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**AN INDEPENDENCE PROPERTY OF CENTRAL  
POLYNOMIALS**

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## AN INDEPENDENCE PROPERTY OF CENTRAL POLYNOMIALS

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Let  $\Phi_n$  be the ring of  $n \times n$  matrices over a commutative field  $\Phi$ . Let  $f_i(x_1, \dots, x_m)$  and  $g_i(y_1, \dots, y_m)$  ( $i = 1, \dots, k$ ) be polynomials with coefficients in  $\Phi$  and with noncommuting indeterminates in the disjoint sets  $\{x_1, \dots, x_m\}$  and  $\{y_1, \dots, y_m\}$ . Assume that  $f_1(x_1, \dots, x_m), \dots, f_k(x_1, \dots, x_m)$  are  $\Phi$ -independent modulo the  $T$ -ideal of polynomial identities of  $\Phi_n$ . Consider the following two statements: (1) whenever  $\sum_{i=1}^k f_i(x_1, \dots, x_m)g_i(y_1, \dots, y_m)$  is central on  $\Phi_n$ , then so is each  $g_i(y_1, \dots, y_m)$  ( $i = 1, \dots, k$ ); (2) whenever  $\sum_{i=1}^k f_i(x_1, \dots, x_m)g_i(y_1, \dots, y_m)$  is a polynomial identity for  $\Phi_n$ , then so is each  $g_i(y_1, \dots, y_m)$  ( $i = 1, \dots, k$ ). It is shown here that statement (2) is always true and that statement (1) holds but for the exceptional case:  $n = 2$  and  $\Phi$  is the ring of integers modulo 2.

**I. Results.** Throughout,  $\Phi$  always denotes a (commutative) field and, for  $n \geq 1$ ,  $\Phi_n$  denotes the ring consisting of all  $n \times n$  matrices over  $\Phi$ . Let  $Z$  be an infinite set of *noncommuting* indeterminates and let  $\Phi\{Z\}$  be the free  $\Phi$ -algebra generated by the set  $Z$ . By a *polynomial*, in noncommuting indeterminates in the set  $Z$  and with its coefficients in the field  $\Phi$ , we mean an element of the free  $\Phi$ -algebra  $\Phi\{Z\}$ . A polynomial  $f(z_1, \dots, z_m) \in \Phi\{Z\}$  is said to be a *polynomial identity* of  $\Phi_n$  if for any  $a_1, \dots, a_m \in \Phi_n$ ,  $f(a_1, \dots, a_m) = 0$ . A polynomial  $f(z_1, \dots, z_m) \in \Phi\{Z\}$  is said to be *central* on  $\Phi_n$ , if for any  $a_1, \dots, a_m \in \Phi_n$ ,  $f(a_1, \dots, a_m)$  is always in the center of  $\Phi_n$ . We let  $\mathcal{S}_n$  denote the set of all polynomial identities of  $\Phi_n$ . Then  $\mathcal{S}_n$  is a  $T$ -ideal in  $\Phi\{Z\}$ .

As we will consider polynomials in indeterminates in two disjoint sets, we make this notion precise as follows: Let  $X$  and  $Y$  be two disjoint sets of noncommuting indeterminates. Polynomials in  $\Phi\{X\}$  and polynomials in  $\Phi\{Y\}$  are said to be in noncommuting indeterminates in the disjoint sets  $X$  and  $Y$  respectively. Set  $Z = X \cup Y$ . The free  $\Phi$ -algebras  $\Phi\{X\}$  and  $\Phi\{Y\}$  can be regarded as  $\Phi$ -subalgebras of  $\Phi\{Z\}$  in a natural way. Hence the products and sums of elements in  $\Phi\{X\} \cup \Phi\{Y\}$  can be taken in  $\Phi\{Z\}$ .

Assume that  $\Phi$  is an infinite field. Let  $f(x_1, \dots, x_m)$  and  $g(y_1, \dots, y_m)$  be two polynomials in noncommuting indeterminates in the two disjoint sets  $\{x_1, \dots, x_m\}$  and  $\{y_1, \dots, y_m\}$  respectively. It is proved in [2] by Regev that, if  $f(x_1, \dots, x_m)g(y_1, \dots, y_m)$  is central on  $\Phi_n$ , then both  $f(x_1, \dots, x_m)$  and  $g(y_1, \dots, y_m)$  must be also central. Our primary objective here is to prove the following natural generalization

**THEOREM.** *Let  $\Phi_n$  be the ring of  $n \times n$  matrices over a field  $\Phi$  and let  $\mathcal{I}_n$  be the  $T$ -ideal of polynomial identities of  $\Phi_n$ . For  $i = 1, \dots, k$ , let  $f_i(x_1, \dots, x_m)$  and  $g_i(y_1, \dots, y_m)$  be polynomials with their coefficients in  $\Phi$  and in noncommuting indeterminates in the disjoint sets  $\{x_1, \dots, x_m\}$  and  $\{y_1, \dots, y_m\}$  respectively. Assume that the polynomial  $\sum_{i=1}^k f_i(x_1, \dots, x_m)g_i(y_1, \dots, y_m)$  is central on  $\Phi_n$ . Then, except only when  $k \geq 2$ ,  $n = 2$  and  $\Phi$  is the Galois field with only two elements, the following hold:*

