FOURIER COEFFICIENTS OF NONHOLOMORPHIC MODULAR FORMS AND SUMS OF KLOOSTERMAN SUMS

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This paper studies Fourier coefficients of non-holomorphic modular forms and sums of Kloosterman sums.

1. Introduction. Let \( \Gamma = \text{PSL}(2, \mathbb{Z}) \) and \( H^+ = \{ x + iy | y > 0 \} \). Consider the Hilbert space \( \mathcal{L}^2(H^+/\Gamma) \) of function \( u(z) \) satisfying:
\[
u(\gamma z) = u(z) \quad (\gamma \in \Gamma)
\]
and
\[
\langle u, u \rangle = \iint_{H^+ / \Gamma} |u(z)|^2 \frac{dx \, dy}{y^2} < +\infty.
\]
Consider the Laplacian \( \Delta \) on \( \mathcal{L}^2(H^+/\Gamma) \):
\[
\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).
\]
A function \( u(z) \) in \( \mathcal{L}^2(H^+/\Gamma) \) is called a cusp form if the constant term in the Fourier expansion of \( u(z) \) vanishes. It is known that the Laplacian \( \Delta \) has a complete discrete spectral decomposition on the subspace of cusp forms. The Maass wave forms \( u_j(z) \) defined by
\[
(1) \quad \Delta u_j(z) = \lambda_j u_j(z), \quad \langle u_j, u_j \rangle = 1,
\]
where \( \lambda_1 \leq \lambda_1 \leq \lambda_3 \leq \cdots \) are the discrete eigenvalues of \( \Delta \), constitute an orthonormal basis for the subspace of cusp forms. Note that \( \lambda_1 > \frac{3}{2}\pi^2 \). From (1) we have the Fourier expansion:
\[
(2) \quad u_j(z) = \sqrt{y} \sum_{n \neq 0} \rho_j(n) K_{ik_j}(2\pi |n| e(nx), \quad e(\theta) = e^{2\pi i \theta}
\]
where \( \lambda_j = \frac{1}{4} + k_j^2 \) and \( K_{ik_j}(\cdot) \) is the Whittaker function. We have
\[
(3) \quad \# \{ k_j | |k_j| \leq X \} = \frac{1}{12} X^2 + cX \log X + O(X)
\]
where \( c \) is a constant; cf. Venkov [7].
An important problem in the theory of non-holomorphic modular form is to estimate the Fourier coefficients \( \rho_j(n) \). The Ramanujan-Peterson conjecture states that for large \( |n| \)

\[
\rho_j(n) \ll |n|^{\varepsilon} \quad (\varepsilon > 0).
\]

A method to study the Fourier coefficients \( \rho_j(n) \) of \( u_j(z) \) is the non-holomorphic Poincaré series introduced by Selberg [5]:

\[
P_m(z, s) = \sum_{\gamma \in \Gamma/\Gamma_\infty} (\text{Im} y z)^s e(m y z) \quad (\text{Re} s > 1),
\]

where \( m \geq 1 \) is an integer and \( \Gamma_\infty \) is the subgroup of translations. The Poincaré series belongs to \( \mathcal{L}^2(H^+ / \Gamma) \), and its inner product against a function \( u(z) \in \mathcal{L}^2(H^+ / \Gamma) \) gives the \( m \)th Fourier coefficient of \( u(z) \). Selberg [5] obtained the meromorphic continuation of \( P_m(z, s) \) to the entire complex \( s \)-plane. By considering the inner product of two Poincaré series, Kuznetsov [4] developed summation formulas connecting the Fourier coefficients \( \rho_j(n) \) and the Kloosterman sum

\[
S(m, n; c) = \sum_{\substack{d \equiv 1 \pmod{c} \\ ad \equiv 1 \pmod{c}}} e \left( \frac{am + dn}{c} \right).
\]

One of the summation formulas useful to us is equation (9) below. By using the summation formulas, Kuznetsov [4] proved that

\[
(5) \quad \sum_{0 \leq k_j < X} \frac{|\rho_j(n)|^2}{ch \pi k_j} = \frac{1}{\pi^2} X^2 + O(X \log X + X n^{\varepsilon} + n^{1/2+\varepsilon}),
\]

and

\[
(6) \quad \sum_{c < T} \frac{S(m, n; c)}{c} \ll T^{1/6} \log^{1/3} T.
\]

The Weil estimate gives

\[
|S(m, n; c)| \leq (m, n, c)^{1/2}d(c)c^{1/2},
\]

which yields a trivial bound \( O(T^{1/2+\varepsilon}) \) for the sum in (6).

