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**FOURIER COEFFICIENTS OF NONHOLOMORPHIC
MODULAR FORMS AND SUMS OF KLOOSTERMAN SUMS**

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This paper studies Fourier coefficients of non-holomorphic modular forms and sums of Kloosterman sums.

1. Introduction. Put $\Gamma = \text{PSL}(2, Z)$ and $H^+ = \{x + iy | y > 0\}$. Consider the Hilbert space $\mathcal{L}^2(H^+/\Gamma)$ of function $u(z)$ satisfying:

$$u(\gamma z) = u(z) \quad (\gamma \in \Gamma)$$

and

$$\langle u, u \rangle = \iint_{H^+/\Gamma} |u(z)|^2 \frac{dx dy}{y^2} < +\infty.$$

Consider the Laplacian Δ on $\mathcal{L}^2(H^+/\Gamma)$:

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

A function $u(z)$ in $\mathcal{L}^2(H^+/\Gamma)$ is called a cusp form if the constant term in the Fourier expansion of $u(z)$ vanishes. It is known that the Laplacian Δ has a complete discrete spectral decomposition on the subspace of cusp forms. The Maass wave forms $u_j(z)$ defined by

$$(1) \quad \Delta u_j(z) = \lambda_j u_j(z), \quad \langle u_j, u_j \rangle = 1,$$

where $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ are the discrete eigenvalues of Δ , constitute an orthonormal basis for the subspace of cusp forms. Note that $\lambda_1 > \frac{3}{2}\pi^2$. From (1) we have the Fourier expansion:

$$(2) \quad u_j(z) = \sqrt{y} \sum_{n \neq 0} \rho_j(n) K_{ik_j}(2\pi|n|) e(nx), \quad e(\theta) = e^{2\pi i \theta}$$

where $\lambda_j = \frac{1}{4} + k_j^2$ and $K_{ik_j}(\cdot)$ is the Whittaker function. We have

$$(3) \quad \#\{k_j | |k_j| \leq X\} = \frac{1}{12} X^2 + cX \log X + O(X)$$

where c is a constant; cf. Venkov [7].

An important problem in the theory of non-holomorphic modular form is to estimate the Fourier coefficients $\rho_j(n)$. The Ramanujan-Peterson conjecture states that for large $|n|$

$$\rho_j(n) \ll_{\varepsilon, j} |n|^\varepsilon \quad (\varepsilon > 0).$$

A method to study the Fourier coefficients $\rho_j(n)$ of $u_j(z)$ is the non-holomorphic Poincaré series introduced by Selberg [5]:

$$P_m(z, s) = \sum_{\gamma \in \Gamma/\Gamma_\infty} (\text{Im } \gamma z)^s e(m\gamma z) \quad (\text{Re } s > 1),$$

where $m \geq 1$ is an integer and Γ_∞ is the subgroup of translations. The Poincaré series belongs to $\mathcal{L}^2(H^+/\Gamma)$, and its inner product against a function $u(z) \in \mathcal{L}^2(H^+/\Gamma)$ gives the m th Fourier coefficient of $u(z)$. Selberg [5] obtained the meromorphic continuation of $P_m(z, s)$ to the entire complex s -plane. By considering the inner product of two Poincaré series, Kuznietsov [4] developed summation formulas connecting the Fourier coefficients $\rho_j(n)$ and the Kloosterman sum

$$S(m, n; c) = \sum_{\substack{d=1 \\ ad \equiv 1 \pmod{c}}}^c e\left(\frac{am + dn}{c}\right).$$

One of the summation formulas useful to us is equation (9) below. By using the summation formulas, Kuznietsov [4] proved that

$$(5) \quad \sum_{0 < k_j < X} \frac{|\rho_j(n)|^2}{ch\pi k_j} = \frac{1}{\pi^2} X^2 + O(X \log X + Xn^\varepsilon + n^{1/2+\varepsilon}),$$

and

$$(6) \quad \sum_{c < T} \frac{S(m, n; c)}{c} \ll_{m, n} T^{1/6} \log^{1/3} T.$$

The Weil estimate gives

$$|S(m, n; c)| \leq (m, n, c)^{1/2} d(c) c^{1/2},$$

which yields a trivial bound $O(T^{1/2+\varepsilon})$ for the sum in (6).

