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**HARMONIC MAJORIZATION OF A SUBHARMONIC  
FUNCTION ON A CONE OR ON A CYLINDER**

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## HARMONIC MAJORIZATION OF A SUBHARMONIC FUNCTION ON A CONE OR ON A CYLINDER

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*To Professor N. Yanagihara on his 60th birthday*

**For a subharmonic function  $u$  defined on a cone or on a cylinder which is dominated on the boundary by a certain function, we generalize the classical Phragmén-Lindelöf theorem by making a harmonic majorant of  $u$  and show that if  $u$  is non-negative in addition, our harmonic majorant is the least harmonic majorant. As an application, we give a result concerning the classical Dirichlet problem on a cone or on a cylinder with an unbounded function defined on the boundary.**

**1. Introduction.** Let  $\mathbb{R}$  and  $\mathbb{R}_+$  be the sets of all real numbers and all positive real numbers, respectively. The  $m$ -dimensional Euclidean space is denoted by  $\mathbb{R}^m$  ( $m \geq 2$ ) and  $O$  denote the origin of it. By  $\partial S$  and  $\bar{S}$ , we denote the boundary and the closure of a set  $S$  in  $\mathbb{R}^m$ . Let  $|P - Q|$  denote the Euclidean distance between two points  $P, Q \in \mathbb{R}^m$ . A point on  $\mathbb{R}^m$  ( $m \geq 2$ ) is represented by  $(X, y)$ ,  $X = (x_1, x_2, \dots, x_{m-1})$ . We introduce the spherical coordinates  $(r, \Theta)$ ,  $\Theta = (\theta_1, \theta_2, \dots, \theta_{m-1})$ , in  $\mathbb{R}^m$  which are related to the coordinates  $(X, y)$  by

$$\begin{cases} x_1 = r \left( \prod_{j=1}^{m-1} \sin \theta_j \right), & y = r \cos \theta_1, \\ x_{m+1-k} = r \left( \prod_{j=1}^{k-1} \sin \theta_j \right) \cos \theta_k & (m \geq 3, 2 \leq k \leq m-1), \\ x_1 = r \cos \theta_1, & y = r \sin \theta_1 \quad (m = 2), \end{cases}$$

where  $0 \leq r < +\infty$  and  $-\frac{1}{2}\pi \leq \theta_{m-1} < \frac{3}{2}\pi$  ( $m \geq 2$ ),  $0 \leq \theta_j \leq \pi$  ( $m \geq 3, 1 \leq j \leq m-2$ ). The unit sphere and the surface area  $2\pi^{m/2}\{\Gamma(m/2)\}^{-1}$  of it are denoted by  $\mathbb{S}^{m-1}$  and  $s_m$  ( $m \geq 2$ ), respectively. The upper half unit sphere  $\{(1, \Theta) \in \mathbb{S}^{m-1}; 0 \leq \theta_1 < \frac{\pi}{2}$  (if  $m = 2$ , then  $0 < \theta_1 < \pi\}$  is also denoted by  $\mathbb{S}_+^{m-1}$  ( $m \geq 2$ ). For simplicity, a point  $(1, \Theta)$  on  $\mathbb{S}^{m-1}$  and a set  $S, S \subset \mathbb{S}^{m-1}$ , are often identified with  $\Theta$  and  $\{\Theta; (1, \Theta) \in S\}$ , respectively. For two

sets  $E_1 \subset \mathbb{R}_+$  and  $E_2 \subset \mathbb{S}^{m-1}$ , the set

$$\{(r, \Theta) \in \mathbb{R}^m; r \in E_1, (1, \Theta) \in E_2\}$$

in  $\mathbb{R}^m$  is denoted by  $E_1 \times E_2$ . Given a domain  $\Omega$  on  $\mathbb{S}^{m-1}$  ( $m \geq 2$ ), the set  $\mathbb{R}_+ \times \Omega$  is called a cone and denoted by  $C(\Omega)$ . The special cone  $C(\mathbb{S}_+^{m-1})$  ( $m \geq 2$ ) called the half-space will be denoted by  $\mathbb{T}_m$ . For a positive number  $r$ , the set  $\{r\} \times \mathbb{S}^{m-1}$  is denoted by  $S_m(r)$  and  $S_m(r) \cap \mathbb{T}_m$  by  $S_m^+(r)$ .

In our previous paper [12, Theorem 5.1], we gave a harmonic majorant of a certain subharmonic function  $u(P)$  defined on a cone  $C(\Omega)$  with a domain  $\Omega$  having smooth boundary, such that

$$(1.1) \quad \overline{\lim}_{P \in C(\Omega), P \rightarrow Q} u(P) \leq 0$$

for every  $Q \in \partial C(\Omega) - \{O\}$ . It can be regarded as one of the generalizations of the classical Phragmén-Lindelöf theorem. We also showed in [12, Corollary 5.2] that if the function  $u(P)$  is non-negative in addition, our harmonic majorant is the least harmonic majorant. In this paper, we shall consider generalizations of these results, by replacing 0 of (1.1) with a general function  $g(Q)$  on  $\partial C(\Omega) - \{O\}$ . They were motivated by the following Theorems A, B, C and D, which are special cases of our results (see Remark 5).

Nevanlinna [10] proved

**THEOREM A.** *Let  $g(t)$  be a continuous function on  $\mathbb{R}$  such that*

$$(1.2) \quad \int_{-\infty}^{\infty} \frac{|g(t)| + |g(-t)|}{t^2} dt < +\infty$$

*and let  $f(z)$  be a regular function on  $\mathbb{T}_2$  such that*

$$\overline{\lim}_{\text{Im}(z) > 0, z \rightarrow t} \log |f(z)| \leq g(t)$$

*for any  $t \in \partial \mathbb{T}_2$ . If*

$$(1.3) \quad \lim_{r \rightarrow \infty} \frac{1}{r} \int_0^\pi \log^+ |f(re^{i\theta})| \sin \theta d\theta = 0,$$

*then*

$$(1.4) \quad \log |f(z)| \leq \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{g(t)}{(t-x)^2 + y^2} dt$$

*for any  $z = x + iy \in \mathbb{T}_2$ .*

In the slightly different form from Theorem A, Boas [2, pp. 92–93] also stated

**THEOREM B.** *Make the same assumption as in Theorem A. If*

$$\overline{\lim}_{r \rightarrow \infty} \frac{1}{r} M_{\log |f|}(r) < +\infty \quad \left( M_{\log |f|}(r) = \sup_{|z|=r, \operatorname{Im}(z) > 0} \log |f(z)| \right),$$

then

$$(1.5) \quad \log |f(z)| \leq \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{g(t)}{(t-x)^2 + y^2} dt + a_f y$$

for any  $z = x + iy \in \mathbb{T}_2$ , where

$$a_f = \frac{2}{\pi} \lim_{r \rightarrow \infty} \frac{1}{r} \int_0^\pi \log |f(re^{i\theta})| \sin \theta d\theta.$$

Keller [7] proved an analogous result for a harmonic function on  $\mathbb{T}_3$ .

**THEOREM C.** *Let  $g(Q)$  be a continuous function on  $\partial\mathbb{T}_3$  such that*

$$\int^\infty r^{-2} \left( \int_{-\pi/2}^{3\pi/2} \left| g \left( r, \frac{\pi}{2}, \theta_2 \right) \right| d\theta_2 \right) dr < +\infty$$

$$\left( Q = \left( r, \frac{\pi}{2}, \theta_2 \right) \in \partial\mathbb{T}_3 \right).$$

Let  $h(P)$  be a harmonic function on  $\mathbb{T}_3$  such that

$$\overline{\lim}_{P \in \mathbb{T}_3, P \rightarrow Q} h(P) \leq g(Q)$$

for any  $Q \in \partial\mathbb{T}_3$ .

(I) *There exists*

$$b_{h^+} = \lim_{r \rightarrow \infty} \frac{1}{r} \int_{S_3^+(r)} h^+(P) \cos \theta_1 d\sigma_{\hat{P}}, \quad 0 \leq b_{h^+} \leq +\infty,$$

where  $h^+(P) = \max\{h(P), 0\}$  ( $P \in S_3^+(r)$ ) and  $d\sigma_{\hat{P}} = \sin \theta_1 d\theta_1 d\theta_2$  is the surface element on  $\mathbb{S}^2$  at the radial projection  $\hat{P} = (1, \theta_1, \theta_2)$  of  $P = (r, \theta_1, \theta_2) \in S_3^+(r)$ .

(II) *For any  $P \in \mathbb{T}_3$ ,*

$$h(P) \leq \frac{y}{2\pi} \int_{\partial\mathbb{T}_3} g(Q) |P - Q|^{-3} dQ + \frac{3}{2\pi} b_{h^+} y,$$

where  $dQ$  is the area element on  $\partial\mathbb{T}_3$ .

With respect to the least harmonic majorant of a subharmonic function on  $\mathbb{T}_m$ , Kuran [8, Theroem 3] proved

**THEOREM D.** *Let  $c < 0$  and let  $u(X, y)$  be subharmonic on*

$$\{(X, y) \in \mathbb{R}^m; X \in \mathbb{R}^{m-1}, y > c\}$$

*such that  $u \geq 0$  on  $\mathbb{T}_m$ .*

(I) *If*

$$(1.6) \quad \int_{\mathbb{R}^{m-1}} (1 + |X|^2)^{-1/2m} u(X, 0) dX < +\infty,$$

*then there exists the limit*

$$l_u = \lim_{r \rightarrow \infty} 2ms_m^{-1} r^{-m-1} \int_{S_m^+(r)} yu(Q) d\sigma_Q, \quad 0 \leq l_u \leq +\infty,$$

*where  $|X| = \sqrt{x_1^2 + \dots + x_{m-1}^2}$ ,  $dX$  is the  $(m - 1)$ -dimensional volume element at  $X = (x_1, \dots, x_{m-1}) \in \mathbb{R}^{m-1}$  ( $m \geq 2$ ) and  $d\sigma_Q$  is the surface element of the sphere  $S_m(r)$  at  $Q = (X, y) \in S_m^+(r)$ . Further if*

$$(1.7) \quad l_u < +\infty,$$

*then*

$$(1.8) \quad l_u y + 2s_m^{-1} y \int_{\mathbb{R}^{m-1}} |P - Q|^{-m} u(X, 0) dX$$

$$(P = (X, y) \in \mathbb{T}_m, Q = (X, 0) \in \partial\mathbb{T}_m)$$

*is the least harmonic majorant of  $u(P)$  on  $\mathbb{T}_m$ .*

(II) *If  $u$  possesses a harmonic majorant on  $\mathbb{T}_m$ , then (1.6) and (1.7) hold.*

As an application, we shall give a result concerning the classical Dirichlet problem on a cone with an unbounded function defined on the boundary. Our method in this paper can be applied to a subharmonic function  $u(X, y)$  defined on an infinite cylinder

$$\{(X, y) \in \mathbb{R}^m; X \in D, y \in \mathbb{R}\},$$

where  $D$  is a bounded domain in  $\mathbb{R}^{m-1}$  ( $m \geq 2$ ). We shall state some results in the cylindrical case.