The Linnik-Selberg conjecture states that

\[
(7) \quad \sum_{c \leq T} \frac{S(m, n; c)}{c} \ll T^{\varepsilon} \quad (T > (m, n)^{1/2}, \varepsilon > 0).
\]
To deal with the estimate of $\rho_j(n)$, Selberg [5] introduced the above conjecture.

Another method to study the sum of Kloosterman sum in (6) is by the Kloosterman zeta function introduced by Selberg [5]:

$$Z_{m,n}(s) = \sum_{c=1}^{\infty} \frac{S(m, n; c)}{c^{2s}} \quad \left(\text{Re} \ s > \frac{3}{4}\right).$$

Selberg [5] obtained the meromorphic continuation of $Z_{m,n}(s)$ to the entire complex plane. A useful characterization of $Z_{m,n}(s)$ may be found in (7.26) of Kuznietsov [4].

Goldfeld and Sarnak [3] have given a very simple proof of the bound $O(T^{1/6+\varepsilon})$ for the sum in (6) by proving a good bound on $Z_{m,n}(s)$ in the critical strip.

Equation (5) means that on the average $|\rho_j(n)|^2/ch\pi k_j$ is bounded with respect to the indices $k_j$ from 0 to $X$. In this paper, we will show the following:

**Theorem 1.** We have for $n^{1+\varepsilon} \ll t$ ($\varepsilon > 0$),

$$\sum_{|k_j-t|<1} \frac{|\rho_j(n)|^2}{ch\pi k_j} \ll t \quad (t \to +\infty).$$

Theorem 1 means that on the average $|\rho_j(n)|^2/ch\pi k_j$ is bounded with respect to $k_j$ in short interval.

With Theorem 1, we will show furthermore

**Theorem 2.** For any $f(t) \to +\infty$ and $f(t) = o(t)$ as $t \to +\infty$, and $n^{1+\varepsilon} \ll t$ ($\varepsilon > 0$), we have

$$\sum_{|k_j-t|<f(t)} \frac{|\rho_j(n)|^2}{ch\pi k_j} \sim \frac{4}{\pi^2} tf(t) \quad (t \to +\infty)$$

and

**Theorem 3.** For $Y \geq 10$, we have

$$\int_{Y}^{eY} \left( \sum_{c \leq x} \frac{S(m, n; c)}{d} \right)^2 \frac{dx}{x} \ll \log Y.$$
It may be interesting to note that we get as a by-product of the proof of Theorem 2 the following:

**Theorem 4.** For any $\sigma \in \mathbb{C}$, we have

$$
\int_{-\infty}^{\infty} \Gamma\left(\sigma - \frac{1}{2} - ir\right) \Gamma\left(\sigma - \frac{1}{2} + ir\right) \, dr = \pi 2^{2-2\sigma} \Gamma(2\sigma - 1).
$$

Theorem 4 would follow immediately from the proof of Theorem 2. In view of (3), it may be interesting to compare Kuznietsov's estimate (5) with Theorems 1 and 2. Theorem 3 means that the sum in (6) is "very small" for almost all $x$ and for most of the time better than the Linnik-Selberg conjecture. More precisely, for $Y \geq 10$ and $f(x) \not\rightarrow \infty$, let $M_Y \subset [Y, eY]$ such that

$$
\left| \sum_{c < x} \frac{S(m, n; c)}{c} \right| \geq f(x) \log^{1/2} x \quad (x \in M_Y).
$$

Then Theorem 3 shows that the Lebesgue measure of $M_Y$ is $O(f(Y)^{-2}Y)$.

By putting $\sigma = \frac{3}{4} + 1/\log n$ in Lemma 1, Theorem 1 follows immediately. We prove Theorem 3 by establishing Lemma 2, which is analogous to the explicit formula in the theory of prime number, and by using Gallagher's mean-value inequality for exponential sum which is Lemma 3. The method imitates an idea of Gallagher [2].