The Linnik-Selberg conjecture states that

$$(7) \quad \sum_{c \leq T} \frac{S(m, n; c)}{c} \ll_\varepsilon T^\varepsilon \quad (T > (m, n)^{1/2}, \varepsilon > 0).$$

To deal with the estimate of $\rho_j(n)$, Selberg [5] introduced the above conjecture.

Another method to study the sum of Kloosterman sum in (6) is by the Kloosterman zeta function introduced by Selberg [5]:

$$(8) \quad Z_{m,n}(s) = \sum_{c=1}^{\infty} \frac{S(m, n; c)}{c^{2s}} \quad \left(\operatorname{Re} s > \frac{3}{4} \right).$$

Selberg [5] obtained the meromorphic continuation of $Z_{m,n}(s)$ to the entire complex plane. A useful characterization of $Z_{m,n}(s)$ may be found in (7.26) of Kuznetsov [4].

Goldfeld and Sarnak [3] have given a very simple proof of the bound $O(T^{1/6+\varepsilon})$ for the sum in (6) by proving a good bound on $Z_{m,n}(s)$ in the critical strip.

Equation (5) means that on the average $|\rho_j(n)|^2/ch\pi k_j$ is bounded with respect to the indices k_j from 0 to X . In this paper, we will show the following:

THEOREM 1. *We have for $n^{1+\varepsilon} \ll t$ ($\varepsilon > 0$),*

$$\sum_{|k_j-t|<1} \frac{|\rho_j(n)|^2}{ch\pi k_j} \ll t \quad (t \rightarrow +\infty).$$

Theorem 1 means that on the average $|\rho_j(n)|^2/ch\pi k_j$ is bounded with respect to k_j in short interval.

With Theorem 1, we will show furthermore

THEOREM 2. *For any $f(t) \rightarrow +\infty$ and $f(t) = o(t)$ as $t \rightarrow +\infty$, and $n^{1+\varepsilon} \ll t$ ($\varepsilon > 0$), we have*

$$\sum_{|k_j-t|<f(t)} \frac{|\rho_j(n)|^2}{ch\pi k_j} \sim \frac{4}{\pi^2} t f(t) \quad (t \rightarrow +\infty)$$

and

THEOREM 3. *For $Y \geq 10$, we have*

$$\int_Y^{eY} \left(\sum_{c \leq x} \frac{S(m, n; c)}{d} \right)^2 \frac{dx}{x} \ll_{m,n} \log Y.$$

It may be interesting to note that we get as a by-product of the proof of Theorem 2 the following:

THEOREM 4. *For any $\sigma \in \mathbb{C}$, we have*

$$\int_{-\infty}^{\infty} \Gamma\left(\sigma - \frac{1}{2} - ir\right) \Gamma\left(\sigma - \frac{1}{2} + ir\right) dr = \pi 2^{2-2\sigma} \Gamma(2\sigma - 1).$$

Theorem 4 would follow immediately from the proof of Theorem 2. In view of (3), it may be interesting to compare Kuznietsov’s estimate (5) with Theorems 1 and 2. Theorem 3 means that the sum in (6) is “very small” for almost all x and for most of the time better than the Linnik-Selberg conjecture. More precisely, for $Y \geq 10$ and $f(x) \nearrow \infty$, let $M_Y \subset [Y, eY]$ such that

$$\left| \sum_{c < x} \frac{S(m, n; c)}{c} \right| \geq f(x) \log^{1/2} x \quad (x \in M_Y).$$

Then Theorem 3 shows that the Lebesgue measure of M_Y is $O(f(Y)^{-2}Y)$.

By putting $\sigma = \frac{3}{4} + 1/\log n$ in Lemma 1, Theorem 1 follows immediately. We prove Theorem 3 by establishing Lemma 2, which is analogous to the explicit formula in the theory of prime number, and by using Gallagher’s mean-value inequality for exponential sum which is Lemma 3. The method imitates an idea of Gallagher [2].

2. Lemmas. The proof of Lemma 1 is based on the following equation (9) which follows by putting $s_1 = \sigma + it$ and $s_2 = \sigma - it$ in the lemmas in §4.1 and §4.4 of Kuznietsov [4].