(1) *If  $f_i(x_1, \dots, x_m)$ ,  $i = 1, \dots, k$ , are  $\Phi$ -independent modulo  $\mathcal{I}_n$ , then all  $g_i(y_1, \dots, y_m)$ ,  $i = 1, \dots, k$ , must be central on  $\Phi_n$ .*

(2) *If  $g_i(y_1, \dots, y_m)$ ,  $i = 1, \dots, k$ , are  $\Phi$ -independent modulo  $\mathcal{I}_n$ , then all  $f_i(x_1, \dots, x_m)$ ,  $i = 1, \dots, k$ , must be central on  $\Phi_n$ .*

(3) *If both the sets  $\{f_i(x_1, \dots, x_m): i = 1, \dots, k\}$  and  $\{g_i(y_1, \dots, y_m): i = 1, \dots, k\}$  are  $\Phi$ -independent modulo  $\mathcal{I}_n$ , then all  $f_i(x_1, \dots, x_m)$  and  $g_i(y_1, \dots, y_m)$ ,  $i = 1, \dots, k$ , must be central on  $\Phi_n$ .*

Unlike the result of [2], our field  $\Phi$  need *not* be infinite. The only exception in our theorem is the ring of  $2 \times 2$  matrices over  $\text{GF}(2)$ , the integers modulo 2, and even in this exceptional ring, our theorem above still holds when  $k = 1$ . Thus, the special instance of our theorem above when  $k = 1$  already generalizes the result of [2] by removing the assumption that  $\Phi$  is infinite.

An interesting immediate consequence is the following

**COROLLARY.** *Let  $\Phi_n$ ,  $\mathcal{I}_n$ ,  $f_i(x_1, \dots, x_m)$  and  $g_i(y_1, \dots, y_m)$ ,  $i = 1, \dots, k$ , be as explained in the theorem above. Assume that  $\sum_{i=1}^k f_i(x_1, \dots, x_m)g_i(y_1, \dots, y_m) \in \mathcal{I}_n$ . Then, without any exception on  $k$ ,  $n$  and  $\Phi$ , the following hold always:*

(1) *If  $f_i(x_1, \dots, x_m)$ ,  $i = 1, \dots, k$ , are  $\Phi$ -independent modulo  $\mathcal{I}_n$ , then  $g_i(y_1, \dots, y_m) \in \mathcal{I}_n$  for all  $i = 1, \dots, k$ .*

(2) *If  $g_i(y_1, \dots, y_m)$ ,  $i = 1, \dots, k$ , are  $\Phi$ -independent modulo  $\mathcal{I}_n$ , then  $f_i(x_1, \dots, x_m) \in \mathcal{I}_n$  for all  $i = 1, \dots, k$ .*

It is interesting to observe that, in the notation of the corollary above, if both the sets  $\{f_i(x_1, \dots, x_m): i = 1, \dots, k\}$  and  $\{g_i(y_1, \dots, y_m): i = 1, \dots, k\}$  are  $\Phi$ -independent modulo  $\mathcal{S}_n$ , then the polynomial  $\sum_{i=1}^k f_i(x_1, \dots, x_m)g_i(y_1, \dots, y_m)$  can *never* be an identity of  $\Phi_n$ .

Before proceeding to the proofs, let us give an example showing that the exceptional case of our theorem above really happens:

**EXAMPLE.** Let  $\Phi$  the Galois field with only two elements 0 and 1. Let  $\lambda$  be a new indeterminate intended to range over  $\Phi_2$ . The possible minimum polynomials for elements in  $\Phi_2$  are  $\lambda, \lambda - 1, \lambda^2 - \lambda, \lambda^2, (\lambda - 1)^2$  and  $\lambda^2 + \lambda + 1$ . Let  $h(\lambda) = \lambda^2(\lambda - 1)^2$ . If the minimum polynomial of  $a \in \Phi_2$  is  $\lambda, \lambda - 1, \lambda^2 - \lambda, \lambda^2$  or  $(\lambda - 1)^2$ , then  $h(a) = 0$ . If the minimum polynomial of  $a \in \Phi$  is  $\lambda^2 + \lambda + 1$ , then, since  $h(\lambda) = (\lambda^2 + \lambda + 1)^2 + 1$ , we have  $h(a) = 1$ . Hence  $h(\lambda)$  is a central polynomial of  $\Phi$ , and, for  $a \in \Phi_2$ ,  $h(a) = 1$  when and only when the minimum polynomial of  $a$  is  $\lambda^2 + \lambda + 1$ . It is also easy to see that there are only two elements whose minimum polynomials are  $\lambda^2 + \lambda + 1$ , namely,  $a_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$  and  $a_2 = 1 + a_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ . Let  $x, y$  be two distinct indeterminates. Set  $f_1(x) = xh(x), f_2(x) = (f_1(x))^2, g_1(y) = yh(y)$  and  $g_2(y) = (g_1(y))^2$ . Observe that, for  $a \in \Phi_2$ ,

$$ah(a) = \begin{cases} a, & \text{if } a = a_1 \text{ or } a = a_2, \\ 0, & \text{otherwise.} \end{cases}$$