**2. Lemmas.** The proof of Lemma 1 is based on the following equation (9) which follows by putting $s_1 = \sigma + it$ and $s_2 = \sigma - it$ in the lemmas in §4.1 and §4.4 of Kuznietsov [4].

**Proposition.** For $s = \sigma + it$, $\frac{3}{4} < \sigma < \frac{5}{4}$, and any integer $n \geq 1$, we have

$$
\pi \left\{ \sum_{j=1}^{\infty} |\rho_j(n)|^2 \Lambda(s; k_j)
+ \frac{1}{\pi} \int_{-\infty}^{\infty} |\sigma_{2ir}(n)|^2 \Lambda(s; r) \frac{\chi_{\pi r}}{|\zeta(1 + 2ir)|^2} \, dr \right\}
= \Gamma(2\sigma - 1) + (4\pi n)^{2\sigma - 1}
\times \left\{ \frac{2^{3-2\sigma}}{ish2\pi t} \sum_{c=1}^{\infty} \frac{S(n, n; c)}{c^{2\sigma}} \Phi \left( s, \frac{4\pi n}{c} \right) \right\}.
$$
where
\[ \Lambda(s ; r) = \frac{|\Gamma(s - \frac{1}{2} + ir)\Gamma(s - \frac{1}{2} - ir)|^2}{|\Gamma(s)|^2}, \quad \sigma_{2ir}(n) = \sum_{d|n} d^{2ir} \]
and for \( x > 0 \)
\[ \Phi(s, x) = -\pi \int_{1}^{\infty} \left( u - \frac{1}{u} \right)^{2\sigma - 2} \{ (\sin \pi s) J_{2it}(xu) + (\sin \pi \bar{s}) J_{-2it}(xu) \} \frac{du}{u}, \]
and \( J_{2it}(u) \) is the Bessel function.

We need the following estimate for the Bessel function:
\[ J_{it}(u) \ll e^{\pi t/2} (t^2 + u^2)^{-1/4} \quad (t \in \mathbb{R}) \]
uniformly in \( u > 0 \) for \( |t| \to +\infty \).

**Lemma 1.** We have for \( \frac{3}{4} < \sigma < \frac{5}{6} \)
\[ \sum_{|t-k_j|<1} \frac{|\rho_j(n)|^2}{c \pi k_j} \ll t + \sqrt{n} 2^{\sigma - 1} \left( \sigma - \frac{3}{4} \right)^{-2} \quad (t \to +\infty). \]

**Proof.** We take \( \frac{3}{4} < \sigma < \frac{5}{6} \) in the Proposition. With the bound in (11), we see from (10) that
\[ \Phi(s, x) \ll e^{2\pi t} \int_{1}^{\infty} \left( u - \frac{1}{u} \right)^{2\sigma - 2} (t^2 + x^2 u^2)^{-1/4} \frac{du}{u} \]
\[ \ll t^{-1/2} e^{2\pi t} \int_{1}^{\infty} \left( u - \frac{1}{u} \right)^{2\sigma - 2} \left(1 + \left( \frac{x}{t} \right)^2 u^2 \right)^{-1/4} \frac{du}{u} \]
\[ \ll t^{-1/2} e^{2\pi t}, \quad \text{since} \quad \frac{3}{4} < \sigma < \frac{5}{6}. \]

On considering Weil's bound for \( S(m, n; c) \) and (13), the second term on the right-hand side of (9) is then
\[ \ll t^{-1/2} n^{2\sigma - 1} \left( \sigma - \frac{3}{4} \right)^{-2}. \]
On the other hand, the integral in (9) is non-negative, and the series in (9) is

\[ \sum_{|k_j - t| < 1} |\rho_j(n)|^2 |\frac{\Gamma(s - \frac{1}{2} + ik_j)\Gamma(s - \frac{1}{2} - ik_j)}{|\Gamma(s)|^2}| \]

\[ \gg \frac{1}{t} \sum_{|k_j - t| < 1} \frac{|\rho_j(n)|^2}{ch\pi k_j}, \]

since \( \Gamma(s) = \sqrt{2\pi}e^{-(\pi/2)|t|}t^{\sigma - 1/2}(1 + O(|t|^{-1})) \), and \( |\Gamma(s - \frac{1}{2} - ik_j)| \gg 1 \) for \( |t - k_j| < 1 \).