PROPOSITION. *For $s = \sigma + it$, $\frac{3}{4} < \sigma < \frac{5}{4}$, and any integer $n \geq 1$, we have*

$$\begin{aligned} (9) \quad & \pi \left\{ \sum_{j=1}^{\infty} |\rho_j(n)|^2 \Lambda(s; k_j) \right. \\ & \left. + \frac{1}{\pi} \int_{-\infty}^{\infty} |\sigma_{2ir}(n)|^2 \Lambda(s; r) \frac{ch\pi r}{|\zeta(1 + 2ir)|^2} dr \right\} \\ & = \Gamma(2\sigma - 1) + (4\pi n)^{2\sigma-1} \\ & \quad \times \left\{ \frac{2^{3-2\sigma}}{ish2\pi t} \sum_{c=1}^{\infty} \frac{S(n, n; c)}{c^{2\sigma}} \Phi\left(s, \frac{4\pi n}{c}\right) \right\}, \end{aligned}$$

where

$$\Lambda(s; r) = \frac{|\Gamma(s - \frac{1}{2} + ir)\Gamma(s - \frac{1}{2} - ir)|^2}{|\Gamma(s)|^2}, \quad \sigma_{2ir}(n) = \sum_{d|n} d^{2ir}$$

and for $x > 0$

$$(10) \quad \Phi(s, x) = -\pi \int_1^\infty \left(u - \frac{1}{u}\right)^{2\sigma-2} \{(\sin \pi s)J_{2it}(xu) + (\sin \pi \bar{s})J_{-2it}(xu)\} \frac{du}{u},$$

and $J_{2it}(u)$ is the Bessel function.

We need the following estimate for the Bessel function:

$$(11) \quad J_{it}(u) \ll e^{\pi t/2}(t^2 + u^2)^{-1/4} \quad (t \in \mathbb{R})$$

uniformly in $u > 0$ for $|t| \rightarrow +\infty$.

LEMMA 1. We have for $\frac{3}{4} < \sigma < \frac{5}{6}$

$$(12) \quad \sum_{|t-k_j|<1} \frac{|\rho_j(n)|^2}{ch\pi k_j} \ll t + \sqrt{t}n^{2\sigma-1} \left(\sigma - \frac{3}{4}\right)^{-2} \quad (t \rightarrow +\infty).$$

Proof. We take $\frac{3}{4} < \sigma < \frac{5}{6}$ in the Proposition. With the bound in (11), we see from (10) that

$$(13) \quad \begin{aligned} \Phi(s, x) &\ll e^{2\pi t} \int_1^\infty \left(u - \frac{1}{u}\right)^{2\sigma-2} (t^2 + x^2u^2)^{-1/4} \frac{du}{u} \\ &\qquad\qquad\qquad \left(x = \frac{4\pi n}{c}\right) \\ &\ll t^{-1/2} e^{2\pi t} \int_1^\infty \left(u - \frac{1}{u}\right)^{2\sigma-2} \left(1 + \left(\frac{x}{t}\right)^2 u^2\right)^{-1/4} \frac{du}{u} \\ &\ll t^{-1/2} e^{2\pi t}, \quad \text{since } \frac{3}{4} < \sigma < \frac{5}{6}. \end{aligned}$$

On considering Weil's bound for $S(m, n; c)$ and (13), the second term on the right-hand side of (9) is then

$$(14) \quad \ll t^{-1/2} n^{2\sigma-1} \left(\sigma - \frac{3}{4}\right)^{-2}.$$

On the other hand, the integral in (9) is non-negative, and the series in (9) is

$$\begin{aligned} &\geq \sum_{|k_j-t|<1} |\rho_j(n)|^2 \frac{|\Gamma(s - \frac{1}{2} + ik_j)\Gamma(s - \frac{1}{2} - ik_j)|^2}{|\Gamma(s)|^2} \\ &\gg \frac{1}{t} \sum_{|k_j-t|<1} \frac{|\rho_j(n)|^2}{ch\pi k_j}, \end{aligned}$$

since $\Gamma(s) = \sqrt{2\pi}e^{-(\pi/2)|t|}t^{\sigma-1/2}(1 + O(|t|^{-1}))$, and $|\Gamma(s - \frac{1}{2} - ik_j)| \gg 1$ for $|t - k_j| < 1$.

This proves Lemma 1.

