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**ON THE ROMANOV KERNEL AND KURANISHI'S
 L^2 -ESTIMATE FOR $\bar{\partial}_b$ OVER A BALL IN THE STRONGLY
PSEUDO CONVEX BOUNDARY**

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As is proved by Kerzman-Stein, over a compact strongly pseudo convex boundary in C^n , Szegő projection S is the operator defined by Henkin-Ramirez *modulo compact operators*. While, over a *special ball*, U_ε , in the strongly pseudo convex boundary, in order to obtain a local embedding theorem of CR-structures, Kuranishi constructed the Neumann type operator N_b for $\bar{\partial}_b$ and so we have a *local Szegő operator* by

$$S_{U_\varepsilon} = \text{id} - \bar{\partial}_b^* N_b \bar{\partial}_b \quad \text{on } U_\varepsilon,$$

where $\bar{\partial}_b^*$ means the adjoint operator of $\bar{\partial}_b$. There might be a relation between S_{U_ε} and the Romanov kernel like the case of the Szegő operator and the Henkin-Ramirez kernel. We study this problem and show some estimates for the Romanov kernel.

0. Introduction. Let $(M, \circ T'')$ be an abstract strongly pseudo convex CR-manifold. Then as is well known, if $\dim_R M = 2n - 1 \geq 7$, $(M, \circ T'')$ is locally embeddable in a complex euclidean space $C^n((Ak3), (K))$. In the proof of this local embedding theorem, it is shown that: over a *special ball* in the strongly pseudo convex boundary, an L^2 -estimate for $\bar{\partial}_b$, which is stronger than the standard L^2 -estimate, is established and so the L^2 -solution operator for $\bar{\partial}_b$ is obtained. This operator plays an essential role in our local embedding theorem. Therefore it must be important to study this solution operator for $\bar{\partial}_b$ precisely.

In order to get a solution operator, there exists another method. By using an integral formula, a local solution operator for $\bar{\partial}_b$ is constructed explicitly by Henkin and Harvey-Polking. Obviously, these solution operators are different. And it seems quite interesting to study the relation between the L^2 -solution for $\bar{\partial}_b$ and the explicit solution, obtained by using an integral formula. We recall the $\bar{\partial}$ -case over a strongly pseudo convex domain in C^n . In this case, the explicit solution, constructed by Lieb and Range, is a certain kind of the essential part of the Kohn's L^2 -solution. Therefore we could hope for a similar result in the $\bar{\partial}_b$ case over a *special ball* in the strongly pseudo

convex boundary. As mentioned already, our L^2 -a priori estimate is different from the standard L^2 -estimate. Therefore in the above sense, it seems to be natural to consider that the explicit solution operator would satisfy the similar L^2 -estimate. In this paper, we discuss this point over rigid hypersurfaces in C^n (for the definition, see §3 in this paper). And we prove our a priori estimate (Main Theorem in §5 in this paper) for the explicit solution operator.

1. CR-structure and $\bar{\partial}_b$ -operator. Let M be a real hypersurface in C^n . Let p be a reference point of M . We assume that p is a smooth point, namely let ρ be a defining function of M in a neighborhood of p in C^n , i.e., there is a neighborhood $V(p)$ of p satisfying:

$$M \cap V(p) = \{q : q \in V(p), \rho(q) = 0\}$$

and

$$d\rho \neq 0 \quad \text{over } M \cap V(p).$$

Then over $M \cap V(p)$, we can introduce an CR-structure induced from C^n . Namely, let

$${}^\circ T'' = T'' C^n \cap C \otimes TM \quad \text{over } M \cap V(p).$$

Then this ${}^\circ T''$ satisfies

$$(1-1) \quad {}^\circ T'' \cap {}^\circ \bar{T}'' = 0, \quad f\text{-dim}_C(C \otimes TM / ({}^\circ T'' + {}^\circ \bar{T}'')) = 1,$$

$$(1-2) \quad [\Gamma(M \cap V(p), {}^\circ T''), \Gamma(M \cap V(p), {}^\circ \bar{T}'')] \subset \Gamma(M \cap V(p), {}^\circ T'').$$

This pair $(M \cap V(p), {}^\circ T'')$ is called a CR-structure, or a CR-manifold.