This proves Lemma 1.

**Lemma 2.** We have for \( T < \frac{1}{2}x \)

\[ \sum_{c \leq x} \frac{S(m, n; c)}{c} = \sum_{|k_j| < T} \rho_j(n)\rho_j(m) \frac{\Gamma(2ik_j)}{2ik_j} x^{2ik_j} + O \left( \frac{x^{1/2}\log^2 x}{T} \right), \]

the implicit constant here depends on \( m, n \).

Before proceeding with the proof of Lemma 2, we need several analytic properties of \( Z_{m,n}(s) \). On the half plane \( \text{Re } s > 0 \), the poles of \( Z_{m,n}(s) \) are located at \( s = \frac{1}{2} + ik_j \), and as \( t \to \infty \)

\[ Z_{m,n}(s) \ll \frac{|s|^{1/2}}{|\sigma - \frac{1}{2}|} \quad (s = \sigma + it, \sigma \neq \frac{1}{2}). \]

Estimate (15) is obvious by using the result and the same method as in the proof of Theorem 1 of Goldfeld and Sarnak [3]. On the other hand, by the Lemma of §7.3 of Kuznetsov [4], we have the representation for \( Z_{m,n}(s) \) \( (s \in \mathbb{C}) \):

\[ (2\pi\sqrt{mn})^{2s-1} Z_{m,n}(s) \]

\[ = \sum_{j=1}^{\infty} \rho_j(n)\rho_j(m) \frac{\Gamma(2ik_j)}{2ik_j} h(k_j, s) - \frac{\delta_{m,n}}{2\pi} \frac{\Gamma(s)}{\Gamma(1-s)} \]

\[ + \sum_{l=0}^{\infty} p_{m,n}(l) \frac{\Gamma(s+l)}{\Gamma(2-s+l)} + L_{m,n}(s), \]
where \( L_{m,n}(s) \) denotes the analytic continuation of the function which is defined in the half plane \( \Re s > \frac{1}{2} \) by the integral

\[
L_{m,n}(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{n}{m} \right)^{it} \sigma_{2ir}(n) \sigma_{-2ir}(m) \frac{h(r,s)}{\xi(1+2ir)\xi(1-2ir)} dr,
\]

and

\[
h(r,s) = \frac{1}{2} \sin(\pi s) \Gamma \left( s - \frac{1}{2} + ir \right) \Gamma \left( s - \frac{1}{2} - ir \right),
\]

and

\[
p_{m,n}(l) = (2l+1) \sum_{c=1}^{\infty} S(m,n;c) \frac{J_{2l+1} \left( \frac{4\pi \sqrt{mn}}{c} \right)}{c}.
\]

By (16), we see that

\[
\Re s \quad Z_{m,n}(s) = \frac{(2\pi \sqrt{mn})^{-2ikj}}{2} \Gamma(2ikj) \rho_j(n) \rho_j(m).
\]

Consider \( s = \sigma + it \) with

\[
\left| \sigma - \frac{1}{2} \right| \leq \frac{\delta}{\log(|t|+2)}
\]

for a small \( \delta > 0 \). Deforming suitably the integral path in the integral of \( L_{m,n}(s) \), we have for \( s \) satisfying (18)

\[
L_{m,n}(s) = O(\log^2 |t|)
\]

since \( \zeta(x + iy) \neq 0 \) and \( \zeta(x + iy) \ll \log(|y|+2) \) in the region \( x > 1 - (c/\log(|y|+2)) \) \( (c > 0) \).