LEMMA 2. We have for $T < \frac{1}{2}x$

$$\sum_{c \leq x} \frac{S(m, n; c)}{c} = \sum_{|k_j| < T} \frac{\rho_j(n)\overline{\rho_j(m)}}{ch\pi k_j} \frac{\Gamma(2ik_j)}{2ik_j} x^{2ik_j} + O\left(\frac{x^{1/2} \log^2 x}{T}\right),$$

the implicit constant here depends on m, n .

Before proceeding with the proof of Lemma 2, we need several analytic properties of $Z_{m,n}(s)$. On the half plane $\text{Re } s > 0$, the poles of $Z_{m,n}(s)$ are located at $s = \frac{1}{2} + ik_j$, and as $t \rightarrow \infty$

$$(15) \quad Z_{m,n}(s) \ll_{m,n} \frac{|s|^{1/2}}{|\sigma - \frac{1}{2}|} \quad (s = \sigma + it, \sigma \neq \frac{1}{2}).$$

Estimate (15) is obvious by using the result and the same method as in the proof of Theorem 1 of Goldfeld and Sarnak [3]. On the other hand, by the Lemma of §7.3 of Kuznietsov [4], we have the representation for $Z_{m,n}(s)$ ($s \in \mathbb{C}$):

$$\begin{aligned} (16) \quad &(2\pi\sqrt{mn})^{2s-1} Z_{m,n}(s) \\ &= \sum_{j=1}^{\infty} \frac{\rho_j(n)\overline{\rho_j(m)}}{ch\pi k_j} h(k_j, s) - \frac{\delta_{m,n}}{2\pi} \frac{\Gamma(s)}{\Gamma(1-s)} \\ &\quad + \sum_{l=0}^{\infty} p_{m,n}(l) \frac{\Gamma(s+l)}{\Gamma(2-s+l)} + L_{m,n}(s) \end{aligned}$$

where $L_{m,n}(s)$ denotes the analytic continuation of the function which is defined in the half plane $\text{Re } s > \frac{1}{2}$ by the integral

$$L_{m,n}(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{n}{m}\right)^{it} \sigma_{2ir}(n)\sigma_{-2ir}(m) \frac{h(r,s)}{\zeta(1+2ir)\zeta(1-2ir)} dr,$$

and

$$h(r,s) = \frac{1}{2} \sin(\pi s) \Gamma\left(s - \frac{1}{2} + ir\right) \Gamma\left(s - \frac{1}{2} - ir\right),$$

and

$$p_{m,n}(l) = (2l+1) \sum_{c=1}^{\infty} \frac{S(m,n;c)}{c} J_{2l+1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

By (16), we see that

$$(17) \quad \text{Re } s = \frac{1}{2} + ik_j \quad Z_{m,n}(s) = \frac{(2\pi\sqrt{mn})^{-2ik_j}}{2} \Gamma(2ik_j) \rho_j(n) \overline{\rho_j(m)}.$$

Consider $s = \sigma + it$ with

$$(18) \quad \left| \sigma - \frac{1}{2} \right| \leq \frac{\delta}{\log(|t| + 2)}$$

for a small $\delta > 0$. Deforming suitably the integral path in the integral of $L_{m,n}(s)$, we have for s satisfying (18)

$$(19) \quad L_{m,n}(s) = O(\log^2 |t|)$$

since $\zeta(x + iy) \neq 0$ and $\zeta(x + iy) \ll \log(|y| + 2)$ in the region $x > 1 - (c/\log(|y| + 2))$ ($c > 0$).

Also for s satisfying (18), we have

$$(20) \quad \frac{\Gamma(s)}{\Gamma(1-s)} \ll 1$$

and

$$(21) \quad \sum_{l=0}^{\infty} p_{m,n}(l) \frac{\Gamma(s+l)}{\Gamma(2-s+l)} \ll mn.$$

By using the estimate on Bessel function

$$|J_k(y)| \leq \min\left(1, \frac{(y/2)^k}{(k-1)!}\right),$$

we prove (21) as follows: note first that $(2l+1) \Gamma(s+l)/\Gamma(2-s+l) \ll 1$. Thus

