Proof. Since $T \not\equiv 0$ and T is order continuous there is $x_1 \in X$ such that $y_1 = Tx_1 \neq 0$ and $\mu(E_1) < \infty$ where $E_1 = \operatorname{supp}(x)$. Choose a set E_2 of finite measure for which $y_1 \chi_{E_2} \neq 0$. Now put $E = E_1 \cup E_2 \cup D$ and define T_E by $T_E x = \chi_E T(x\chi_E)$. Obviously, T_E

is a nonzero order continuous disjointness preserving operator from the r.i. ideal X_E into the r.i. ideal Y_E. By Theorem 1, $X_E \subseteq Y_E$. In particular, $X_D \subset Y$. □

We have treated the case of real spaces only, but the results remain true for complex spaces as well.

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