Also for \( s \) satisfying (18), we have

\[
\frac{\Gamma(s)}{\Gamma(1-s)} \ll 1
\]

and

\[
\sum_{l=0}^{\infty} p_{m,n}(l) \frac{\Gamma(s+l)}{\Gamma(2-s+l)} \ll mn.
\]

By using the estimate on Bessel function

\[
|J_k(y)| \leq \min \left( 1, \frac{(y/2)^k}{(k-1)!} \right),
\]
we prove (21) as follows: note first that \( (2l + 1) \frac{\Gamma(s + l)}{\Gamma(2 - s + l)} \ll 1 \). Thus

\[
\sum_{l=0}^{\infty} p_{m, n}(l) \frac{\Gamma(s + l)}{\Gamma(2 - s + l)} \ll \sum_{l=0}^{\infty} \sum_{1 \leq c \leq 20\sqrt{mn}} \frac{|S(m, n; c)|}{c} |J_{2l+1}\left(\frac{4\sqrt{mn}}{c}\right)|
\]

\[
+ \sum_{l=0}^{\infty} \sum_{c > 20\sqrt{mn}} \frac{|S(m, n; c)|}{c} |J_{2l+1}\left(\frac{4\sqrt{mn}}{c}\right)|
\]

\[
\ll \sum_{0 \leq l \leq 20\sqrt{mn}} \sum_{1 \leq c \leq 20\sqrt{mn}} \frac{|S(m, n; c)|}{c} |J_{2l+1}\left(\frac{4\sqrt{mn}}{c}\right)|
\]

\[
+ \sum_{1 \leq c \leq 20\sqrt{mn}} \sum_{l > 20\sqrt{mn}} \frac{|S(m, n; c)|}{c} \sum_{l=0}^{\infty} |J_{2l+1}\left(\frac{4\sqrt{mn}}{c}\right)| 2l+1 \frac{1}{(2l)!}
\]

\[
\ll mn + \sum_{1 \leq c \leq 20\sqrt{mn}} \frac{|S(m, n; c)|}{c} \sum_{l > 20\sqrt{mn}} \left(\frac{2\sqrt{mn}}{c}\right)^{2l+1} \frac{1}{(2l)!}
\]

\[
+ \sum_{c > 20\sqrt{mn}} \frac{|S(m, n; c)|}{c} \sum_{l=0}^{\infty} \left(\frac{2\sqrt{mn}}{c}\right)^{2l+1} \frac{1}{(2l)!}
\]

\[
\ll mn + \sum_{l \leq c \leq 20\sqrt{mn}} \frac{|S(m, n; c)|}{c}
\]

\[
+ \sum_{c > 20\sqrt{mn}} \frac{|S(m, n; c)|}{c} \times \frac{2\sqrt{mn}}{c}
\]

\[
\ll mn .
\]

This proves (21). Estimate (21) is obviously not the best, but we are satisfied with this presently.

Also by using Theorem 1 and (5), we have that

\[
(22) \sum_{|k_j - t| > 1} \frac{\rho_j(n)\rho_j(m)}{c\pi k_j} h(k_j, x) = O(|t|)
\]

for \( s \) satisfying (18) and \( \max\{m^{1+\varepsilon}, n^{1+\varepsilon}\} \ll |t| \). Thus by (19), (20),
(21) and (22), equation (16) becomes

\[ Z_{m,n}(s) = (2\pi \sqrt{mn})^{1-2s} \sum_{|k_j-t|<1} \frac{\rho_j(n)\rho_j(m)}{c\pi k_j} h(k_j, s) + O(|t|) \]

for \( s \) satisfying (18) and \( mn \ll |t| \) and \( \max\{m^{1+\epsilon}, n^{1+\epsilon}\} \ll |t| \).

We are now in a position to prove Lemma 2.

**Proof of Lemma 2.** Choose \( 0 < \epsilon \leq \delta / \log(|t| + 2) \) for small \( \delta > 0 \). By (15) and the Lindelöf-Phragmen principle it follows that

\[ |Z_{m,n}(s)| \ll \frac{|t|^{3/2 - 2\sigma + 2\epsilon}}{\epsilon^2} \]

for \( \frac{1}{2} + \epsilon \leq \sigma \leq \frac{3}{4} + \epsilon \), since \( Z_{m,n}(\frac{3}{4} + \epsilon) \ll \epsilon^{-2} \) by (8); and obviously

\[ |Z_{m,n}(s)| \ll \frac{|t|^{1/2}}{\epsilon - \sigma} \]

for \( \frac{1}{10} \leq \sigma \leq \frac{1}{2} - \epsilon \).