Thus none of  $f_1(x), f_2(x), g_1(y), g_2(y)$  can be central on  $\Phi_2$ . Also if  $a = a_1$  or if  $a = a_2$ , then  $f_1(a) = g_1(a) = a$  and  $f_2(a) = g_2(a) = a^2$ . Since  $a$  and  $a^2$  are  $\Phi$ -independent, both the sets  $\{f_1(x), f_2(x)\}$  and  $\{g_1(y), g_2(y)\}$  are  $\Phi$ -independent modulo  $\mathcal{S}_2$ . We show that the polynomial  $f_1(x)g_1(y) + f_2(x)g_2(y)$  is central. If  $a \neq a_1, a_2$  or if  $b \neq a_1, a_2$ , then  $f_1(a) = f_2(a) = 0$  or  $g_1(b) = g_2(b) = 0$  respectively and hence  $f_1(a)g_1(b) + f_2(a)g_2(b) = 0$ . If  $a = b = a_1$ , then  $f_1(a)g_1(a) + f_2(b)g_2(b) = a_1^2 + a_1^4 = (a_1 + a_1^2)^2 = 1^2 = 1$ . Similarly, if  $a = b = a_2$ , then  $f_1(a)g_1(a) + f_2(b)g_2(b) = a_2^2 + a_2^4 = 1$  also. If  $a = a_1$  and  $b = a_2$ , then  $f_1(a)g_1(b) + f_2(a)g_2(b) = a_1a_2 + a_1^2a_2^2 = 0$ . Similarly, if  $a = a_2$  and  $b = a_1$ , then  $f_1(a)g_1(b) + f_2(a)g_2(b) = 0$  also. Thus we have constructed a counterexample for  $k = 2$ . For  $k > 2$ , we pick new indeterminates  $x_3, \dots, x_k, y_3, \dots, y_k$ , so that they are distinct from each other and also distinct from  $x, y$ . Set  $f_i(x_i) = h(x_i)$  and  $g_i(y_i) = h(y_i)$  for  $i = 3, \dots, k$ . Since all  $f_i(x_i)$  and  $g_i(y_i)$  ( $i = 3, \dots, k$ ) thus defined are central on  $\Phi_2$  and since  $f_1(x)g_1(y) + f_2(x)g_2(y)$  has been already shown to be central on  $\Phi_2$ , so must be  $f_1(x)g_1(y) + f_2(x)g_2(y) + \sum_{i=3}^k f_i(x_i)g_i(y_i)$ .

Since  $f_i(x_i)$  ( $i = 3, \dots, k$ ) involve indeterminates distinct from each other and also from  $x, y$ ,  $f_i(x_i)$  ( $i = 3, \dots, k$ ) must be  $\Phi$ -independent from each other and also from  $f_1(x), f_2(x)$  modulo  $\mathcal{S}_2$ . So  $f_1(x), f_2(x), f_3(x_3), \dots, f_k(x_k)$  are  $\Phi$ -independent modulo  $\mathcal{S}_2$ . Similarly,  $g_1(y), g_2(y), g_3(y_3), \dots, g_k(y_k)$  are also  $\Phi$ -independent modulo  $\mathcal{S}_2$ . We have constructed the desired example for any  $k \geq 2$ .

**II. Proofs.** As our results are trivial when  $n = 1$ , we assume throughout that  $n > 1$ . We will let  $e_{ij} \in \Phi_n$  denote the  $n \times n$  matrix unit with 1 in its  $(i, j)$ -entry and 0 elsewhere. Our argument is based on the following two simple facts:

*Fact 1.* Assume that  $n > 2$  or  $\Phi$  contains more than two elements. If  $a \in \Phi_n$  is not central, then there exist finitely many invertible elements  $u_1, \dots, u_s$  such that  $e_{12} = \sum_{i=1}^s u_i a u_i^{-1}$ .

*Proof.* Let  $A$  be the additive subgroup generated by all conjugates of  $a$ . Then the set  $A$  is obviously invariant under conjugations by invertible elements of  $\Phi_n$  and  $A$  is also noncentral since  $a \in A$  is noncentral. Since  $[\Phi_n, \Phi_n]$  is the only proper (noncentral) Lie ideal of  $\Phi_n$ ,  $A$  must contain  $[\Phi_n, \Phi_n]$  by Theorem 1 [1] and Theorem 2 [1] together. But  $e_{12} = e_{12}e_{22} - e_{22}e_{12} \in [\Phi_n, \Phi_n] \subseteq A$ . So there exist finitely many invertible elements  $u_1, \dots, u_s \in \Phi_n$  such that  $e_{12} = \sum_{i=1}^s u_i a u_i^{-1}$ .

For  $a \in \Phi_n$ , the centralizer of  $a$ , denoted by  $C(a)$ , is defined to be the set  $\{x \in \Phi_n : ax = xa\}$ . For simplicity of notation, we denote the center of  $\Phi_n$  by  $\Phi$ .

*Fact 2.* Assume that  $n > 2$  or  $\Phi$  contains more than two elements. If  $a \in \Phi_n$  is such that  $uau^{-1} - a \in \Phi$  for all invertible elements  $u \in C(e_{12})$ , then  $a = \alpha + \beta e_{12}$  for some  $\alpha, \beta \in \Phi$ .