**2. Preliminaries.** Let  $\Lambda_m$  be the spherical part of the Laplace operator

$$\Delta_m = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_{m-1}^2} + \frac{\partial^2}{\partial y^2} \quad (m \geq 2)$$

relative to the system of spherical coordinates:

$$\Delta_m = \frac{m-1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + r^{-2} \Lambda_m.$$

Given a domain  $\Omega$  on  $S^{m-1}$ , consider the Dirichlet problem

$$(2.1) \quad \begin{aligned} (\Lambda_m + \lambda)F &= 0 \quad \text{on } \Omega, \\ F &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

We denote the least positive eigenvalue of it by  $\lambda_\Omega^{(1)}$  and write  $f_\Omega(\Theta)$  for the normalized positive eigenfunction corresponding to  $\lambda_\Omega^{(1)}$ , when they exist. Thus

$$(2.2) \quad \int_\Omega f_\Omega^2(\Theta) d\sigma_\Theta = 1,$$

where  $d\sigma_\Theta$  is the surface element on  $S^{m-1}$ . Two solutions of the equation

$$t^2 + (m-2)t - \lambda_\Omega^{(1)} = 0$$

are denoted by  $\alpha_\Omega, -\beta_\Omega$  ( $\alpha_\Omega, \beta_\Omega > 0$ ).

Let  $\Phi(r, \Theta)$  be a function on  $C(\Omega)$ . For any given  $r$  ( $r \in \mathbb{R}_+$ ), the integral

$$\int_\Omega \Phi(r, \Theta) f_\Omega(\Theta) d\sigma_\Theta$$

is denoted by  $N_\Phi(r)$ , when it exists. The finite or infinite limits

$$\lim_{r \rightarrow \infty} r^{-\alpha_\Omega} N_\Phi(r) \quad \text{and} \quad \lim_{r \rightarrow 0} r^{\beta_\Omega} N_\Phi(r)$$

are denoted by  $\mu_\Phi$  and  $\eta_\Phi$ , respectively, when they exist. The maximum modulus  $M_\Phi(r)$  ( $0 < r < +\infty$ ) of  $\Phi(r, \Theta)$  is defined as

$$M_\Phi(r) = \sup_{\Theta \in \Omega} \Phi(r, \Theta).$$

We denote  $\max\{\Phi(P), 0\}$  and  $\max\{-\Phi(P), 0\}$  by  $\Phi^+(P)$  and  $\Phi^-(P)$ , respectively.

This paper is essentially based on some results in Yoshida [11]. Hence, in the subsequent consideration, we make the same assumption on  $\Omega$  as in it: if  $m \geq 3$ , then  $\Omega$  is a  $C^{2,\sigma}$ -domain ( $0 < \sigma < 1$ ) on  $S^{m-1}$  surrounded by a finite number of mutually disjoint closed hypersurfaces (e.g., see Gilbarg and Trudinger [4, pp. 88–89] for the definition of  $C^{2,\sigma}$ -domain). Then there exist two positive constants  $L_1$  and  $L_2$  such that

$$(2.3) \quad L_1 \operatorname{dis}(\Theta, \partial\Omega) \leq f_\Omega(\Theta) \leq L_2 \operatorname{dis}(\Theta, \partial\Omega) \quad (\Theta \in \Omega)$$

(by modifying Miranda’s method [9, pp. 7–8], we can prove this inequality).

REMARK 1. Let  $\Omega = \mathbb{S}_+^{m-1}$ . Then  $\alpha_\Omega = 1$ ,  $\beta_\Omega = m - 1$  and

$$\begin{aligned} f_\Omega(\Theta) &= \begin{pmatrix} (2ms_m^{-1})^{1/2} \cos \theta_1 & (m \geq 3) \\ \frac{2}{\pi} \sin \theta & (m = 2) \end{pmatrix} \\ &= (2m_m^{-1})^{1/2} \frac{y}{r} \quad (m \geq 2). \end{aligned}$$

Let  $X = (x_1, x_2, \dots, x_{m-1})$  be a point of  $\mathbb{R}^{m-1}$  ( $m \geq 2$ ). Given a bounded domain  $D$  in  $\mathbb{R}^{m-1}$  ( $m \geq 2$ ), consider the Dirichlet problem

$$\begin{aligned} (\Delta_{m-1} + \lambda)F &= 0 \quad \text{on } D, \\ F &= 0 \quad \text{on } \partial D. \end{aligned}$$

Let  $\lambda_D$  be the least positive eigenvalue of it and let  $f_D(X)$  be the normalized eigenfunction corresponding to  $\lambda_D$ . As in the conical case, we assume that the boundary  $\partial D$  of  $D \subset \mathbb{R}^{m-1}$  ( $m \geq 3$ ) is sufficiently smooth. The set

$$D \times \mathbb{R} = \{(X, y) \in \mathbb{R}^m; X \in D, y \in \mathbb{R}\}$$

in  $\mathbb{R}^m$  is called a cylinder and denoted by  $\Gamma(D)$  ( $m \geq 2$ ). Let  $\Psi(X, y)$  be a function on  $\Gamma(D)$ . The integral

$$\int_D \Psi(X, y) f_D(X) dX$$

of  $\Psi(X, y)$  is denoted by  $N_\Psi^\Gamma(y)$  when it exists, where  $dX$  denotes the  $(m - 1)$ -dimensional volume element. The finite or infinite limits

$$\lim_{y \rightarrow \infty} e^{-\sqrt{\lambda_D} y} N_\Psi(y) \quad \text{and} \quad \lim_{y \rightarrow -\infty} e^{\sqrt{\lambda_D} y} N_\Psi(y)$$

are denoted by  $\mu_\Psi^\Gamma$  and  $\eta_\Psi^\Gamma$ , respectively, when they exist.

Let  $G_\Omega(P, Q)$  (resp.  $G_D(P, Q)$ ) be the Green function of a cone  $C(\Omega)$  (resp. a cylinder  $\Gamma(D)$ ) with pole at  $P \in C(\Omega)$  (resp.  $P \in \Gamma(D)$ ), and let  $\partial G_\Omega(P, Q)/\partial n$  (resp.  $\partial G_D(P, Q)/\partial n$ ) be the differentiation at  $Q \in \partial C(\Omega) - \{O\}$  (resp.  $Q \in \partial \Gamma(D)$ ) along the inward normal into  $C(\Omega)$  (resp.  $\Gamma(D)$ ). It follows from our assumption on  $\Omega$  (resp.  $D$ ) that  $\partial G_\Omega(P, Q)/\partial n$  (resp.  $\partial G_D(P, Q)/\partial n$ ) is continuous on  $\partial C(\Omega) - \{O\}$  (resp.  $\partial \Gamma(D)$ ) (see Gilbarg and Trudinger [4, Theorem 6.15]).

Let  $g(Q)$  be a locally integrable function on  $\partial C(\Omega) - \{O\}$  (resp.  $\partial\Gamma(D)$ ) such that

$$(2.4) \quad \int_0^{+\infty} r^{-\alpha_\Omega-1} \left( \int_{\partial\Omega} |g(r, \Theta)| d\sigma_\Theta \right) dr < +\infty,$$

$$\int_0^{+\infty} r^{\beta_\Omega-1} \left( \int_{\partial\Omega} |g(r, \Theta)| d\sigma_\Theta \right) dr < +\infty,$$

(resp.

$$(2.5) \quad \int_{-\infty}^{+\infty} e^{-\sqrt{\lambda_D}|y|} \left( \int_{\partial D} |g(X, y)| d\sigma_X \right) dy < +\infty),$$

where  $d\sigma_\Theta$  (resp.  $d\sigma_X$ ) is the surface area element of  $\partial\Omega$  (resp.  $\partial D$ ) at  $\Theta \in \partial\Omega$  (resp.  $X \in \partial D$ ). If  $m = 2$  and  $\Omega = (\gamma, \delta)$  (resp.  $D = (\gamma, \delta)$ ), then

$$\int_{\partial\Omega} |g(r, \Theta)| d\sigma_\Theta \quad \left( \text{resp.} \quad \int_{\partial D} |g(X, y)| d\sigma_X \right)$$

$$= |g(r, \gamma)| + |g(r, \delta)| \quad \left( \text{resp.} \quad |g(\gamma, y)| + |g(\delta, y)| \right).$$

The Poisson integral  $PI_g(P)$  (resp.  $PI_\Gamma(P)$ ) of  $g$  relative to  $C(\Omega)$  (resp.  $\Gamma(D)$ ) is defined as follows:

$$PI_g(P) = \frac{1}{c_m} \int_{\partial C(\Omega) - \{O\}} g(Q) \frac{\partial}{\partial n} G_\Omega(P, Q) d\sigma_Q$$

$$\left( \text{resp.} \quad PI_\Gamma(P) = \frac{1}{c_m} \int_{\partial\Gamma(D)} g(Q) \frac{\partial}{\partial n} G_D(P, Q) d\sigma_Q \right),$$

where

$$c_m = \begin{cases} 2\pi & (m = 2), \\ (m - 2)s_m & (m \geq 3) \end{cases}$$

and  $d\sigma_Q$  is the surface area element on  $\partial C(\Omega) - \{O\}$  (resp.  $\partial\Gamma(D)$ ).