Consider the integral

\[ I(T) = \frac{1}{2\pi i} \int_{\eta-iT}^{\eta+iT} Z_{m,n}(s) \frac{x^{2s-1}}{2s-1} \, ds \quad (\eta = \frac{3}{4} + \epsilon) \]

with \( T > 0 \) not an ordinate of a pole of \( Z_{m,n}(s) \). Now by Lemma 3.12 of Titchmarsh [6], we get

\[ \sum_{c \leq x} S(m, n; c) = \frac{1}{2\pi i} \int_{\eta-iT}^{\eta+iT} Z_{m,n}(s) \frac{x^{2s-1}}{2s-1} \, ds \]

\[ + O_{m,n} \left( \frac{x^{\eta}}{Te^2} \right) . \]

Computations of residues yield

\[ I(T) = \sum_{|k_j|<T} \xi_j \frac{x^{2ik_j}}{2ik_j} + \frac{1}{2\pi i} \int_{1/10-iT}^{1/10+iT} Z_{m,n}(s) \frac{x^{2s-1}}{2s-1} \, ds \]

\[ + \frac{1}{2\pi i} \int_{1/10+iT}^{\eta+iT} Z_{m,n}(s) \frac{x^{2s-1}}{2s-1} \, ds \]

where \( \xi_j \) is the residue of \( Z_{m,n}(s) \) at \( s = \frac{1}{2} + ik_j \). Using (17), we
see that

\[ \xi_j \ll \frac{|\rho_j(n)\rho_j(m)|}{\text{ch} \pi k_j}|k_j|^{-1/2}. \]

Now we estimate the integrals in (28). By (25), we have first

\[ \int_{1/10-iT}^{1/10+iT} Z_{m, n}(s) \frac{x^{2s-1}}{2s-1} ds \ll x^{-4/5}T^{1/2} \]

and

\[ \int_{1/10-iT}^{1/10+iT} Z_{m, n}(s) \frac{x^{2s-1}}{2s-1} ds \ll \frac{x^{-2\epsilon T^{-1/2}}}{\epsilon}. \]

By (24), we have

\[ \int_{1/2+\epsilon+iT}^{\eta+iT} Z_{m, n}(s) \frac{x^{2s-1}}{2s-1} ds \ll \frac{x^{1/2+\epsilon}}{\epsilon^2|t|} \frac{1}{|\log T_x|} \]

for \(|\log T_x| \gg 1\).

Finally, by (23) we have

\[ \int_{1/2-\epsilon+iT}^{1/2+\epsilon+iT} Z_{m, n}(s) \frac{x^{2s-1}}{2s-1} ds \]

\[ = \sum_{|k_j+T|<1} \frac{\rho_j(n)\rho_j(m)}{\text{ch} \pi k_j} \int_{1/2+\epsilon+iT}^{1/2+\epsilon+iT} (2\pi \sqrt{mn})^{1-2s} h(k_j, s) \frac{x^{2s-1}}{2s-1} ds + O_{m, n}(\epsilon). \]

Noting that \(|\Gamma(s)| \gg |s|^{-1}\) for \(\epsilon \ll |s| \ll 1\) and by suitably deforming the integral path on the right-hand side of (33) to an upper or lower semi-circle according as \(1/2+ikj\) stays below or above the integral path, we get

\[ \int_{1/2-\epsilon+iT}^{1/2+\epsilon+iT} (2\pi \sqrt{mn})^{1-2s} h(k_j, s) \frac{x^{2s-1}}{2s-1} ds \ll T^{-3/2} \]

since \(|k_j+T| < 1\), so the right-hand side of (33) is

\[ \ll \sum_{m, n} \frac{|\rho_j(n)\rho_j(m)|}{\text{ch} \pi k_j} T^{-3/2} + \epsilon \]

\[ \ll T^{-1/2} + \epsilon, \]

by Theorem 1.
Putting (30), (31), (32), and (34) together, equation (28) becomes, for \( \log \frac{x}{T} \gg 1 \) and \( \varepsilon = \delta \log^{-1} T \),
\[
I(T) = \sum_{|k_j| < T} \xi_j \frac{x^{2ik_j}}{2ik_j} + O_m, n \left( \frac{x^{1/2} \log^2 x T}{T} \right)
\]
for \( T \leq \frac{1}{2} x \), which combined with (27) yield
\[
\sum_{c \leq x} S(m, n; c) c^{-1} \ddr < x \log^4 x \]
This completes the proof of Lemma 2.