$$\begin{aligned}
 & \sum_{l=0}^{\infty} p_{m,n}(l) \frac{\Gamma(s+l)}{\Gamma(2-s+l)} \\
 & \ll \sum_{l=0}^{\infty} \sum_{1 \leq c \leq 20\pi\sqrt{mn}} \frac{|S(m,n;c)|}{c} \left| J_{2l+1} \left(\frac{4\pi\sqrt{mn}}{c} \right) \right| \\
 & \quad + \sum_{l=0}^{\infty} \sum_{c > 20\pi\sqrt{mn}} \frac{|S(m,n;c)|}{c} \left| J_{2l+1} \left(\frac{4\pi\sqrt{mn}}{c} \right) \right| \\
 & \ll \sum_{0 \leq l \leq 20\pi\sqrt{mn}} \sum_{1 \leq c \leq 20\pi\sqrt{mn}} \frac{|S(m,n;c)|}{c} \left| J_{2l+1} \left(\frac{4\pi\sqrt{mn}}{c} \right) \right| \\
 & \quad + \sum_{1 \leq c \leq 20\pi\sqrt{mn}} \sum_{l > 20\pi\sqrt{mn}} \frac{|S(m,n;c)|}{c} \left| J_{2l+1} \left(\frac{4\pi\sqrt{mn}}{c} \right) \right| \\
 & \quad + \sum_{l > 20\pi\sqrt{mn}} \frac{|S(m,n;c)|}{c} \sum_{l=0}^{\infty} \left| J_{2l+1} \left(\frac{4\pi\sqrt{mn}}{c} \right) \right| \\
 & \ll mn + \sum_{1 \leq c \leq 20\pi\sqrt{mn}} \frac{|S(m,n;c)|}{c} \sum_{l > 20\pi\sqrt{mn}} \left(\frac{2\pi\sqrt{mn}}{c} \right)^{2l+1} \frac{1}{(2l)!} \\
 & \quad + \sum_{c > 20\pi\sqrt{mn}} \frac{|S(m,n;c)|}{c} \sum_{l=0}^{\infty} \left(\frac{2\pi\sqrt{mn}}{c} \right)^{2l+1} \frac{1}{(2l)!} \\
 & \ll mn + \sum_{l \leq c \leq 20\pi\sqrt{mn}} \frac{|S(m,n;c)|}{c} \\
 & \quad + \sum_{c > 20\pi\sqrt{mn}} \frac{|S(m,n;c)|}{c} \times \frac{2\pi\sqrt{mn}}{c} \\
 & \ll mn.
 \end{aligned}$$

This proves (21). Estimate (21) is obviously not the best, but we are satisfied with this presently.

Also by using Theorem 1 and (5), we have that

$$(22) \quad \sum_{|k_j - t| > 1} \frac{\rho_j(n) \overline{\rho_j(m)}}{c h \pi k_j} h(k_j, x) = O(|t|)$$

for s satisfying (18) and $\max\{m^{1+\varepsilon}, n^{1+\varepsilon}\} \ll |t|$. Thus by (19), (20),

(21) and (22), equation (16) becomes

$$(23) \quad Z_{m,n}(s) = (2\pi\sqrt{mn})^{1-2s} \sum_{|k_j-t|<1} \frac{\rho_j(n)\overline{\rho_j(m)}}{ch\pi k_j} h(k_j, s) + O(|t|)$$

for s satisfying (18) and $mn \ll |t|$ and $\max\{m^{1+\varepsilon}, n^{1+\varepsilon}\} \ll |t|$.

We are now in a position to prove Lemma 2.

Proof of Lemma 2. Choose $0 < \varepsilon \leq \delta / \log(|t| + 2)$ for small $\delta > 0$. By (15) and the Lindelöf-Phragmen principle it follows that

$$(24) \quad |Z_{m,n}(s)| \ll_{m,n} \frac{|t|^{3/2-2\sigma+2\varepsilon}}{\varepsilon^2}$$

for $\frac{1}{2} + \varepsilon \leq \sigma \leq \frac{3}{4} + \varepsilon$, since $Z_{m,n}(\frac{3}{4} + \varepsilon) \ll_{m,n} \varepsilon^{-2}$ by (8); and obviously

$$(25) \quad |Z_{m,n}(s)| \ll_{m,n} \frac{|t|^{1/2}}{\frac{1}{2} - \sigma}$$

for $\frac{1}{10} \leq \sigma \leq \frac{1}{2} - \varepsilon$.