REMARK 2. Let  $\Omega = \mathbb{S}_+^{m-1}$ . Then

$$G_\Omega(P, Q) = \begin{cases} |P - Q|^{2-m} - |P - \bar{Q}|^{2-m} & (m \geq 3), \\ -\log|P - Q| + \log|P - \bar{Q}| & (m = 2), \end{cases}$$

where  $\bar{Q} = (X, -y)$ , that is,  $\bar{Q}$  is the mirror image of  $Q = (X, y)$  with respect to  $\partial\mathbb{T}_m$ . Hence, for two points  $P = (X, y) \in \mathbb{T}_m$  and  $Q \in \partial\mathbb{T}_m$ ,

$$\frac{\partial}{\partial n} G_\Omega(P, Q) = \begin{cases} 2(m - 2)|P - Q|^{-m}y & (m \geq 3), \\ 2|P - Q|^{-2}y & (m = 2). \end{cases}$$

**3. Statement of results.** The following Theorem 1 is a fundamental result in this paper.

**THEOREM 1.** Let  $g(Q)$  be a locally integrable function on  $\partial C(\Omega) - \{O\}$  satisfying (2.4) and let  $u(P)$  be a subharmonic function on  $C(\Omega)$  such that

$$(3.1) \quad \overline{\lim}_{P \in C(\Omega), P \rightarrow Q} \{u(P) - \text{PI}_g(P)\} \leq 0$$

and

$$(3.2) \quad \overline{\lim}_{P \in C(\Omega), P \rightarrow Q} \{u^+(P) - \text{PI}_{|g|}(P)\} \leq 0$$

for any  $Q \in \partial C(\Omega) - \{O\}$ . Then all of the limits  $\mu_{u^+}$ ,  $\eta_{u^+}$ ,  $\mu_u$  and  $\eta_u$  ( $0 \leq \mu_{u^+}$ ,  $\eta_{u^+} \leq +\infty$ ,  $-\infty < \mu_u$ ,  $\eta_u \leq +\infty$ ) exist, and if

$$(3.3) \quad \mu_{u^+} < +\infty \quad \text{and} \quad \eta_{u^+} < +\infty,$$

then

$$(3.4) \quad u(P) \leq \text{PI}_g(P) + (\mu_u r^{\alpha_\Omega} + \eta_u r^{-\beta_\Omega}) f_\Omega(\Theta)$$

for any  $P = (r, \Theta) \in C(\Omega)$ .

**REMARK 3.** It is evident that (3.3) follows from

$$(3.5) \quad \underline{\lim}_{r \rightarrow \infty} r^{-\alpha_\Omega} M_u(r) < +\infty \quad \text{and} \quad \underline{\lim}_{r \rightarrow 0} r^{\beta_\Omega} M_u(r) < +\infty.$$

It is proved in Yoshida [12, Remark 9.1] that if

$$\overline{\lim}_{P \in C_m(\Omega), P \rightarrow Q} u(P) \leq 0,$$

for any  $Q \in \partial C(\Omega) - \{O\}$ , (3.5) also follows from (3.3).

**REMARK 4.** If  $u(P)$  is a positive harmonic function on  $C(\Omega)$ , then (3.3) is always satisfied. To see it, apply (I) of Lemma 2 (which will be stated in §4) to  $-u(P)$ . We immediately obtain that  $-\infty < \mu_{-u}$ ,  $\eta_{-u} \leq +\infty$ , so that  $\mu_{u^+} = \mu_u < +\infty$  and  $\eta_{u^+} = \eta_u < +\infty$ .

The following Theorem 2 generalizes a result of Yoshida [11, Theorem 5].

**THEOREM 2.** Let  $g(Q)$  be a continuous function on  $\partial C(\Omega) - \{\overline{O}\}$  satisfying (2.4) and let  $u(P)$  be a subharmonic function on  $C(\Omega)$  such that

$$(3.6) \quad \overline{\lim}_{P \in C(\Omega), P \rightarrow Q} u(P) \leq g(Q)$$

for any  $Q \in \partial C(\Omega) - \{O\}$ . Then all of the limits  $\mu_{u^+}$ ,  $\eta_{u^+}$ ,  $\mu_u$  and  $\eta_u$  ( $0 \leq \mu_{u^+}$ ,  $\eta_{u^+} \leq +\infty$ ,  $-\infty < \mu_u$ ,  $\eta_u \leq +\infty$ ) exist, and if

$$(3.7) \quad \mu_{u^+} < +\infty \quad \text{and} \quad \eta_{u^+} < +\infty,$$

then

$$(3.8) \quad u(P) \leq \text{PI}_g(P) + (\mu_u r^{\alpha_\Omega} + \eta_u r^{-\beta_\Omega}) f_\Omega(\Theta)$$

for any  $P = (r, \Theta) \in C(\Omega)$ .

**COROLLARY 1.** Let  $g(Q)$  be a continuous function on  $\partial \mathbb{T}_m$  ( $m \geq 2$ ) such that

$$(3.9) \quad \int^{+\infty} r^{-2} \left( \int_{\partial \mathbb{S}_+^{m-1}} |g(r, \Theta)| d\sigma_\Theta \right) dr < +\infty.$$

Let  $u(P)$  be a subharmonic function on  $\mathbb{T}_m$  such that

$$(3.10) \quad \overline{\lim}_{P \in \mathbb{T}_m, P \rightarrow Q} u(P) \leq g(Q)$$

for any  $Q \in \partial \mathbb{T}_m$ . Then both of the limits  $\mu_{u^+}$  ( $0 \leq \mu_{u^+} \leq +\infty$ ) and  $\mu_u$  ( $-\infty < \mu_u \leq +\infty$ ) exist, and

$$(3.11) \quad u(P) \leq 2s_m^{-1} \int_{\partial \mathbb{T}_m} g(Q) |P - Q|^{-m} d\sigma_Q + (2ms_m^{-1})^{1/2} \mu_{u^+} y$$

for any  $P = (X, y) \in \mathbb{T}_m$ . If

$$\underline{\lim}_{r \rightarrow \infty} r^{-1} M_u(r) < +\infty,$$

then

$$(3.12) \quad u(P) \leq 2s_m^{-1} \int_{\partial \mathbb{T}_m} g(Q) |P - Q|^{-m} d\sigma_Q + (2ms_m^{-1})^{1/2} \mu_u y$$

for any  $P = (X, y) \in \mathbb{T}_m$ .

**REMARK 5.** Let  $f(z)$  be a regular function on  $\mathbb{T}_2$ . Put  $m = 2$  and  $u(P) = \log |f(z)|$  in Corollary 1. Then (3.9) is equal to (1.2). Since (1.3) gives

$$\mu_{\log^+ |f|} = 0,$$

(1.4) follows from (3.11). Since

$$\mu_{\log |f|} = \frac{2}{\pi} \lim_{r \rightarrow \infty} \frac{1}{r} \int_0^\pi \log |f(re^{i\theta})| \sin \theta d\theta = \frac{\pi}{2} a_f,$$

(3.12) gives (1.5). Thus we obtain Theorems A and B.

Next, to obtain Theorem C, put  $m = 3$  and  $u = h$  in Corollary 1. From (3.11), we have

$$h(P) \leq \frac{y}{2\pi} \int_{\partial T_3} g(Q)|P - Q|^{-3} d\sigma_{\Theta} + \left(\frac{3}{2\pi}\right)^{1/2} \mu_{h^+y}$$

for any  $P = (X, y) \in T_3$ . Since

$$\mu_{h^+} = \left(\frac{3}{2\pi}\right)^{1/2} b_{h^+}$$

(Remark 1 with  $m = 3$ ), we immediately obtain Theorem C.

EXAMPLE 1. Let  $\lambda_{\Omega}^{(2)}$  be the second least positive eigenvalue of (2.1) and let  $F_{\Omega}(\Theta)$  be a normalized eigenfunction corresponding to  $\lambda_{\Omega}^{(2)}$ . Let  $A_{\Omega}$  be the positive solution of the equation

$$t^2 + (m - 2)t - \lambda_{\Omega}^{(2)} = 0.$$

The harmonic function

$$H(P) = r^{A_{\Omega}} F_{\Omega}(\Theta) \quad (P = (r, \Theta) \in C_m(\Omega))$$

on  $\partial C(\Omega)$  has the property

$$(3.13) \quad \lim_{P \in C(\Omega), P \rightarrow Q} H(P) = 0,$$

for any  $Q \in \partial C(\Omega) - \{O\}$ . Since  $\lambda_{\Omega}^{(2)} > \lambda_{\Omega}^{(1)}$ , it is evident that

$$\lim_{r \rightarrow \infty} r^{-\alpha_{\Omega}} M_H(r) = +\infty.$$

Hence it follows from Remark 3 that

$$(3.14) \quad \mu_{H^+} = +\infty.$$

This  $H(P)$  shows that (3.6) with a continuous function on  $\partial C(\Omega) - \{O\}$  satisfying (2.4) does not always give (3.7). Further, let  $g(Q)$  be a continuous function on  $\partial C(\Omega) - \{O\}$  satisfying (2.4). Put

$$I(P) = H(P) + \text{PI}_g(P)$$

on  $C(\Omega)$ . Then we see from (3.13) that  $I(P)$  is a harmonic function on  $C(\Omega)$  satisfying

$$\lim_{P \in C(\Omega), P \rightarrow Q} I(P) = g(Q)$$

for any  $Q \in \partial C(\Omega) - \{O\}$  (see Lemma 3 and Lemma 6). Hence (3.6) is valid for the function  $g(Q)$  on  $\partial C(\Omega) - \{O\}$ . However it is easy to see that (3.8) is not true. Since  $F_\Omega(\Theta)$  is orthogonal to  $f_\Omega(\Theta)$  and

$$N_H(r) = 0 \quad (0 < r < +\infty),$$

it follows from Lemma 3 that

$$\mu_I = \mu_H + \mu_{PI_g} = 0, \quad \eta_I = \eta_H + \eta_{PI_g} = 0.$$

Since

$$I^+(P) \geq H^+(P) - PI_{|g|}(P)$$

on  $C(\Omega)$ , we see from (3.14) and Lemma 3 that

$$\mu_{I^+} \geq \mu_{H^+} = +\infty.$$

Hence this  $I(P)$  shows that (3.8) does not always follow without (3.7).