*Proof.* Let  $a = \sum_{s,t=1}^n \alpha_{st} e_{st} \in \Phi_n$  be such that, for all invertible elements  $u \in C(e_{12})$ ,  $uau^{-1} - a \in \Phi$ , that is,  $ua - au = \gamma u$  for some  $\gamma \in \Phi$ . First, consider the case  $n \geq 3$ . For  $j \geq 2$ , since  $e_{1j} \in C(e_{12})$  and  $(e_{1j})^2 = 0$ ,  $1 + e_{1j}$  is an invertible element in  $C(e_{12})$ . So  $e_{1j}a - ae_{1j} = (1 + e_{1j})a - a(1 + e_{1j}) = \gamma(1 + e_{1j})$  for some  $\gamma \in \Phi$ . Since both  $e_{1j}a$  and  $ae_{1j}$  are of rank at most one,  $e_{1j}a - ae_{1j}$  is of rank at most two and hence cannot be invertible in  $\Phi_n$  ( $n \geq 3$ ).

So  $e_{1j}a - ae_{1j} = 0$ . By direct computation,

$$e_{1j}a - ae_{1j} = \sum_{t=1}^n \alpha_{jt}e_{1t} - \sum_{s=1}^n \alpha_{s1}e_{sj} = 0.$$

By comparing the coefficients of both sides, we have  $\alpha_{jj} = \alpha_{11}$  and  $\alpha_{jt} = 0$  for all  $t \neq j$ . Now, consider  $e_{i2}$ , where  $i \neq 2$ . As before, we have  $0 = e_{i2}a - ae_{i2} = \sum_{t=1}^n \alpha_{2t}e_{it} - \sum_{s=1}^n \alpha_{si}e_{s2}$  and hence, by comparing the coefficients,  $\alpha_{si} = 0$  for all  $s \neq i$ . In particular,  $\alpha_{1i} = 0$  for all  $i \geq 3$ . Combining all these together, we have  $a = \alpha + \beta e_{12}$ , where  $\alpha = \alpha_{11} = \alpha_{22} = \dots = \alpha_{nn}$  and  $\beta = \alpha_{12}$ .

Now consider the case  $n = 2$ . By our assumption,  $\Phi$  contains an element, say  $\delta$ , other than 0 and 1. Since both  $1 + e_{12}$  and  $\delta + e_{12}$  are invertible elements in  $C(e_{12})$ , we have  $e_{12}a - ae_{12} = (1 + e_{12})a - a(1 + e_{12}) = \gamma(1 + e_{12})$  for some  $\gamma \in \Phi$ , and similarly,  $e_{12}a - ae_{12} = (\delta + e_{12})a - a(\delta + e_{12}) = \gamma'(\delta + e_{12})$  for some  $\gamma' \in \Phi$ . Hence  $\gamma(1 + e_{12}) = e_{12}a - ae_{12} = \gamma'(\delta + e_{12})$ . This can happen only when  $\gamma = \gamma' = 0$ , since  $1 + e_{12}$  and  $\delta + e_{12}$  are obviously  $\Phi$ -independent. Now, as before, let

$$a = \sum_{s,t=1}^2 \alpha_{st}e_{st} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}.$$

Then

$$e_{12}a - ae_{12} = \begin{pmatrix} \alpha_{21} & \alpha_{22} - \alpha_{11} \\ 0 & -\alpha_{21} \end{pmatrix} = 0.$$

So  $\alpha_{21} = 0$  and  $\alpha_{11} = \alpha_{22}$ . That is,  $a = \alpha + \beta e_{12}$ , where  $\alpha = \alpha_{11} = \alpha_{22}$  and  $\beta = \alpha_{12}$ .

For brevity, we introduce the following definition:

**DEFINITION.** For  $a_1, \dots, a_k, b_1, \dots, b_k \in \Phi_n$ , we write

$$\langle a_1, \dots, a_k \rangle * \langle b_1, \dots, b_k \rangle \in \Phi$$

if the following condition (\*) is satisfied:

$$(*) \quad \sum_{i=1}^k (ua_iu^{-1})(vb_iv^{-1}) \in \Phi$$

for any invertible elements  $u, v \in \Phi_n$ .

By conjugation (\*) by  $u^{-1}$ , we have  $\sum_{i=1}^k a_i(u^{-1}v)b_i(u^{-1}v)^{-1} \in \Phi$ . Since  $u^{-1}v$  also ranges over all invertible elements of  $\Phi_n$ , (\*) is

equivalent to:

$$(*)' \quad \sum_{i=1}^k a_i (v b_i v^{-1}) \in \Phi \quad \text{for all invertible elements } v \in \Phi_n.$$

Symmetrically,  $(*)$  is also equivalent to

$$(*)'' \quad \sum_{i=1}^k (u a_i u^{-1}) b_i \in \Phi \quad \text{for all invertible elements } u \in \Phi_n.$$

In the following fact, we collect some simple properties about the condition  $(*)$ , which will be needed in the sequel:

*Fact 3.* Assume that  $a_1, \dots, a_k, b_1, \dots, b_k, b'_1, \dots, b'_k \in \Phi_n$ .

(1) If both  $\langle a_1, \dots, a_k \rangle * \langle b_1, \dots, b_k \rangle \in \Phi$  and  $\langle a_1, \dots, a_k \rangle * \langle b'_1, \dots, b'_k \rangle \in \Phi$ , then  $\langle a_1, \dots, a_k \rangle * \langle b_1 + b'_1, \dots, b_k + b'_k \rangle \in \Phi$ .

(2) If  $\langle a_1, \dots, a_k \rangle * \langle b_1, \dots, b_k \rangle \in \Phi$ , then  $\langle a_1, \dots, a_k \rangle * \langle v b_1 v^{-1}, \dots, v b_k v^{-1} \rangle \in \Phi$  for any invertible element  $v \in \Phi_n$ .