Consider the integral

$$(26) \quad I(T) = \frac{1}{2\pi i} \int_{\eta-iT}^{\eta+iT} Z_{m,n}(s) \frac{x^{2s-1}}{2s-1} ds \quad (\eta = \frac{3}{4} + \varepsilon)$$

with $T > 0$ not an ordinate of a pole of $Z_{m,n}(s)$. Now by Lemma 3.12 of Titchmarsh [6], we get

$$(27) \quad \sum_{c \leq x} \frac{S(m, n; c)}{c} = \frac{1}{2\pi i} \int_{\eta-iT}^{\eta+iT} Z_{m,n}(s) \frac{x^{2s-1}}{2s-1} ds + O_{m,n} \left(\frac{x^\eta}{T\varepsilon^2} \right).$$

Computations of residues yield

$$(28) \quad I(T) = \sum_{|k_j|<T} \xi_j \frac{x^{2ik_j}}{2ik_j} + \frac{1}{2\pi i} \int_{1/10-iT}^{1/10+iT} Z_{m,n}(s) \frac{x^{2s-1}}{2s-1} ds + \frac{1}{2\pi i} \int_{1/10\pm iT}^{\eta\pm iT} Z_{m,n} \frac{x^{2s-1}}{2s-1} ds$$

where ξ_j is the residue of $Z_{m,n}(s)$ at $s = \frac{1}{2} + ik_j$. Using (17), we

see that

$$(29) \quad \xi_j \ll \frac{|\rho_j(n)\rho_j(m)|}{ch\pi k_j} |k_j|^{-1/2}.$$

Now we estimate the integrals in (28). By (25), we have first

$$(30) \quad \int_{1/10-iT}^{1/10+iT} Z_{m,n}(s) \frac{x^{2s-1}}{2s-1} ds \ll_{m,n} x^{-4/5} T^{1/2}$$

and

$$(31) \quad \int_{1/10\pm iT}^{1/2-\varepsilon\pm iT} Z_{m,n}(s) \frac{x^{2s-1}}{2s-1} ds \ll_{m,n} \frac{x^{-2\varepsilon} T^{-1/2}}{\varepsilon}.$$

By (24), we have

$$(32) \quad \int_{1/2+\varepsilon\pm iT}^{\eta\pm iT} Z_{m,n}(s) \frac{x^{2s-1}}{2s-1} ds \ll_{m,n} \frac{x^{1/2+\varepsilon}}{\varepsilon^2 |t|} \frac{1}{|\log \frac{T}{x}|}$$

for $|\log \frac{T}{x}| \gg 1$.

Finally, by (23) we have

$$(33) \quad \int_{1/2-\varepsilon\pm iT}^{1/2+\varepsilon\pm iT} Z_{m,n}(s) \frac{x^{2s-1}}{2s-1} ds \\ = \sum_{|k_j \mp T| < 1} \frac{\rho_j(n)\overline{\rho_j(m)}}{ch\pi k_j} \\ \cdot \int_{1/2-\varepsilon\pm iT}^{1/2+\varepsilon\pm iT} (2\pi\sqrt{mn})^{1-2s} h(k_j, s) \frac{x^{2s-1}}{2s-1} ds + O_{m,n}(\varepsilon).$$

Noting that $|\Gamma(s)| \gg |s|^{-1}$ for $\varepsilon \ll |s| \ll 1$ and by suitably deforming the integral path on the right-hand side of (33) to an upper or lower semi-circle according as $\frac{1}{2} + ik_j$ stays below or above the integral path, we get

$$\int_{1/2-\varepsilon\pm iT}^{1/2+\varepsilon\pm iT} (2\pi\sqrt{mn})^{1-2s} h(j_k, s) \frac{x^{2s-1}}{2s-1} ds \ll_{m,n} T^{-3/2}$$

since $|k_j \mp T| < 1$, so the right-hand side of (33) is

$$(34) \quad \ll_{m,n} \sum_{|k_j \mp T| < 1} \frac{|\rho_j(n)\rho_j(m)|}{ch\pi k_j} T^{-3/2} + \varepsilon \\ \ll_{m,n} T^{-1/2} + \varepsilon,$$

by Theorem 1.