EXAMPLE 2. There exists a subharmonic function  $u(P)$  such that (3.7) is satisfied and (3.6) holds for no locally integrable function  $g(Q)$  on  $\partial C(\Omega) - \{O\}$  satisfying (2.4). Let  $\xi$  be a number satisfying  $0 < \xi < \frac{\pi}{2}$  and let

$$\Omega = \left\{ \Theta = (\theta_1, \theta_2, \dots, \theta_{m-1}) \in \mathbb{S}^{m-1}; |\theta_1| < \xi < \frac{\pi}{2} \right\}.$$

Consider the subharmonic function

$$v(r, \Theta) = r^{\alpha_\Omega}$$

on  $C(\Omega)$  and any locally integrable function  $g(Q)$  on  $\partial C(\Omega) - \{O\}$  such that

$$\overline{\lim}_{P \in C(\Omega), P \rightarrow Q} v(r, \Theta) \leq g(Q)$$

at every  $Q = (r, \Theta) \in \partial C(\Omega) - \{O\}$ . Then we always have

$$\int^{+\infty} r^{-\alpha_\Omega - 1} \left( \int_{\partial\Omega} |g(r, \Theta)| d\sigma_\Theta \right) dr = +\infty.$$

On the other hand, we have that

$$\lim_{r \rightarrow \infty} r^{-\alpha_\Omega} M_v(r) = 1,$$

so that  $\mu_{v^+} < +\infty$ .

Let  $W$  be a domain in  $\mathbb{R}^m$  and let  $g(Q)$  be a function on  $\partial W$ . A harmonic function on  $W$  satisfying

$$\lim_{P \in W, P \rightarrow Q} h(P) = g(Q)$$

for any  $Q \in \partial W$  is called the solution of the *classical Dirichlet problem* on  $W$  with  $g$ . In comparison with a result of Keller [7, Satz in p. 25], from Theorem 2 we obtain the following Theorem 3 which gives a kind of uniqueness of solutions of the classical Dirichlet problem on an unbounded domain  $C(\Omega)$ . It must be remarked that the classical Dirichlet problem on unbounded domains has no unique solution (e.g. see Helms [6, p. 42 and p. 158]).

**THEOREM 3.** *Let  $g(Q)$  be a continuous function on  $\partial C(\Omega) - \{O\}$  satisfying (2.4)*

(I) *The Poisson integral  $PI_g(P)$  is a solution of the classical Dirichlet problem on  $C(\Omega)$  with  $g$ .*

(II) *Let  $h(P)$  be any solution of the classical Dirichlet problem on  $C(\Omega)$  with  $g$ . Then all of the limits  $\mu_h, \eta_h$  ( $-\infty < \mu_h, \eta_h \leq +\infty$ ),  $\mu_{|h|}$  and  $\eta_{|h|}$  ( $0 \leq \mu_{|h|}, \eta_{|h|} \leq +\infty$ ) exist, and if*

$$(3.15) \quad \mu_{|h|} < +\infty \quad \text{and} \quad \eta_{|h|} < +\infty,$$

*then*

$$(3.16) \quad h(P) = PI_g(P) + (\mu_h r^{\alpha_\Omega} + \eta_h r^{-\beta_\Omega}) f_\Omega(\Theta)$$

*for any  $P = (r, \Theta) \in C(\Omega)$ .*

**REMARK 6.** The harmonic function  $I(P)$  in Example 1 is one of the solutions of the classical Dirichlet problem on  $C(\Omega)$ , which do not satisfy (3.15). In fact, (3.14) gives

$$\mu_{|I|} = \mu_{|PI_g + H|} = +\infty,$$

because

$$\mu_{|PI_g|} = 0$$

from Lemma 3 and

$$\mu_{|PI_g + H|} \geq \mu_{|H|} - \mu_{|PI_g|} \geq \mu_{H^+} - \mu_{|PI_g|} = \mu_{H^+}.$$

**COROLLARY 2.** *Let  $g(Q)$  be a continuous function on  $\partial C(\Omega) - \{O\}$  satisfying (2.4). If  $h(P)$  is a positive harmonic function on  $C(\Omega)$  which is the solution of the classical Dirichlet problem on  $C(\Omega)$  with  $g$ , then (3.16) holds.*

The following Theorem 4 generalizes a result of Yoshida [12, Corollary 5.2].

**THEOREM 4.** *Let  $u$  be subharmonic on a domain containing  $\overline{C(\Omega)} - \{O\}$  and let*

$$u \geq 0 \quad \text{on } C(\Omega).$$

(I) *If  $\tilde{u} = u|_{\partial C(\Omega) - \{O\}}$  (the restriction of  $u$  to  $\partial C(\Omega) - \{O\}$ ) satisfies (2.4), then both of the limits  $\mu_u$  and  $\eta_u$  ( $0 \leq \mu_u, \eta_u \leq +\infty$ ) exist. Further, if*

$$(3.17) \quad \mu_u < +\infty \quad \text{and} \quad \eta_u < +\infty,$$

*then*

$$h_u(P) = \text{PI}_{\tilde{u}}(P) + (\mu_u r^{\alpha_\Omega} + \eta_u r^{-\beta_\Omega}) f_\Omega(\Theta) \quad (P = (r, \Theta) \in C(\Omega))$$

*is the least harmonic majorant of  $u$  on  $C(\Omega)$ .*

(II) *If  $u$  possesses a harmonic majorant on  $C(\Omega)$ , then  $\tilde{u}$  satisfies (2.4) and (3.17) holds.*

**REMARK 7.** When  $u(P)$  satisfies the additional condition

$$\lim_{P \in C(\Omega), P \rightarrow Q} u(P) = 0$$

for any  $Q \in \partial C(\Omega) - \{O\}$ , we extend  $u(P)$  to  $\mathbb{R}^m - \{O\}$  by defining  $u(P) = 0$  for any  $P \in \mathbb{R}^m - C(\Omega) - \{O\}$ . Then  $u(P)$  is subharmonic on  $\mathbb{R}^m - \{O\}$ . From Remark 3 and (I) of Theorem 4, we obtain a result of Yoshida [12, Corollary 5.2].

**COROLLARY 3.** *Let  $u$  be subharmonic on a domain containing  $\overline{\mathbb{T}_m}$  ( $m \geq 2$ ) and let*

$$u \geq 0 \quad \text{on } \mathbb{T}_m.$$

(I) *If  $\tilde{u} = u|_{\partial \mathbb{T}_m}$  satisfies*

$$(3.18) \quad \int^{+\infty} r^{-2} \left( \int_{\partial \mathbb{S}_+^{m-1}} \tilde{u}(r, \Theta) d\sigma_\Theta \right) dr < +\infty,$$

*then the limit  $\mu_u$  ( $0 \leq \mu_u \leq +\infty$ ) exists. Further, if*

$$(3.19) \quad \mu_u < +\infty,$$

*then*

$$(3.20) \quad 2s_m^{-1}y \int_{\partial \mathbb{T}_m} \tilde{u}(Q) |P - Q|^{-m} d\sigma_Q + (2ms_m^{-1})^{1/2} \mu_u y$$

*is the least harmonic majorant of  $u$  on  $\mathbb{T}_m$ .*

(II) If  $u$  possesses a harmonic majorant on  $\mathbb{T}_m$ , then  $\tilde{u}$  satisfies (3.18) and (3.19) holds.

REMARK 8. Theorem D immediately follows from Corollary 3. In fact,  $u$  is bounded above on any compact subset of  $\overline{T}_m$ . Hence (3.19) is equivalent to (1.6). We also see from Remark 1 that

$$l_u = (2ms_m^{-1})^{1/2} \mu_u$$

and (3.20) is equal to (1.8).

Finally we shall state some results in the cylindrical case.

THEOREM 5. Let  $g(Q)$  be a continuous function on  $\partial\Gamma(D)$  satisfying (2.5) and let  $u(P)$  be a subharmonic function on  $\Gamma(D)$  such that

$$\overline{\lim}_{P \in \Gamma(D), P \rightarrow Q} u(P) \leq g(Q)$$

for any  $Q \in \partial\Gamma(D)$ . Then all of the limits  $\mu_{u^+}^\Gamma, \eta_{u^+}^\Gamma, \mu_u^\Gamma$  and  $\eta_u^\Gamma$  ( $0 \leq \mu_{u^+}^\Gamma, \eta_{u^+}^\Gamma \leq +\infty, -\infty < \mu_u^\Gamma, \eta_u^\Gamma \leq +\infty$ ) exist, and if

$$\mu_{u^+}^\Gamma < +\infty \quad \text{and} \quad \eta_{u^+}^\Gamma < +\infty$$

then

$$u(P) \leq \text{PI}_g(P) + (\mu_u^\Gamma e^{\sqrt{\lambda_D}y} + \eta_u^\Gamma e^{-\sqrt{\lambda_D}y}) f_D(X)$$

for any  $P = (X, y) \in \Gamma(D)$ .

THEOREM 6. Let  $g(Q)$  be a continuous function on  $\partial\Gamma(D)$  satisfying (2.5).

(I) The Poisson integral  $\text{PI}_g^\Gamma(P)$  is a solution of the classical Dirichlet problem on  $\Gamma(D)$  with  $g$ .

(II) Let  $h(P)$  be any solution of the classical Dirichlet problem on  $\Gamma(D)$  with  $g$ . Then all of the limits  $\mu_h^\Gamma, \eta_h^\Gamma$  ( $-\infty < \mu_h^\Gamma, \eta_h^\Gamma \leq +\infty$ ),  $\mu_{|h|}^\Gamma$  and  $\eta_{|h|}^\Gamma$  ( $0 \leq \mu_{|h|}^\Gamma, \eta_{|h|}^\Gamma \leq +\infty$ ) exist, and if

$$\mu_{|h|}^\Gamma < +\infty \quad \text{and} \quad \eta_{|h|}^\Gamma < +\infty,$$

then

$$(3.21) \quad h(P) = \text{PI}_g^\Gamma(P) + (\mu_h^\Gamma e^{\sqrt{\lambda_D}y} + \eta_h^\Gamma e^{-\sqrt{\lambda_D}y}) f_D(X)$$

for any  $P = (X, y) \in \Gamma(D)$ .