(3) If  $\langle a_1, \dots, a_k \rangle * \langle b_1, \dots, b_k \rangle \in \Phi$  and  $b_k = \sum_{i=1}^{k-1} \beta_i b_i$ , where  $\beta_i \in \Phi$  ( $i = 1, \dots, k-1$ ), then  $\langle a_1 + \beta_1 a_k, a_2 + \beta_2 a_k, \dots, a_{k-1} + \beta_{k-1} a_k \rangle * \langle b_1, b_2, \dots, b_{k-1} \rangle \in \Phi$ .

*Proof.* Immediate.

Our Fact 3 above is concerned about variations of  $\langle b_1, \dots, b_k \rangle$  in the condition  $(*)$ . The corresponding properties concerning about variations of  $\langle a_1, \dots, a_k \rangle$  in the condition  $(*)$  can be formulated and proved similarly.

We start the proof of our main theorem with the following:

**LEMMA 1.** Assume that  $n \neq 2$  or the field  $\Phi$  contains more than two elements. Let  $b_1, \dots, b_k \in \Phi_n$  be  $\Phi$ -independent. For  $a_1, \dots, a_k \in \Phi_n$ , if  $\langle a_1, \dots, a_k \rangle * \langle b_1, \dots, b_k \rangle \in \Phi$ , then  $a_1, \dots, a_k \in \Phi$ .

*Proof.* If  $n = 1$ , then all elements of  $\Phi_n$  are central and Lemma 1 holds trivially. So let  $n \geq 2$ . Assume on the contrary that Lemma 1 is false. Let  $k$  be the minimal integer such that the assertion of Lemma 1 fails. First, assume that  $k = 1$ . Write  $a = a_1$  and  $b = b_1$  for brevity. By our assumption,  $a$  is not central and, since  $ab \in \Phi$ ,  $b$  cannot be central either. By Fact 1, there exist invertible elements  $u_1, \dots, u_s, v_1, \dots, v_t \in \Phi_n$  such that  $e_{12} = \sum_{i=1}^s u_i a_1 u_i^{-1} = \sum_{j=1}^t v_j b_1 v_j^{-1}$ . By Fact 3,  $\langle \sum_{i=1}^s u_i a u_i^{-1} \rangle * \langle \sum_{j=1}^t v_j b v_j^{-1} \rangle \in \Phi$ . So we

have  $\langle e_{12} \rangle * \langle e_{12} \rangle \in \Phi$ . Let

$$v = e_{12} + e_{21} + \sum_{j>2}^n e_{jj} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Then  $v$  is an invertible element of  $\Phi_n$  and  $ve_{12}v^{-1} = e_{21}$ . Since  $\langle e_{12} \rangle * \langle e_{12} \rangle \in \Phi$ , we have  $e_{11} = e_{12}e_{21} = e_{12}(ve_{12}v^{-1}) \in \Phi$ , a contradiction.

Now, assume  $k \geq 2$ . By reindexing  $a_i, b_i$  ( $i = 1, \dots, k$ ) if necessary, we may assume that  $a_k$  is *not* central. By Fact 1, there exist invertible elements  $v_1, \dots, v_s \in \Phi_n$  such that  $e_{12} = \sum_{j=1}^s v_j a_k v_j^{-1}$ . By Fact 3,  $\langle \sum_{j=1}^s v_j a_1 v_j^{-1}, \dots, \sum_{j=1}^s v_j a_k v_j^{-1} \rangle * \langle b_1, \dots, b_k \rangle \in \Phi$ . Replacing each  $a_1, \dots, a_k$  by  $\sum_{j=1}^s v_j a_1 v_j^{-1}, \dots, \sum_{j=1}^s v_j a_k v_j^{-1}$  respectively, we may assume that  $a_k = e_{12}$  to start with. Let  $u$  be an invertible element of  $C(e_{12})$ . By Fact 3,  $\langle ua_1 u^{-1}, \dots, ua_k u^{-1} \rangle * \langle b_1, \dots, b_k \rangle \in \Phi$  and hence also  $\langle ua_1 u^{-1} - a_1, \dots, ua_k u^{-1} - a_k \rangle * \langle b_1, \dots, b_k \rangle \in \Phi$ . Since  $ua_k u^{-1} - a_k = 0$ , we have  $\langle ua_1 u^{-1} - a_1, \dots, ua_{k-1} u^{-1} - a_{k-1} \rangle * \langle b_1, \dots, b_{k-1} \rangle \in \Phi$ . By the minimality of  $k$ ,  $ua_1 u^{-1} - a_1, \dots, ua_{k-1} u^{-1} - a_{k-1}$  are all central. Since this holds for any invertible elements  $u \in C(e_{12})$ , and since  $n > 2$  or  $|\Phi| > 2$ , by Fact 2,  $a_1 = \alpha_1 + \beta_1 e_{12}, \dots, a_{k-1} = \alpha_{k-1} + \beta_{k-1} e_{12}$  for some  $\alpha_1, \dots, \alpha_{k-1}, \beta_1, \dots, \beta_{k-1} \in \Phi$ . By (3) of Fact 3,  $\langle 1, e_{12} \rangle * \langle \alpha_1 b_1 + \dots + \alpha_{k-1} b_{k-1}, \beta_1 b_1 + \dots + \beta_{k-1} b_{k-1} + b_k \rangle \in \Phi$ . Set  $b'_1 = \alpha_1 b_1 + \dots + \alpha_{k-1} b_{k-1}$  and  $b'_2 = \beta_1 b_1 + \dots + \beta_{k-1} b_{k-1} + b_k$ . Then  $\langle 1, e_{12} \rangle * \langle b'_1, b'_2 \rangle \in \Phi$ . Since  $b_1, \dots, b_k$  are assumed to be  $\Phi$ -independent,  $b'_2$  is  $\Phi$ -independent of  $b'_1$  and, in particular, must be nonzero. Let  $v$  be an arbitrary invertible element of  $\Phi_n$ . By Fact 3 again,  $\langle v(1)v^{-1} - 1, ve_{12}v^{-1} - e_{12} \rangle * \langle b'_1, b'_2 \rangle \in \Phi$ , that is,  $\langle ve_{12}v^{-1} - e_{12} \rangle * \langle b'_2 \rangle \in \Phi$ . By our result for the case  $k = 1$  in the previous paragraph,  $ve_{12}v^{-1} - e_{12} \in \Phi$ . Now let  $v = 1 + e_{21}$ . Then  $v^{-1} = 1 - e_{12}$ . We compute  $ve_{12}v^{-1} - e_{12} = (1 + e_{21})e_{12}(1 - e_{21}) - e_{12} = -e_{11} - e_{21} + e_{22}$ . But, obviously,  $-e_{11} - e_{21} + e_{22}$  cannot be central, a contradiction. This completes our proof of Lemma 1.