Putting (30), (31), (32), and (34) together, equation (28) becomes, for $\log|\frac{T}{x}| \gg 1$ and $\varepsilon = \delta \log^{-1} T$,

$$(35) \quad I(T) = \sum_{|k_j| < T} \xi_j \frac{x^{2ik_j}}{2ik_j} + O_{m,n} \left(\frac{x^{1/2} \log^2 xT}{T} \right)$$

for $T \leq \frac{1}{2}x$, which combined with (27) yield

$$(36) \quad \sum_{c \leq x} \frac{S(m, n; c)}{c} = \sum_{|k_j| < T} \xi_j \frac{x^{2ik_j}}{2ik_j} + O_{m,n} \left(\frac{x^{1/2} \log^2 xT}{T} \right).$$

This completes the proof of Lemma 2.

By putting $T = x^{1/3} \log^{4/3} x$ in (36), we get $O(x^{1/6} \log^{2/3} x)$ on the right-hand side of (36) which is slightly inferior to Kuznietsov’s bound (6).

LEMMA 3. *Let $A(u) = \sum_v c(v)e^{ivu}$ be an absolutely convergent series with complex coefficients $c(v)$ and real indices v . Then for $T > 0$*

$$\int_{-T}^T \left| \sum_v c(v)e^{ivu} \right|^2 du \ll \int_{-\infty}^{\infty} \left| T \sum_{t < v < t+T^{-1}} c(v) \right|^2 dt.$$

Proof. This is Lemma 1 of Gallagher [1].

3. Proofs of Theorems. We prove first Theorem 2. Take $\sigma = \frac{3}{4} + 1/\log n$. Then the Proposition of §2 gives

$$(37) \quad \sum_{|k_j - t| < 1} \frac{|\rho_j(n)|^2}{ch\pi k_j} \ll t \quad (n^{1+\varepsilon} \ll t),$$

which is the assertion of Theorem 1.

In view of $|\zeta(1 + ir)|^{-1} \ll \log|r|$ ($|r| \rightarrow +\infty$), a rough estimate gives

$$(38) \quad \int_{-\infty}^{\infty} |\sigma_{2ir}(n)|^2 \Lambda(s; r) \frac{ch\pi r}{|\zeta(1 + 2ir)|^2} dr \ll t^{-1} \log t d^2(n).$$

We have, for $k_j \geq t + \sqrt{f(t)}$,

$$(39) \quad \Lambda(s, k_j) = \frac{|\Gamma(s - 1/2 + ik_j)\Gamma(s - 1/2 - ik_j)|^2}{|\Gamma(s)|^2} \ll e^{\pi t} t^{1-2\sigma} e^{-2\pi k_j} |t + k_j|^{2\sigma-2} |k_j - t|^{2\sigma-2},$$

and for $k_j \leq t - \sqrt{f(t)}$

$$(40) \quad \Lambda(s; k_j) \ll e^{-\pi t} t^{1-2\sigma} |t + k_j|^{2\sigma-2} |t - k_j|^{2\sigma-2}.$$

On considering (37) and (5), inequalities (39) and (40) give rise to

$$(41) \quad \sum_{|t-k_j| \geq \sqrt{f(t)}} |\rho_j(n)|^2 \Lambda(s; k_j) = o(1) \quad (t \rightarrow +\infty).$$

Now (14) together with (38) and (41) yield, by virtue of (9),

$$(42) \quad \sum_{|k_j-t| < \sqrt{f(t)}} |\rho_j(n)|^2 \Lambda(s; k_j) = \frac{1}{\pi} \Gamma(2\sigma - 1) + o(1),$$

since $n^{1+\varepsilon} \ll t$. And also for $|k_j - t| < \sqrt{f(t)}$

$$\Lambda(s; k_j) = 2^{2\sigma-2} t^{-1} e^{-\pi k_j} \left| \Gamma \left(s - \frac{1}{2} - ik_j \right) \right|^2 (1 + o(1)).$$

Substituting this into (42), we obtain

$$(43) \quad \sum_{|k_j-t| < \sqrt{f(t)}} |\rho_j(n)|^2 e^{-\pi k_j} \left| \Gamma \left(s - \frac{1}{2} - ik_j \right) \right|^2 = \frac{2^{2-2\sigma}}{\pi} \Gamma(2\sigma - 1) t + o(t),$$

since $\sqrt{f(t)} = o(t)$.