By putting \( T = x^{1/3} \log^{4/3} x \) in (36), we get \( O(x^{1/6} \log^{2/3} x) \) on the right-hand side of (36) which is slightly inferior to Kuznetsof's bound (6).

**LEMMA 3.** Let \( A(u) = \sum_v c(v) e^{ivu} \) be an absolutely convergent series with complex coefficients \( c(v) \) and real indices \( v \). Then for \( T > 0 \)
\[
\int_{-T}^{T} \left| \sum_v c(v) e^{ivu} \right|^2 du \ll \int_{-\infty}^{\infty} \left| \sum_{|t| < t + T^{-1}} c(v) \right|^2 dt.
\]
**Proof.** This is Lemma 1 of Gallagher [1].

3. **Proofs of Theorems.** We prove first Theorem 2. Take \( \sigma = \frac{3}{4} + 1/\log n \). Then the Proposition of §2 gives
\[
\sum_{|k_j - t| < 1} \frac{|\rho_j(n)|^2}{ch \pi k_j} \ll t \quad (n^{1+\varepsilon} \ll t),
\]
which is the assertion of Theorem 1.

In view of \( |\zeta(1 + ir)|^{-1} \ll \log |r| \ (|r| \to +\infty) \), a rough estimate gives
\[
\int_{-\infty}^{\infty} |\sigma_{2ir}(n)|^2 \Lambda(s; r) \frac{ch \pi r}{|\zeta(1 + 2ir)|^2} dr \ll t^{-1} \log t d^2(n).
\]
We have, for \( k_j \geq t + \sqrt{f(t)} \),
\[
\Lambda(s, k_j) = \frac{\Gamma(s - 1/2 + ik_j)\Gamma(s - 1/2 - ik_j)}{\Gamma(s)|^2} \ll e^{\pi t i^{1-2\sigma}} e^{-2\pi k_j |t + k_j|^{2\sigma - 2}|k_j - t|^{2\sigma - 2}},
\]
and for \( k_j \leq t - \sqrt{f(t)} \)

\[
\Lambda(s ; k_j) \ll e^{-\pi t} t^{1-2\sigma} |t + k_j|^{2\sigma - 2} |t - k_j|^{2\sigma - 2}.
\]

On considering (37) and (5), inequalities (39) and (40) give rise to

\[
\sum_{|t - k_j| \geq \sqrt{f(t)}} |\rho_j(n)|^2 \Lambda(s ; k_j) = o(1) \quad (t \to +\infty).
\]

Now (14) together with (38) and (41) yield, by virtue of (9),

\[
\sum_{|k_j - t| < \sqrt{f(t)}} |\rho_j(n)|^2 \Lambda(s ; k_j) = \frac{1}{\pi} \Gamma(2\sigma - 1) + o(1),
\]

since \( n^{1+\epsilon} \ll t \). And also for \( |k_j - t| < \sqrt{f(t)} \)

\[
\Lambda(s ; k_j) = 2^{2\sigma - 2} t^{-1} e^{-\pi k_j} \left| \Gamma\left(s - \frac{1}{2} - ik_j\right)\right|^2 (1 + o(1)).
\]

Substituting this into (42), we obtain

\[
\sum_{|k_j - t| < \sqrt{f(t)}} |\rho_j(n)|^2 e^{-\pi k_j} \left| \Gamma\left(s - \frac{1}{2} - ik_j\right)\right|^2 = \frac{2^{2-2\sigma}}{\pi} \Gamma(2\sigma - 1) t + o(t),
\]

since \( \sqrt{f(t)} = o(t) \).