LEMMA 2. Assume that  $n \neq 2$  or  $\Phi$  contains more than two elements. Let  $f_1(x_1, \dots, x_m), \dots, f_k(x_1, \dots, x_m) \in \Phi\{x_1, \dots, x_m\}$  be  $\Phi$ -independent modulo  $\mathcal{I}_n$ . For any  $b_1, \dots, b_k \in \Phi_n$ , if

$\sum_{i=1}^k f_i(x_1, \dots, x_m) b_i \in \Phi$  for any assignment of values in  $\Phi_n$  to  $x_1, \dots, x_m$ , then  $b_1, \dots, b_k \in \Phi$ .

*Proof.* For any invertible element  $u \in \Phi_n$  and for any assignment of values in  $\Phi_n$  to  $x_1, \dots, x_m$ , we have, by our assumption, that

$$\sum_{i=1}^k (u f_i(x_1, \dots, x_m) u^{-1}) b_i = \sum_{i=1}^k f_i(ux_1 u^{-1}, \dots, ux_m u^{-1}) b_i \in \Phi.$$

Hence, for any assignment of values in  $\Phi_n$  to  $x_1, \dots, x_m$ , we have  $\langle f_1(x_1, \dots, x_m), \dots, f_k(x_1, \dots, x_m) \rangle * \langle b_1, \dots, b_k \rangle \in \Phi$ .

First assume that  $b_1, \dots, b_k$  are  $\Phi$ -independent modulo  $\Phi$ , in the sense that for any  $\beta_1, \dots, \beta_k \in \Phi$ ,  $\beta_1 b_1 + \dots + \beta_k b_k \in \Phi$  implies  $\beta_1 = \dots = \beta_k = 0$ , i.e.,  $\{1, b_1, \dots, b_k\}$  are linearly independent. Then, by Lemma 1, for any assignment of values in  $\Phi_n$  to  $x_1, \dots, x_m$ ,  $f_1(x_1, \dots, x_m), \dots, f_k(x_1, \dots, x_m) \in \Phi$  and, by our assumption on the  $\Phi$ -independence of  $b_1, \dots, b_k$  modulo  $\Phi$ ,  $f_1(x_1, \dots, x_m) = \dots = f_k(x_1, \dots, x_m) = 0$ . This is a contradiction to the  $\Phi$ -independence of  $f_1(x_1, \dots, x_m), \dots, f_k(x_1, \dots, x_m)$  modulo  $\mathcal{S}_n$ . Thus  $\{1, b_1, \dots, b_k\}$  are linearly dependent. By reindexing  $b_1, \dots, b_k$  if necessary, we may assume that  $\{1, b_1, \dots, b_s\}$ , where  $0 \leq s < k$ , forms a  $\Phi$ -basis of the  $\Phi$ -subspace spanned by  $\{1, b_1, \dots, b_k\}$ . For  $j = s+1, \dots, k$ , write  $b_j = \sum_{i=1}^s \beta_i^{(j)} b_i + \gamma^{(j)}$ , where, for  $i = 1, \dots, s$ ,  $\beta_i^{(j)}, \gamma^{(j)} \in \Phi$ . Hence

$$\begin{aligned} & f_1(x_1, \dots, x_m) b_1 + \dots + f_k(x_1, \dots, x_m) b_k \\ &= \left( f_1(x_1, \dots, x_m) + \sum_{j=s+1}^k f_j(x_1, \dots, x_m) \beta_1^{(j)} \right) b_1 + \dots \\ &+ \left( f_s(x_1, \dots, x_m) + \sum_{j=s+1}^k f_j(x_1, \dots, x_m) \beta_s^{(j)} \right) b_s \\ &+ \left( \sum_{j=s+1}^k f_j(x_1, \dots, x_m) \gamma^{(j)} \right) 1. \end{aligned}$$