Taking integrals on both sides of (43) yields

$$(44) \quad \int_{t-f(t)}^{t+f(t)} \sum_{|k_j-r| < \sqrt{f(r)}} |\rho_j(n)|^2 e^{-\pi k_j} \left| \Gamma \left(\sigma - \frac{1}{2} + i(r - k_j) \right) \right|^2 dr = \frac{2^{3-2\sigma}}{\pi} \Gamma(2\sigma - 1) t f(t) + o(t f(t)).$$

Interchanging the order of summation and integral in (44), the left-hand side of (44) becomes

$$(45) \quad \sum_{|k_j-t| < f(t)} |\rho_j(n)|^2 e^{-\pi k_j} \int_{k_j-\sqrt{f(t)}}^{k_j+\sqrt{f(t)}} \left| \Gamma \left(\sigma - \frac{1}{2} + i(r - k_j) \right) \right|^2 dr + o(t f(t)),$$

by using (37). Note further that

$$\begin{aligned} \int_{k_j - \sqrt{f(t)}}^{k_j + \sqrt{f(t)}} \left| \Gamma \left(\sigma - \frac{1}{2} + i(r - k_j) \right) \right|^2 dr \\ = \int_{-\infty}^{\infty} \left| \Gamma \left(\sigma - \frac{1}{2} + ir \right) \right|^2 dr + O(e^{-\pi\sqrt{f(t)}}). \end{aligned}$$

From this and (44) and (45) it follows that

$$\begin{aligned} (46) \quad \sum_{|k_j - t| < f(t)} |\rho_j(n)|^2 e^{-\pi k_j} \\ \sim \frac{2^{3-2\sigma}}{\pi} \left(\int_{-\infty}^{\infty} \left| \Gamma \left(\sigma - \frac{1}{2} + ir \right) \right|^2 dr \right)^{-1} \Gamma(2\sigma - 1) t f(t) \end{aligned}$$

for $\sigma = \frac{3}{4} + 1/\log n$ and $n^{1+\varepsilon} \ll t$.

Now if we fix n , then we see from the proof that (46) holds good uniformly for σ in an interval $I \subset (\frac{3}{4}, \infty)$. By analytic continuation, there is a constant ξ for which

$$(47) \quad \xi \int_{-\infty}^{\infty} \Gamma \left(\sigma - \frac{1}{2} + ir \right) \Gamma \left(\sigma - \frac{1}{2} - ir \right) dr = 2^{2-2\sigma} \Gamma(2\sigma - 1) \quad (\sigma \in \mathbb{C}).$$

Indeed $\xi = \frac{1}{\pi}$, since

$$\int_{-\infty}^{\infty} \left| \Gamma \left(\frac{1}{2} + ir \right) \right|^2 dr = \int_{-\infty}^{\infty} \frac{\pi}{ch\pi r} dr = \pi.$$

This completes the proof of Theorem 1, and equation (47) gives the proof of Theorem 4.

Finally we prove Theorem 3. For $Y \geq 10$, $Y \leq x \leq eY$, and $Y^{2/3} \leq T \leq \frac{1}{2}Y$, Lemma 2 gives

$$(48) \quad \sum_{c \leq x} \frac{S(m, n; c)}{c} = \sum_{|k_j| < T} \xi_j \frac{x^{2ik_j}}{2ik_j} + o(1).$$

On applying Lemma 3 to (48), we get

$$\begin{aligned}
 & \int_Y^{eY} \left(\sum_{c \leq x} \frac{S(m, n; c)}{c} \right)^2 \frac{dx}{x} \\
 & \ll \int_Y^{eY} \left| \sum_{|k_j| < T} \frac{\xi_j}{2k_j} x^{2ik_j} \right|^2 \frac{dx}{x} + o(1) \\
 & = \int_{\log Y}^{1+\log Y} \left| \sum_{|k_j| < T} \frac{\xi_j}{2k_j} e^{2ik_j u} \right|^2 du + o(1) \\
 & \ll \int_{-T-1}^{T+1} \left| \sum_{t < k_j < t+1} \frac{\xi_j}{2k_j} \right|^2 dt + o(1) \\
 & \ll \int_1^{T+1} \left(\sum_{|k_j - t| < 1} \frac{|\rho_j(n)\rho_j(m)|}{ch\pi k_j} k_j^{-3/2} \right)^2 dt + o(1), \quad \text{by (29),} \\
 & \ll_{m,n} \int_1^{T+1} t^{-1} dt + o(1) \quad \text{by Theorem 1,} \\
 & \ll_{m,n} \log T \\
 & \ll_{m,n} \log Y.
 \end{aligned}$$

This completes the proof of Theorem 3.

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