Taking integrals on both sides of (43) yields

\[
\int_{t-f(t)}^{t+f(t)} \sum_{|k_j - r| < \sqrt{f(r)}} |\rho_j(n)|^2 e^{-\pi k_j} \left| \Gamma\left(s - \frac{1}{2} + i(r - k_j)\right)\right|^2 dr = \frac{2^{3-2\sigma}}{\pi} \Gamma(2\sigma - 1) tf(t) + o(tf(t)).
\]

Interchanging the order of summation and integral in (44), the left-hand side of (44) becomes

\[
\sum_{|k_j - t| < f(t)} |\rho_j(n)|^2 e^{-\pi k_j} \int_{k_j - \sqrt{f(t)}}^{k_j + \sqrt{f(t)}} \left| \Gamma\left(s - \frac{1}{2} + i(r - k_j)\right)\right|^2 dr + o(tf(t)),
\]
by using (37). Note further that

\[
\int_{k_j - \sqrt{f(t)}}^{k_j + \sqrt{f(t)}} \left| \Gamma \left( \sigma - \frac{1}{2} + i(r - k_j) \right) \right|^2 dr = \int_{-\infty}^{\infty} \left| \Gamma(\sigma - \frac{1}{2} + ir) \right|^2 dr + O(e^{-\pi \sqrt{f(t)}}).
\]

From this and (44) and (45) it follows that

\[
(46) \quad \sum_{|k_j - t| < f(t)} \vert \rho_j(n) \vert^2 e^{-\pi k_j} \sim \frac{2^{3-2\sigma}}{\pi} \left( \int_{-\infty}^{\infty} \left| \Gamma \left( \sigma - \frac{1}{2} + ir \right) \right|^2 dr \right)^{-1} \Gamma(2\sigma - 1)tf(t)
\]

for \( \sigma = \frac{3}{4} + 1/\log n \) and \( n^{1+\varepsilon} \ll t \).

Now if we fix \( n \), then we see from the proof that (46) holds good uniformly for \( \sigma \) in an interval \( I \subset (\frac{3}{4}, \infty) \). By analytic continuation, there is a constant \( \xi \) for which

\[
(47) \quad \xi \int_{-\infty}^{\infty} \Gamma \left( \sigma - \frac{1}{2} + ir \right) \Gamma \left( \sigma - \frac{1}{2} - ir \right) dr = 2^{2-2\sigma} \Gamma(2\sigma - 1)
\]

(\( \sigma \in \mathbb{C} \)).

Indeed \( \xi = \frac{1}{\pi} \), since

\[
\int_{-\infty}^{\infty} \left| \Gamma \left( \frac{1}{2} + ir \right) \right|^2 dr = \int_{-\infty}^{\infty} \frac{\pi}{c h \pi r} dr = \pi.
\]

This completes the proof of Theorem 1, and equation (47) gives the proof of Theorem 4.

Finally we prove Theorem 3. For \( Y \geq 10 \), \( Y \leq x \leq eY \), and \( Y^{2/3} \leq T \leq \frac{1}{2} Y \), Lemma 2 gives

\[
(48) \quad \sum_{c \leq x} \frac{S(m, n; c)}{c} = \sum_{|k_j| < T} \xi_j \frac{x^{2i k_j}}{2 i k_j} + o(1).
\]
On applying Lemma 3 to (48), we get

\[ \int_{e^Y}^{e^Y} \left( \sum_{c \leq x} \frac{S(m, n; c)}{c} \right)^2 \frac{dx}{x} \]

\[ \ll \int_{e^Y}^{e^Y} \left| \sum_{|k_j| < T} \xi_j \frac{x^{2ik_j}}{2k_j} \right|^2 \frac{dx}{x} + o(1) \]

\[ = \int_{\log Y}^{1+\log Y} \left| \sum_{|k_j| < T} \xi_j \frac{\xi_j e^{2ik_j u}}{2k_j} \right|^2 d\xi_j + o(1) \]

\[ \ll \int_{-T-1}^{T+1} \left| \sum_{t < k_j < t+1} \xi_j \frac{\xi_j}{2k_j} \right|^2 dt + o(1) \]

\[ \ll \int_{1}^{T+1} \left( \sum_{|k_j - t| < 1} \frac{|\rho_j(n)\rho_j(m)|}{c\chi_k k_j} \right)^2 \frac{k_j^{-3/2}}{dt} + o(1), \text{ by (29)}, \]

\[ \ll \frac{1}{m, n} \int_{m, n}^{T+1} t^{-1} dt + o(1) \text{ by Theorem 1,} \]

\[ \ll \log T \]

\[ \ll \log Y. \]

This completes the proof of Theorem 3.

**References**


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