By Lemma 1 again, for any assignment of values in  $\Phi_n$  to  $x_1, \dots, x_m$ , the matrices  $f_i(x_1, \dots, x_m) + \sum_{j=s+1}^k f_j(x_1, \dots, x_m) \beta_i^{(j)}$ ,

( $i = 1, \dots, s$ ), as well as the matrix  $\sum_{j=s+1}^k f_j(x_1, \dots, x_m)\gamma^{(j)}$ , are all central in  $\Phi_n$  and, since  $1, b_1, \dots, b_s$  are assumed to be  $\Phi$ -independent modulo  $\Phi$ , we have

$$\begin{aligned}
 f_1(x_1, \dots, x_m) + \sum_{j=s+1}^k f_j(x_1, \dots, x_m)\beta_1^{(j)} &= 0, \\
 &\vdots \\
 f_s(x_1, \dots, x_m) + \sum_{j=s+1}^k f_j(x_1, \dots, x_m)\beta_s^{(j)} &= 0.
 \end{aligned}$$

But this contradicts with the  $\Phi$ -independence of  $f_1(x_1, \dots, x_m), \dots, f_k(x_1, \dots, x_m)$  modulo  $\mathcal{S}_n$ . Hence  $s$  must be 0 and the unit 1 spans the  $\Phi$ -subspace spanned by  $\{1, b_1, \dots, b_k\}$ . This is equivalent to the fact that  $b_1, \dots, b_k \in \Phi$ , as desired.

As with Lemma 1, there is also a symmetrical version of Lemma 2, which can be formulated and proved analogously.

Our last lemma treats the special case when  $n = 2$  and  $\Phi$  contains only two elements.

**LEMMA 3.** *Let  $\Phi = \{0, 1\}$  be the ring of integers modulo 2.*

(1) *For  $a, b \in \Phi_2 \setminus \{0\}$ , if  $\langle a \rangle * \langle b \rangle \in \Phi$ , then  $a, b \in \Phi$ .*

(2) *Let  $f_1(x_1, \dots, x_m), \dots, f_k(x_1, \dots, x_m) \in \Phi\{x_1, \dots, x_m\}$  be  $\Phi$ -independent modulo  $\mathcal{S}_2$ . For  $b_1, \dots, b_k \in \Phi_2$ , if*

$$\sum_{i=1}^k f_i(x_1, \dots, x_m)b_i = 0$$

*for all  $x_1, \dots, x_m \in \Phi_2$ , then  $b_1 = \dots = b_k = 0$ .*

*Proof.* (1) Let us determine the conjugacy classes of  $\Phi_2$ . Two elements in  $\Phi_2$  are similar if and only if they have the same minimum polynomials. The possible minimum polynomials in  $\Phi_2$  are  $\lambda, \lambda + 1, \lambda^2, \lambda^2 + 1, \lambda^2 + \lambda, \lambda^2 + \lambda + 1$ . For a polynomial  $\phi(\lambda)$ , in the indeterminate  $\lambda$  and with coefficients in  $\Phi$ , let  $B(\phi(\lambda))$  denote the set consisting of all elements in  $\Phi_2$  whose minimum polynomials are  $\phi(\lambda)$  and also let  $A(\phi(\lambda))$  denote the additive subgroup generated by

$B(\phi(\lambda))$ . By direct computation, we have the following list of all possible  $B(\phi(\lambda))$  and  $A(\phi(\lambda))$ :

$$\begin{aligned}
 B(\lambda) &= \{0\}, \\
 B(\lambda + 1) &= \{1\}, \\
 B(\lambda^2) &= \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}, \\
 B(\lambda^2 + 1) &= \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}, \\
 B(\lambda^2 + \lambda) &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \right. \\
 &\quad \left. \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}, \\
 B(\lambda^2 + \lambda + 1) &= \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}, \\
 A(\lambda) &= \{0\}, \\
 A(\lambda + 1) &= \{0, 1\}, \\
 A(\lambda^2) &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \right. \\
 &\quad \left. \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \\
 A(\lambda^2 + 1) &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}, \\
 A(\lambda^2 + \lambda) &= \Phi_2, \\
 A(\lambda^2 + \lambda + 1) &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.
 \end{aligned}$$

For  $x, y \in \Phi_2$ , if  $\langle x \rangle * \langle y \rangle \in \Phi_2$  and if one of  $x, y$  is central, then the other must also be central. So let us assume, towards a contradiction, that neither of  $a$  and  $b$  is central. Since  $\langle a \rangle * \langle b \rangle \in \Phi$ ,  $\langle a' \rangle * \langle b' \rangle \in \Phi$  for any  $a'$  in the additive subgroup generated by the conjugates of  $a$  and for any  $b'$  in the additive subgroup generated by the conjugates of  $b$ . Observe that in the above list of  $A(\phi(\lambda))$ , all but  $A(\lambda)$  and  $A(\lambda^2 + 1)$  contain the identity 1. If the minimum polynomial of  $b$  is not  $\lambda^2 + 1$ , then  $\langle a \rangle * \langle 1 \rangle \in \Phi$  and hence  $a$  must be central, a contradiction. So the minimum polynomial of  $b$  is  $\lambda^2 + 1$ . Similarly, the minimum polynomial of  $a$  is also  $\lambda^2 + 1$ . But then  $\langle a' \rangle * \langle b' \rangle \in \Phi$

for any  $a', b' \in A(\lambda^2 + 1)$ . This is absurd: For instance,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in A(\lambda^2 + 1) \quad \text{but}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \notin \Phi.$$

This contradiction completes our proof of (1).