COROLLARY 4. Let  $g(Q)$  be a continuous function on  $\partial\Gamma(D)$  satisfying (2.5). If  $h(P)$  is a positive harmonic function on  $\Gamma(D)$  which

is the solution of the classical Dirichlet problem on  $\Gamma(D)$  with  $g$ , then (3.21) holds.

**THEOREM 7.** Let  $u$  be subharmonic on a domain containing  $\overline{\Gamma(D)}$  and let

$$u \geq 0 \quad \text{on } \Gamma(D).$$

(I) If  $\tilde{u} = u|_{\partial\Gamma(D)}$  (the restriction of  $u$  to  $\partial\Gamma(D)$ ) satisfies (2.5), then both of the limits  $\mu_u^\Gamma$  and  $\eta_u^\Gamma$  ( $0 \leq \mu_u^\Gamma, \eta_u^\Gamma \leq +\infty$ ) exist. Further, if

$$(3.22) \quad \mu_u^\Gamma < +\infty \quad \text{and} \quad \eta_u^\Gamma < +\infty,$$

then

$$PI_u^\Gamma(P) + (\mu_u^\Gamma e^{\sqrt{\lambda_D}y} + \eta_u^\Gamma e^{-\sqrt{\lambda_D}y})f_D(X) \quad (P = (X, y) \in \Gamma(D))$$

is the least harmonic majorant of  $u$  on  $\Gamma(D)$ .

(II) If  $u$  possesses a harmonic majorant on  $\Gamma(D)$ , then  $\tilde{u}$  satisfies (2.5) and (3.22) holds.

**4. Proof of Theorem 1.** For a domain  $\Omega \subset S^{m-1}$  ( $m \geq 2$ ) and a number  $t$  ( $0 < t < +\infty$ ), the sets

$$\begin{aligned} &\{(r, \Theta) \in \mathbb{R}^m; 0 < r \leq t, \Theta \in \partial\Omega\} \quad \text{and} \\ &\{(r, \Theta) \in \mathbb{R}^m; r \geq t, \Theta \in \partial\Omega\} \end{aligned}$$

are denoted by  $S_\Omega^-(t)$  and  $S_\Omega^+(t)$ , respectively. For two numbers  $t_1$  and  $t_2$  ( $0 < t_1 < t_2 < +\infty$ ), let  $S_\Omega(t_1, t_2)$  denote the set

$$\{(r, \Theta) \in \mathbb{R}^m; t_1 \leq r \leq t_2, \Theta \in \partial\Omega\}.$$

For a point  $Q \in \mathbb{R}^m$ , the set  $\{P \in \mathbb{R}^m; |P - Q| < \rho\}$  ( $\rho > 0$ ) is represented by  $U_\rho(Q)$ . We write  $G_\Omega^j(P, Q)$  for the Green function of

$$C^j(\Omega) = (j^{-1}, j) \times \Omega \quad (j \text{ is a positive integer})$$

with pole at  $P$ . For an upper semicontinuous function  $\phi(Q)$  on  $\partial C^j(\Omega)$ , the Perron-Wiener-Brelot solution of the Dirichlet problem with respect to  $C^j(\Omega)$  is denoted by  $H_\phi^j(P)$  (e.g. see Helms [6]). Since the harmonic measure  $\omega(P, E)$  of  $E \subset \partial C^j(\Omega)$  with respect to  $C^j(\Omega)$  is equal to

$$c_m^{-1} \int_E \frac{\partial}{\partial n} G_\Omega^j(P, Q) d\sigma_Q$$

(see Dahlberg [3, Theorem 3]), we know that  $H_\phi^j(P)$  is equal to

$$c_m^{-1} \int_{S^{(j^{-1}, j)} \cup (\{j^{-1}\} \times \Omega) \cup (\{j\} \times \Omega)} \phi(Q) \frac{\partial}{\partial n} G_\Omega^j(P, Q) d\sigma_Q.$$

To prove Theorem 1, we need some lemmas.

LEMMA 1. *There exist two positive constants  $k_1$  and  $k_2$  (resp.  $k_3$  and  $k_4$ ) such that*

$$\begin{aligned} k_1 r^{\alpha_\Omega} t^{-\beta_\Omega - 1} f_\Omega(\Theta) & \quad (\text{resp. } k_3 r^{-\beta_\Omega} t^{\alpha_\Omega - 1} f_\Omega(\Theta)) \\ & \leq \frac{\partial}{\partial n} G_\Omega(P, Q) \leq k_2 r^{\alpha_\Omega} t^{-\beta_\Omega - 1} f_\Omega(\Theta) \\ & \quad (\text{resp. } k_4 r^{-\beta_\Omega} t^{\alpha_\Omega - 1} f_\Omega(\Theta)) \end{aligned}$$

for  $P = (r, \Theta) \in C(\Omega)$  and  $Q = (t, \Phi) \in \partial C(\Omega) - \{O\}$  satisfying  $0 < r < \frac{1}{2}t$  (resp.  $0 < t < \frac{1}{2}r$ ).

*Proof.* These immediately follow from Azarin’s inequalities [1, Lemma 1] and (2.3).

LEMMA 2 (Yoshida [12, Theorem 3.31]). *Let  $u(P)$  be a subharmonic function on  $C(\Omega)$  ( $m \geq 2$ ) such that*

$$\overline{\lim}_{P \in C(\Omega), P \rightarrow Q} u(P) \leq 0$$

for any  $Q \in \partial C(\Omega) - \{O\}$ .

- (I) *Both of the limits  $\mu_u$  and  $\eta_u$  ( $-\infty < \mu_u, \eta_u \leq +\infty$ ) exist.*
- (II) *If  $\eta_u \leq 0$ , then  $r^{-\alpha_\Omega} N_u(r)$  is non-decreasing on  $(0, +\infty)$ .*
- (III) *If  $\mu_u \leq 0$ , then  $r^{\beta_\Omega} N_u(r)$  is non-increasing on  $(0, +\infty)$ .*

LEMMA 3. *Let  $g(Q)$  be a locally integrable function on  $\partial C(\Omega) - \{O\}$  satisfying (2.4). Then  $PI_{|g|}(P)$  (resp.  $PI_g(P)$ ) is a harmonic function on  $C(\Omega)$  such that both of the limits  $\mu_{PI_{|g|}}$  and  $\eta_{PI_{|g|}}$  (resp.  $\mu_{PI_g}$  and  $\eta_{PI_g}$ ) exist, and*

$$\mu_{PI_{|g|}} = \eta_{PI_{|g|}} = 0 \quad (\text{resp. } \mu_{PI_g} = \eta_{PI_g} = 0).$$

*Proof.* Take any  $P = (r, \Theta) \in C(\Omega)$  and two numbers  $R_1, R_2$  ( $R_1 < \frac{1}{2}r, R_2 > 2r$ ). Then by Lemma 1

$$\begin{aligned} (4.1) \quad c_m^{-1} \int_{S_\Omega^+(R_2)} |g(Q)| \frac{\partial}{\partial n} G_\Omega(P, Q) d\sigma_Q \\ \leq k_5 \int_{R_2}^{+\infty} t^{-\alpha_\Omega - 1} \left( \int_{\partial\Omega} |g(t, \Phi)| d\sigma_\Phi \right) dt, \end{aligned}$$

where  $k_5 = k_2 c_m^{-1} r^{\alpha_\Omega} f_\Omega(\Theta)$ , and

$$(4.2) \quad c_m^{-1} \int_{S_\Omega^-(R_1)} |g(Q)| \frac{\partial}{\partial n} G_\Omega(P, Q) d\sigma_Q \leq k_6 \int_0^{R_1} t^{\beta_\Omega - 1} \left( \int_{\partial\Omega} |g(t, \Phi)| d\sigma_\Phi \right) dt$$

where  $k_6 = k_4 c_m^{-1} r^{-\beta_\Omega} f_\Omega(\Theta)$ . Hence we see from (2.4) that  $PI_{|g|}(P)$  and  $PI_g(P)$  are finite for any  $P \in C(\Omega)$ . Thus  $PI_g(P)$  and  $PI_{|g|}(P)$  are harmonic on  $C(\Omega)$ .

Let  $\nu_{R,P}^{(1)}(E)$  and  $\nu_{R,P}^{(2)}(E)$  ( $0 < R < +\infty, P \in C(\Omega)$ ) be two positive measures on  $\partial C(\Omega) - \{O\}$  such that

$$\nu_{R,P}^{(1)}(E) = c_m^{-1} \int_{E \cap S_\Omega^+(R)} \frac{\partial}{\partial n} G_\Omega(P, Q) d\sigma_Q$$

and

$$\nu_{R,P}^{(2)}(E) = c_m^{-1} \int_{E \cap S_\Omega^-(R)} \frac{\partial}{\partial n} G_\Omega(P, Q) d\sigma_Q$$

for every Borel subset  $E$  of  $\partial C(\Omega) - \{O\}$ . Then  $PI_{|g|}(P)$  is the sum of two positive harmonic functions:

$$(4.3) \quad PI_{|g|}(P) = h_{1,R}(P) + h_{2,R}(P),$$

where

$$h_{1,R}(P) = \int_{\partial C(\Omega) - \{O\}} |g| d\nu_{R,P}^{(1)}$$

and

$$h_{2,R}(P) = \int_{\partial C(\Omega) - \{O\}} |g| d\nu_{R,P}^{(2)}.$$

Let  $r_1$  ( $r_1 > 0$ ) be a number and let  $\varepsilon$  be any positive number. From (2.4) we can choose a number  $r^*$  ( $r^* > 2r_1$ ) so large that

$$(4.4) \quad \int_{S_\Omega^+(r^*)} |g(t, \Phi)| t^{-\beta_\Omega - 1} d\sigma_Q \leq \frac{c_m}{2k_2} \varepsilon \quad (Q = (t, \Phi)).$$

By applying Lemma 1, we see from (4.4) that

$$N_{h_{1,r^*}}(r_1) \leq \frac{1}{2} \varepsilon r_1^{\alpha_\Omega}$$

and hence

$$(4.5) \quad r_1^{-\alpha_\Omega} N_{h_{1,r^*}}(r_1) \geq -\frac{1}{2} \varepsilon.$$

Since

$$r^{-\alpha_\Omega} N_{h_{1,r^*}}(r)$$

is non-decreasing from (II) of Lemma 2, (4.5) gives that

$$(4.6) \quad 0 \leq r^{-\alpha_\Omega} N_{h_1, r^*}(r) \leq \frac{1}{2} \varepsilon \quad (r \geq r_1).$$

By using Lemma 1 again, we obtain that

$$N_{h_2, r^*}(r) \leq k_4 r^{-\beta_\Omega} \int_0^{r^*} t^{\beta_\Omega - 1} \left( \int_{\partial\Omega} |g(t, \Phi)| d\Phi \right) dt \quad (r > 2r^*).$$

By (2.4) we can choose a number  $r_2$  ( $r_2 > 2r^*$ ) so large that

$$(4.7) \quad 0 \leq r^{-\alpha_\Omega} N_{h_2, r^*}(r) \leq \frac{1}{2} \varepsilon \quad (r \geq r_2).$$

We finally conclude from (4.3), (4.6) and (4.7) that

$$0 \leq r^{-\alpha_\Omega} N_{\text{PI}_{|g|}}(r) \leq \varepsilon \quad (r \geq r_2),$$

which gives the existence of  $\mu_{\text{PI}_{|g|}}$  and

$$(4.8) \quad \mu_{\text{PI}_{|g|}} = 0.$$

In the same way we can also prove the existence of  $\eta_{\text{PI}_{|g|}}$  and

$$(4.9) \quad \eta_{\text{PI}_{|g|}} = 0.$$

Since

$$N_{\text{PI}_{|g|}}(r) \geq N_{|\text{PI}_g|}(r) \geq |N_{\text{PI}_g}(r)| \quad (0 < r < +\infty),$$

it immediately follows from (4.8) and (4.9) that both limits  $\mu_{\text{PI}_g}$  and  $\eta_{\text{PI}_g}$  exist and are zero.

LEMMA 4 (Yoshida [12, Theorem 5.1] and Remark 3). Let  $u(P)$  be a subharmonic function on  $C(\Omega)$  ( $m \geq 2$ ) such that

$$\overline{\lim}_{P \in C(\Omega), P \rightarrow Q} u(P) \leq 0$$

for every  $Q \in \partial C(\Omega) - \{O\}$ . If (3.3) is satisfied, then

$$u(r, \Theta) \leq (\mu_u r^{\alpha_\Omega} + \eta_u r^{-\beta_\Omega}) f_\Omega(\Theta) \quad \text{on } C(\Omega).$$

*Proof of Theorem 1.* Consider two subharmonic functions

$$U(P) = u(P) - \text{PI}_g(P) \quad \text{and} \quad U^*(P) = u^+(P) - \text{PI}_{|g|}(P)$$

on  $C(\Omega)$ . Then we have from (3.1) and (3.2) that

$$\overline{\lim}_{P \in C(\Omega), P \rightarrow Q} U(P) \leq 0 \quad \text{and} \quad \overline{\lim}_{P \in C(\Omega), P \rightarrow Q} U^*(P) \leq 0$$

for every  $Q \in \partial C(\Omega) - \{O\}$ . Hence it follows from (I) of Lemma 2 that four limits  $\mu_U, \eta_U, \mu_{U^*}$  and  $\eta_{U^*}$  ( $-\infty < \mu_U, \eta_U, \mu_{U^*}, \eta_{U^*} \leq +\infty$ ) exist. Since

$$N_U(r) = N_u(r) - N_{PI_g}(r) \quad \text{and} \quad N_{U^*}(r) = N_{u^+}(r) - N_{PI_{|g|}}(r),$$

Lemma 3 gives the existence of four limits  $\mu_u, \eta_u, \mu_{u^+}$  and  $\eta_{u^+}$ , and that

$$(4.10) \quad \mu_U = \mu_u, \quad \eta_U = \eta_u, \quad \mu_{U^*} = \mu_{u^+}, \quad \eta_{U^*} = \eta_{u^+}.$$

Since

$$U^+(P) \leq u^+(P) + (PI_g)^-(P) \quad \text{on } C(\Omega),$$

it also follows from Lemma 3 and (3.3) that

$$\mu_{U^+} \leq \mu_{u^+} < +\infty, \quad \eta_{U^+} \leq \eta_{u^+} < +\infty.$$

Hence by applying Lemma 4 to  $U$ , we can obtain from (4.10) that

$$U(P) \leq PI_g(P) + (\mu_u r^{\alpha_\Omega} + \eta_u r^{-\beta_\Omega}) f_\Omega(\Theta) \quad \text{on } C(\Omega) \quad (P = (r, \Theta)),$$

which is (3.4).

**5. Proofs of Theorems 2 and 3, Corollaries 1 and 2.** The following lemma is not obvious for unbounded functions.

**LEMMA 5.** *Let  $g(Q)$  be an upper semicontinuous function on  $\partial C(\Omega) - \{O\}$  satisfying (2.4). Then*

$$\overline{\lim}_{P \in C(\Omega), P \rightarrow Q} PI_g(P) \leq g(Q)$$

for any  $Q \in \partial C(\Omega) - \{O\}$ ,

*Proof.* Let  $Q^* = (r^*, \Theta^*)$  be any point of  $\partial C(\Omega) - \{O\}$  and let  $\varepsilon$  be any positive number. Take a number  $\delta$  ( $0 < \delta < r^*$ ). From (2.4), we can choose a number  $R_2^*, R_2^* > 2(r^* + \delta)$  (resp.  $R_1^*, 0 < R_1^* < \frac{1}{2}(r^* - \delta)$ ) so large (resp. small) that

$$\int_{R_2^*}^{+\infty} t^{-\alpha_\Omega - 1} \left( \int_{\partial\Omega} |g(t, \Phi)| d\sigma_\Phi \right) dt < \frac{C_m}{8k_2 K_\Omega} (r^* + \delta)^{-\alpha_\Omega \varepsilon}$$

$$\left( \text{resp. } \int_0^{R_1^*} t^{\beta_\Omega - 1} \left( \int_{\partial\Omega} |g(t, \Phi)| d\sigma_\Phi \right) dt < \frac{C_m}{8k_4 K_\Omega} (r^* - \delta)^{\beta_\Omega \varepsilon} \right),$$

where

$$K_\Omega = \max_{\Theta \in \Omega} f_\Omega(\Theta).$$

From (4.1) and (4.2), we obtain that

$$(5.1) \quad c_m^{-1} \int_{S_{\Omega}^+(R_2^*)} |g(Q)| \frac{\partial}{\partial n} G_{\Omega}(P, Q) d\sigma_Q < \frac{\varepsilon}{8}$$

and

$$(5.2) \quad c_m^{-1} \int_{S_{\Omega}^-(R_1^*)} |g(Q)| \frac{\partial}{\partial n} G_{\Omega}(P, Q) d\sigma_Q < \frac{\varepsilon}{8}$$

for any  $P = (r, \Theta) \in C(\Omega) \cap U_{\delta}(Q^*)$ . Let  $\varphi$  be a continuous function on  $\partial C(\Omega) - \{O\}$  such that  $0 \leq \varphi \leq 1$  on  $\partial C(\Omega) - \{O\}$  and

$$\varphi = \begin{cases} 1 & \text{on } S_{\Omega}(R_1^*, R_2^*), \\ 0 & \text{on } S_{\Omega}^+(2R_2^*) \cup S_{\Omega}^-(\frac{1}{2}R_1^*). \end{cases}$$

Since the positive harmonic function  $G_{\Omega}(P, Q) - G_{\Omega}^j(P, Q)$  on  $C^j(\Omega)$  converges monotonically to 0 as  $j \rightarrow \infty$ , we can find an integer  $j_0$  ( $j_0^{-1} < 2^{-1}R_1^*$ ,  $j_0 > 2R_2^*$ ) such that

$$(5.3) \quad c_m^{-1} \int_{S_{\Omega}(2^{-1}R_1^*, 2R_2^*)} |\varphi(Q)g(Q)| \times \left| \frac{\partial}{\partial n} G_{\Omega}^{j_0}(P, Q) - \frac{\partial}{\partial n} G_{\Omega}(P, Q) \right| d\sigma_Q < \frac{\varepsilon}{4}$$

for any  $P = (r, \Theta) \in C(\Omega) \cap U_{\delta}(Q^*)$ . It follows from (5.1), (5.2) and (5.3) that

$$(5.4) \quad \begin{aligned} & c_m^{-1} \int_{\partial C(\Omega) - \{O\}} g(Q) \frac{\partial}{\partial n} G_{\Omega}(P, Q) d\sigma_Q \\ & \leq c_m^{-1} \int_{S_{\Omega}(2^{-1}R_1^*, 2R_2^*)} \varphi(Q)g(Q) \frac{\partial}{\partial n} G_{\Omega}^{j_0}(P, Q) d\sigma_Q \\ & \quad + \left| c_m^{-1} \int_{S_{\Omega}(2^{-1}R_1^*, 2R_2^*)} \varphi(Q)g(Q) \frac{\partial}{\partial n} G_{\Omega}^{j_0}(P, Q) d\sigma_Q \right. \\ & \quad \left. - c_m^{-1} \int_{S_{\Omega}(2^{-1}R_1^*, 2R_2^*)} \varphi(Q)g(Q) \frac{\partial}{\partial n} G_{\Omega}(P, Q) d\sigma_Q \right| \\ & \quad + 2c_m^{-1} \int_{S_{\Omega}^+(R_2^*)} |g(Q)| \frac{\partial}{\partial n} G_{\Omega}(P, Q) d\sigma_Q \\ & \quad + 2c_m^{-1} \int_{S_{\Omega}^-(R_1^*)} |g(Q)| \frac{\partial}{\partial n} G_{\Omega}(P, Q) d\sigma_Q \\ & < c_m^{-1} \int_{S_{\Omega}(2^{-1}R_1^*, 2R_2^*)} \varphi(Q)g(Q) \frac{\partial}{\partial n} G_{\Omega}^{j_0}(P, Q) d\sigma_Q \\ & \quad + \frac{3}{4}\varepsilon \end{aligned}$$

for any  $P = (r, \Theta) \in C(\Omega) \cap U_\delta(Q^*)$ . Consider the upper semicontinuous function

$$V(Q) = \begin{cases} \varphi(Q)g(Q) & \text{on } S_\Omega(2^{-1}R_1^*, 2R_2^*), \\ 0 & \text{on } Z \end{cases}$$

$$(Z = S_\Omega(j_0^{-1}, 2^{-1}R_1^*) \cup S_\Omega(2R_2^*, j_0) \cup (\{j_0^{-1}\} \times \Omega) \cup (\{j_0\} \times \Omega))$$

on  $\partial C^{j_0}(\Omega)$ . Since

$$\overline{\lim}_{P \in C(\Omega), P \rightarrow Q^*} H_V^j(P) \leq \overline{\lim}_{Q \in \partial C(\Omega) - \{O\}, Q \rightarrow Q^*} V(Q) = g(Q^*)$$

(e.g. see Helms [6, Lemma 8.20]), we finally obtain from (5.4) that

$$\overline{\lim}_{P \in C(\Omega), P \rightarrow Q^*} c_m^{-1} \int_{\partial C(\Omega) - \{O\}} g(Q) \frac{\partial}{\partial n} G_\Omega(P, Q) d\sigma_Q \leq g(Q^*).$$

From Lemma 5, immediately follows

LEMMA 6. *If  $g(Q)$  is a continuous function on  $\partial C(\Omega) - \{O\}$  satisfying (2.4), then*

$$\lim_{P \in C(\Omega), P \rightarrow Q} \text{PI}_g(P) = g(Q)$$

for every  $Q \in \partial C(\Omega) - \{O\}$ .