(2) Suppose not. Let  $k \geq 1$  be the minimal integer such that the assertion of (2) of this lemma fails. By (1) of this lemma,  $k \geq 2$ . We divide the argument, into three cases.

*Case 1. For some  $i = 1, \dots, k$ ,  $b_i = 1$ :* By reindexing if necessary, we may assume  $b_1 = 1$ . For any invertible element  $u \in \Phi_2$ , we have

$$\begin{aligned} & \sum_{i=2}^k f_i(x_1, \dots, x_m)(ub_iu^{-1} - b_i) \\ &= u \left( \sum_{i=1}^k f_i(u^{-1}x_1u, \dots, u^{-1}x_mu) b_i \right) u^{-1} \\ & \quad - \sum_{i=1}^k f_i(x_1, \dots, x_m) b_i \\ &= 0. \end{aligned}$$

By the minimality of  $k$ ,  $ub_iu^{-1} = b_i$  ( $i = 2, \dots, k$ ). Hence, for any invertible element  $u \in \Phi_2$ ,  $ub_i = b_iu$  ( $i = 2, \dots, k$ ). By a direct computation,  $b_2, \dots, b_k$  must be all central. This contradicts with the  $\Phi$ -independence of  $f_1(x_1, \dots, x_m), \dots, f_k(x_1, \dots, x_m)$  modulo  $\mathcal{F}_2$ .

*Case 2. The minimum polynomial of some  $b_i$  ( $i = 1, \dots, k$ ) is not  $\lambda^2 + 1$ :* By reindexing if necessary, we may assume that the minimum polynomial  $\phi(\lambda)$  of  $b_1$  is not  $\lambda^2 + 1$ . By the list of all possible  $A(\phi(\lambda))$  in the proof of (1),  $1 \in A(\phi(\lambda))$ . So there exist invertible elements  $u_1, \dots, u_s \in \Phi_2$  such that  $\sum_{j=1}^s u_j b_1 u_j^{-1} = 1$ . Set  $b'_i = \sum_{j=1}^s u_j b_i u_j^{-1}$  for  $i = 1, \dots, k$ . Then we have  $\sum_{i=1}^k f_i(x_1, \dots, x_m) b'_i = 0$  for any assignment of values in  $\Phi_2$  to  $x_1, \dots, x_m$ . But  $b'_1 = 1$  and, by Case 1, this is impossible.

*Case 3.* The minimum polynomial of each  $b_i$  ( $i = 1, \dots, k$ ) is  $\lambda^2 + 1$ : Without loss of generality, we may assume  $b_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . Since  $b_1$  is invertible, we have

$$\begin{aligned} & \sum_{i=2}^k f_i(x_1, \dots, x_m)(b_1 b_i b_1^{-1} - b_i) \\ &= b_1 \left( \sum_{i=1}^k f_i(b_1^{-1} x_1 b_1, \dots, b_1^{-1} x_m b_1) b_i \right) b_1^{-1} \\ & \quad - \sum_{i=1}^k f_i(x_1, \dots, x_m) b_i \\ &= 0. \end{aligned}$$

By the minimality of  $k$ ,  $b_1 b_i b_1^{-1} = b_i$ , that is,  $b_1 b_i = b_i b$ . Since  $b_i \in B(\lambda^2 + 1)$  and the only element in  $B(\lambda^2 + 1)$  which commutes with  $b_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  is  $b_1$  itself, we have  $b_1 = b_2 = \dots = b_k$ . Hence  $(\sum_{i=1}^k f_i(x_1, \dots, x_m)) b_1 = 0$  for any  $x_1, \dots, x_m \in \Phi$ . By (1) of this lemma,  $\sum_{i=1}^k f_i(x_1, \dots, x_m) = 0$  for any  $x_1, \dots, x_m \in \Phi_2$ . This contradicts with the  $\Phi$ -independence of  $f_1(x_1, \dots, x_m), \dots, f_k(x_1, \dots, x_m)$  modulo  $\mathcal{S}_2$  and completes our proof.

*Proof of Theorem.* Observe that (3) follows from (1) and (2). As (1) and (2) can be proved analogously, we give here only the proof of (1): If  $n \neq 2$  or the field  $\Phi$  contains more than two elements, then our theorem follows immediately from Lemma 2. If  $n = 2$  and  $\Phi$  contains only two elements 0 and 1, then, according to the hypothesis of our theorem,  $k$  must be one and the assertion of our theorem follows immediately from (1) of Lemma 3.

*Proof of Corollary.* If  $n \neq 2$  or  $\Phi$  contains more than two elements, then our corollary follows immediately from our theorem. If  $n = 2$  and  $\Phi$  contains only two elements 0 and 1, then our corollary follows immediately from (2) of Lemma 3.

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## REFERENCES

- [1] C.-L. Chuang, *On invariant additive subgroups*, Israel J. Math., **57** (1987), 116–128.
- [2] Amitai Regev, *A primeness property for central polynomials*, Pacific J. Math., **83** (1979), 269–271.

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