*Proof of Theorem 2.* First, we see from Lemma 6 that

$$\lim_{P \in C(\Omega), P \rightarrow Q} \text{PI}_g(P) = g(Q) \quad \text{and} \quad \lim_{P \in C(\Omega), P \rightarrow Q} \text{PI}_{|g|}(P) = |g(Q)|$$

for every  $Q \in \partial C(\Omega) - \{O\}$ . Hence we see from (3.6) that

$$\overline{\lim}_{P \in C(\Omega), P \rightarrow Q} \{u(P) - \text{PI}_g(P)\} \leq 0$$

and

$$\overline{\lim}_{P \in C(\Omega), P \rightarrow Q} \{u^+(P) - \text{PI}_{|g|}(P)\} \leq 0$$

for every  $Q \in \partial C(\Omega) - \{O\}$ . Theorem 1 immediately gives Theorem 2.

*Proof of Corollary 1.* Put  $\Omega = \mathbb{S}_+^{m-1}$  in Theorem 2. Since  $g(Q)$  is continuous at  $Q = O$  of  $\partial \mathbb{T}_m$ ,  $|g(Q)|$  is bounded in the neighborhood of  $Q = O$ . Hence we see from Remark 1 and (3.9) that  $g(Q)$  is admissible on  $\partial \mathbb{T}_m$  and from (3.10) that  $\eta_u \leq \eta_{u^+} = 0$ . If  $\mu_{u^+} = +\infty$ , then (3.11) is evidently satisfied. When  $\mu_{u^+} < +\infty$ , (3.11) also follows

from (3.8), Remark 1, Remark 2 and the inequality  $\mu_u \leq \mu_{u^+}$ . It is easily seen that Remark 3 and (3.8) give (3.12).

*Proof of Theorem 3.* It follows from Lemma 3 and Lemma 6 that  $\text{PI}_g(P)$  is one of the solutions. To prove (II), put  $u(P) = h(P)$  and  $-h(P)$  in Theorem 2. Then Theorem 2 gives the existence of all limits  $\mu_h, \eta_h, \mu_h^+, \eta_h^+$ ,

$$(5.5) \quad \mu_{(-h)^+} = \mu_{h^-} \quad \text{and} \quad \eta_{(-h)^+} = \eta_{h^-}.$$

Since

$$(5.6) \quad \mu_{h^+} + \mu_{h^-} = \mu_{|h|} \quad \text{and} \quad \eta_{h^+} + \eta_{h^-} = \eta_{|h|},$$

it follows that both limits  $\mu_{|h|}$  and  $\eta_{|h|}$  exist. Suppose that  $h$  satisfies (3.15). Then we see from (5.5) and (5.6) that  $\mu_{h^+}, \mu_{(-h)^+}, \eta_{h^+}$  and  $\eta_{(-h)^+} < +\infty$ . Hence, by applying Theorem 2 to  $u(P) = h(P)$  and  $-h(P)$  again, we obtain from (3.8) that

$$h(P) \leq \text{PI}_g(P) + (\mu_h r^{\alpha_\Omega} + \eta_h r^{-\beta_\Omega}) f_\Omega(\Theta)$$

and

$$h(P) \geq \text{PI}_g(P) + (\mu_h r^{\alpha_\Omega} + \eta_h r^{-\beta_\Omega}) f_\Omega(\Theta),$$

respectively, which give (3.16).

*Proof of Corollary 2.* It follows from Remark 4 that

$$\mu_{|h|} = \mu_{h^+} < +\infty \quad \text{and} \quad \eta_{|h|} = \eta_{h^+} < +\infty.$$

Thus Theorem 3 implies Corollary 2.

## 6. Proof of Theorem 4.

**LEMMA 7.** *Let  $g(Q)$  be a non-negative lower semicontinuous function on  $\partial C(\Omega) - \{O\}$  satisfying (2.4) and let  $u(P)$  be a non-negative subharmonic function on  $C(\Omega)$  such that*

$$(6.1) \quad \overline{\lim}_{P \in C(\Omega), P \rightarrow Q} u(P) \leq g(Q)$$

*for every  $Q \in \partial C(\Omega) - \{O\}$ . Then both of the limits  $\mu_u$  and  $\eta_u$  ( $0 \leq \mu_u, \eta_u \leq +\infty$ ) exist, and if  $\mu_u < +\infty$  and  $\eta_u < +\infty$ , then*

$$u(P) \leq \text{PI}_g(P) + (\mu_u r^{\alpha_\Omega} + \eta_u r^{-\beta_\Omega}) f_\Omega(\Theta)$$

*for any  $P = (r, \Theta) \in C(\Omega)$ .*

*Proof.* To apply Theorem 1, we shall show that (3.1) and (3.2) hold. Since  $-g(Q)$  is upper semicontinuous on  $\partial C(\Omega) - \{O\}$ , it follows from Lemma 5 that

$$(6.2) \quad \lim_{P \in C(\Omega), P \rightarrow Q} \text{PI}_g(P) \geq g(Q)$$

for every  $Q \in \partial C(\Omega) - \{O\}$ . Hence we see from (6.1) and (6.2) that

$$\begin{aligned} & \overline{\lim}_{P \in C(\Omega), P \rightarrow Q} \{u(P) - \text{PI}_g(P)\} \\ & \leq \overline{\lim}_{P \in C(\Omega), P \rightarrow Q} u(P) - \lim_{P \in C(\Omega), P \rightarrow Q} \text{PI}_g(P) \leq g(Q) - g(Q) = 0 \end{aligned}$$

for every  $Q \in \partial C(\Omega) - \{O\}$ , which provides (3.1). Since  $g$  and  $u$  are non-negative, (3.2) also holds. Thus we obtain Lemma 7 from Theorem 1.

LEMMA 8. *Let  $u$  be subharmonic on a domain containing  $\overline{C(\Omega)} - \{O\}$  such that  $\tilde{u} = u|_{\partial C(\Omega) - \{O\}}$  satisfies (2.4) and*

$$u \geq 0 \quad \text{on } C(\Omega).$$

*Then*

$$\text{PI}_{\tilde{u}}(P) \leq h(P) \quad \text{on } C(\Omega)$$

*for every harmonic majorant  $h$  of  $u$  on  $C(\Omega)$ .*

*Proof.* Take any  $P^* = (r^*, \Theta^*) \in C(\Omega)$ . Let  $\varepsilon$  be any positive number. In the same way as in the proof of Lemma 5, we can choose two numbers  $R_1$  and  $R_2$  ( $2R_1 < r < 2^{-1}R_2$ ) such that

$$(6.3) \quad c_m^{-1} \int_{S_{\Omega}^+(R_2)} \tilde{u}(Q) \frac{\partial}{\partial n} G_{\Omega}(P^*, Q) d\sigma_Q < \frac{\varepsilon}{3}$$

and

$$(6.4) \quad c_m^{-1} \int_{S_{\Omega}^-(R_1)} \tilde{u}(Q) \frac{\partial}{\partial n} G_{\Omega}(P^*, Q) d\sigma_Q < \frac{\varepsilon}{3}.$$

Further, take an integer  $j_0$  ( $j_0^{-1} < R_1$  and  $j_0 > R_2$ ) such that

$$(6.5) \quad c_m^{-1} \int_{S_{\Omega}(R_1, R_2)} \tilde{u}(Q) \left\{ \frac{\partial}{\partial n} G_{\Omega}(P^*, Q) - \frac{\partial}{\partial n} G_{\Omega}^{j_0}(P^*, Q) \right\} d\sigma_Q < \frac{\varepsilon}{3}.$$

Since

$$c_m^{-1} \int_{S_\Omega(R_1, R_2)} \tilde{u}(Q) \frac{\partial}{\partial n} G_\Omega^{j_0}(P, Q) d\sigma_Q \leq H_u^{j_0}(P)$$

for any  $P \in C^{j_0}(\Omega)$ , we have from (6.3), (6.4) and (6.5) that

$$\begin{aligned} (6.6) \quad & \text{PI}_{\tilde{u}}(P^*) - H_u^{j_0}(P^*) \\ & \leq c_m^{-1} \int_{S_\Omega(R_1, R_2)} \tilde{u}(Q) \left\{ \frac{\partial}{\partial n} G_\Omega(P^*, Q) \right. \\ & \qquad \qquad \qquad \left. - \frac{\partial}{\partial n} G_\Omega^{j_0}(P^*, Q) \right\} d\sigma_Q \\ & \quad + c_m^{-1} \int_{S_\Omega^+(R_2)} \tilde{u}(Q) \frac{\partial}{\partial n} G_\Omega(P^*, Q) d\sigma_Q \\ & \quad + c_m^{-1} \int_{S_\Omega^-(R_1)} \tilde{u}(Q) \frac{\partial}{\partial n} G_\Omega(P^*, Q) d\sigma_Q < \varepsilon. \end{aligned}$$

Here, note that  $H_u^{j_0}(P)$  is the least harmonic majorant of  $u(P)$  on  $C^{j_0}(\Omega)$  (see Hayman [5, Theorem 3.15]). If  $h$  is a harmonic majorant of  $u$  on  $C(\Omega)$ , then

$$H_u^{j_0}(P^*) \leq h(P^*).$$

Thus we obtain from (6.6) that

$$\text{PI}_{\tilde{u}}(P^*) < h(P^*) + \varepsilon,$$

which gives the conclusion of Lemma 8.

*Proof of Theorem 4.* Let  $P = (r, \Theta)$  be any point of  $C(\Omega)$  and let  $\varepsilon$  be any positive number. By the Vitali-Carathéodory theorem (e.g. see [11, p. 56]), we can find a lower semicontinuous function  $g_\varepsilon(Q)$  on  $\partial C(\Omega) - \{O\}$  such that

$$(6.7) \quad \tilde{u}(Q) \leq g_\varepsilon(Q) \quad \text{on } \partial C(\Omega) - \{O\}$$

and

$$(6.8) \quad \text{PI}_{g_\varepsilon}(P) < \text{PI}_{\tilde{u}}(P) + \varepsilon.$$

Since

$$\overline{\lim}_{P \in C(\Omega), P \rightarrow Q} u(P) \leq \tilde{u}(Q) \leq g_\varepsilon(Q)$$

for any  $q \in \partial C(\Omega) - \{O\}$  from (6.7), it follows from Lemma 7 that two limits  $\mu_u, \eta_u$  exist and if  $\mu_u < +\infty$  and  $\eta_u < +\infty$ , then

$$(6.9) \quad u(P) \leq \text{PI}_{g_\varepsilon}(P) + (\mu_u r^{\alpha_\Omega} + \eta_u r^{-\beta_\Omega}) f_\Omega(\Theta).$$

Hence we have from (6.8) and (6.9) that

$$u(P) \leq \text{PI}_{\tilde{u}}(P) + \varepsilon + (\mu_u r^{\alpha_\Omega} + \eta_u r^{-\beta_\Omega}) f_\Omega(\Theta).$$

Since  $\varepsilon$  was arbitrary, we obtain

$$u(P) \leq \text{PI}_{\tilde{u}}(P) + (\mu_u r^{\alpha_\Omega} + \eta_u r^{-\beta_\Omega}) f_\Omega(\Theta)$$

for any  $P = (r, \Theta) \in C(\Omega)$ . This shows that  $h_u(P)$  is a harmonic majorant of  $u$  on  $C(\Omega)$ .

To prove that  $h_u$  is the least harmonic majorant of  $u$  on  $C(\Omega)$ , let  $h(P)$  be any harmonic function on  $C(\Omega)$  such that

$$(6.10) \quad u(P) \leq h(P) \quad \text{on } C(\Omega).$$

Consider the harmonic function

$$h^*(p) = h_u(P) - h(P) \quad \text{on } C(\Omega).$$

Since

$$h^*(P) \leq h_u(P) \quad \text{on } C(\Omega),$$

we see from Lemma 3 that  $h^*(P)$  satisfies (3.3). We also see from Lemma 8 that

$$\overline{\lim}_{P \in C(\Omega), P \rightarrow Q} h^*(P) = \overline{\lim}_{P \in C(\Omega), P \rightarrow Q} \{\text{PI}_{\tilde{u}}(P) - h(P)\} \leq 0$$

for any  $Q \in \partial C(\Omega) - \{O\}$ . We have from Lemma 3 and (6.10) that

$$\mu_{h^*} = \mu_{h_u} - \mu_h = \mu_u - \mu_h \leq \mu_u - \mu_u = 0$$

and similarly  $\eta_{h^*} \leq 0$ . Thus we obtain from Lemma 4 that

$$h^*(P) \leq 0 \quad \text{on } C(\Omega),$$

which shows that  $h_u(P)$  is the least harmonic majorant of  $u(P)$  on  $C(\Omega)$ .

To prove (II), let  $h_1(P)$  be a harmonic majorant of  $u(P)$  on  $C(\Omega)$ . Since

$$\mu_u \leq \mu_{h_1} < +\infty \quad \text{and} \quad \eta_u \leq \eta_{h_1} < +\infty$$

from Remark 4, we immediately have (3.17). Fix  $P_0 = (1, \Theta_0)$ ,  $\Theta_0 \in \Omega$ . Take any two numbers  $R_1, R_2$  ( $0 < R_1 < 2^{-1}$ ,  $2 < R_2 < +\infty$ ) and choose a sufficiently large integer  $j^*$ ,  $j^* > \text{Max}(R_1^{-1}, R_2)$ , such that

$$c_m^{-1} \int_{S_\Omega(R_1, 2^{-1})} \tilde{u}(Q) \left\{ \frac{\partial}{\partial n} G_\Omega(P_0, Q) - \frac{\partial}{\partial n} G_\Omega^{j^*}(P_0, Q) \right\} d\sigma_Q \leq 1$$

and

$$c_m^{-1} \int_{S_\Omega(2, R_2)} \tilde{u}(Q) \left\{ \frac{\partial}{\partial n} G_\Omega(P_0, Q) - \frac{\partial}{\partial n} G_\Omega^{j^*}(P_0, Q) \right\} d\sigma_Q \leq 1.$$

Since  $H_u^{j^*}(P)$  is the least harmonic majorant of  $u(P)$  on  $C^{j^*}(\Omega)$ ,

$$\begin{aligned} h_1(P_0) &\geq H_u^{j^*}(P) \geq c_m^{-1} \int_{S_\Omega(j^{*-1}, j^*)} \tilde{u}(Q) \frac{\partial}{\partial n} G_\Omega^{j^*}(P_0, Q) d\sigma_Q \\ &\geq \begin{cases} c_m^{-1} \int_{S_\Omega(R_1, 2^{-1})} \tilde{u}(Q) \frac{\partial}{\partial n} G_\Omega^{j^*}(P_0, Q) d\sigma_Q \\ c_m^{-1} \int_{S_\Omega(2, R_2)} \tilde{u}(Q) \frac{\partial}{\partial n} G_\Omega^{j^*}(P_0, Q) d\sigma_Q. \end{cases} \end{aligned}$$

Hence it follows from Lemma 1 that

$$\begin{aligned} +\infty &> h_1(P_0) + 1 \\ &\geq \begin{cases} c_m^{-1} \int_{S_\Omega(R_1, 2^{-1})} \tilde{u}(Q) \frac{\partial}{\partial n} G_\Omega(P_0, Q) d\sigma_Q \\ \geq k_1 \int_{R_1}^{2^{-1}} r^{-\alpha_\Omega-1} \left( \int_{\partial\Omega} \tilde{u}(r, \Theta) d\sigma_\Theta \right) dr \\ c_m^{-1} \int_{S_\Omega(2, R_2)} \tilde{u}(Q) \frac{\partial}{\partial n} G_\Omega(P_0, Q) d\sigma_Q \\ \geq k_3 \int_2^{R_2} r^{\beta_\Omega-1} \left( \int_{\partial\Omega} \tilde{u}(r, \Theta) d\sigma_\Theta \right) dr, \end{cases} \end{aligned}$$

which shows that  $\tilde{u}$  satisfies (2.4).

**7. Proofs of Theorems 5, 6 and 7.** These proofs proceed in the completely parallel way to the proofs of Theorems 2, 3 and 4, on the basis of two results of Yoshida [12, Theorems 7.2 and 7.5] and the following inequality corresponding to Lemma 1:

$$\begin{aligned} k'_1 e^{-\sqrt{\lambda_D}(y^*-y)} f_D(X) \quad (\text{resp. } k'_3 e^{-\sqrt{\lambda_D}(-y^*+y)} f_D(X)) \\ \leq \frac{\partial}{\partial n} G_D(P, Q) \leq k'_2 e^{-\sqrt{\lambda_D}(y^*-y)} f_D(X) \\ (\text{resp. } k'_4 e^{-\sqrt{\lambda_D}(-y^*+y)} f_D(X)) \end{aligned}$$

for  $P = (X, y) \in \Gamma(D)$  and  $Q = (X^*, y^*) \in \partial\Gamma(D)$  satisfying  $y^* > y + 1$  (resp.  $y^* < y - 1$ ), where  $k'_1$  and  $k'_2$  (resp.  $k'_3$  and  $k'_4$ ) are two positive constants.

### REFERENCES

- [1] V. S. Azarin, *Generalization of a theorem of Hayman on subharmonic functions in an  $m$ -dimensional cone*, Mat. Sb., **66** (108) (1965), 248–264; English transl. in Amer. Math. Soc. Transl. (2) **80** (1969), 119–138.
- [2] R. P. Boas, *Entire Functions*, Academic, New York, 1954.

- [3] B. E. J. Dahlberg, *Estimates of harmonic measure*, Arch. Rational Mech. Anal., **65** (1977), 275–288.
- [4] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of the Second Order*, Springer, Berlin, 1977.
- [5] W. K. Hayman and P. W. Kennedy, *Subharmonic Functions*, vol. 1, Academic, London, 1976.
- [6] L. L. Helms, *Introduction to Potential Theory*, Wiley-Interscience, New York, 1969.
- [7] H. Keller, *Über das Anwachsen von Potentialfunktionen im dreidimensionalen Raum*, Ann. Acad. Sci. Fenn. Ser. A, **83** (1950), 35 pp.
- [8] U. Kuran, *Harmonic majorizations in half-balls and half-spaces*, Proc. London Math. Soc., (3) **21** (1970), 614–636.
- [9] C. Miranda, *Partial Differential Equations of Elliptic Type*, Springer, Berlin, 1970.
- [10] R. Nevanlinna, *Über die Eigenschaften meromorpher Funktionen in einem Winkelraum*, Acta Soc. Sci. Fenn., **50** No. 12 (1922), 45 pp.
- [11] W. Rudin, *Real and Complex Analysis*, Third Edition, McGraw-Hill, New York, 1986.
- [12] H. Yoshida, *Nevanlinna norm of a subharmonic function on a cone or on a cylinder*, Proc. London Math. Soc., (3) **54** (1987), 267–299.

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