Let $(M \cap V(p), {}^\circ T'')$ be a CR-manifold. We introduce a C^∞ vector bundle decomposition

$$(1-3) \quad C \otimes TM = {}^\circ T'' + {}^\circ \bar{T}'' + C\zeta,$$

where

$$(1-3-1) \quad \zeta \text{ is a real vector field,}$$

$$(1-3-2) \quad \zeta_q \notin {}^\circ T''_q + {}^\circ \bar{T}''_q \quad \text{for } q \text{ in } M \cap V(p).$$

By using this decomposition, we have a Levi form

$$L(X, Y) = \sqrt{-1}[X, \bar{Y}]_\zeta \quad \text{for } X, Y \text{ in } \Gamma(M \cap V(p), {}^\circ T''),$$

where $[X, \bar{Y}]_\zeta$ means the ζ -part of $[X, \bar{Y}]$ according to (1-3). As is well known, this map L makes sense for elements X, Y in ${}^\circ T''$. And if this Levi form is positive or negative definite, $(M \cap V(p), {}^\circ T'')$

is called a strongly pseudo convex real hypersurface. Next we briefly explain $\bar{\partial}_b$ -complex. For u in $\Gamma(M \cap V(p), C)$, we set

$$\bar{\partial}_b u(x) = Xu \quad \text{for } X \text{ in } {}^\circ T'' ,$$

where $\Gamma(M \cap V(p), c)$ means the spacing consisting of C^∞ functions over $M \cap V(p)$. Namely we have a first order differential operator

$$\bar{\partial}_b: \Gamma(M \cap V(p), c) \rightarrow \Gamma(M \cap V(p), ({}^\circ T'')^*).$$

By the same way as for usual differential forms, we have

$$\bar{\partial}_b^{(p)}: \Gamma(M \cap V(p), \Lambda^p({}^\circ T'')^*) \rightarrow \Gamma(M \cap V(p), \Lambda^{p+1}({}^\circ T'')^*)$$

and so

$$\bar{\partial}_b^{(p+1)} \circ \bar{\partial}_b^{(p)} = 0.$$

2. Kuranishi's L^2 -estimate. Let $(M, {}^\circ T'')$ be a strongly pseudo convex CR manifold, embedded as a real hypersurface in C^n . Let p be a reference point of M . Then by a change of coordinates, we can assume that there is a neighborhood $W(p)$ of p in C^n , satisfying:

$$\begin{aligned} M \cap W(p) &= \{(z_1, \dots, z_n): (z_1, \dots, z_n) \in W(p), \\ &\quad \text{Im } z_n = h(z_1, \dots, z_{n-1}, \text{Re } z_n)\}, \end{aligned}$$

where $z_i(p) = 0$, $1 \leq i \leq n-1$, and h is a real valued C^∞ function, and

$$\begin{aligned} (\partial^2 h / \partial z_i \partial \bar{z}_j)(0) &= \delta_{ij}, \quad 1 \leq i, j \leq n-1, \\ (\partial^2 h / \partial z_i \partial z_j)(0) &= \delta_{ij}, \quad 1 \leq i, j \leq n-1, \\ dh(0, \dots, 0) &= 0. \end{aligned}$$

In this set up, we introduce a neighborhood $M \cap U_\varepsilon(p)$ of p as follows:

$$\begin{aligned} M \cap U_\varepsilon(p) &= \{(z_1, \dots, z_n): (z_1, \dots, z_n) \in W(p), \\ &\quad \text{Im } z_n = h(z_1, \dots, z_{n-1}, \text{Re } z_n), \\ &\quad 2 \text{Re}\{(1/2\sqrt{-1})z_n + z_n^2\} < \varepsilon\}. \end{aligned}$$

Now we briefly sketch Kuranishi's L^2 -estimate over $M \cap U_\varepsilon(p)$. Obviously by the above assumption, our $M \cap U_\varepsilon(p)$ is diffeomorphic to the real $2n-1$ dimensional ball. We denote this diffeomorphism map by h and we fix this. If ε is chosen sufficiently small, there is a system of bases $Y'_1, Y'_2, \dots, Y'_{n-1}$ of ${}^\circ T''$ over $M \cap U_\varepsilon(p)$, where ${}^\circ T''$ means the CR structure over $M \cap U_\varepsilon(p)$ induced from C^n . In our case, we can define a real vector field ζ , dual to

$$\sqrt{-1}\partial\rho,$$

where $\rho = \text{Im } z_n - h(z_1, \dots, z_{n-1}, \text{Re } z_n)$. And by using this ζ , we have a C^∞ vector bundle decomposition and so we have the Levi form. By the Schmidt orthogonal process, form $Y'_1, Y'_2, \dots, Y'_{n-1}$, we have a system of bases Y_1, Y_2, \dots, Y_{n-1} of ${}^\circ T''$ satisfying

$$-\sqrt{-1}[Y_i, \bar{Y}_j]_\zeta = \delta_{ij},$$

where $-\sqrt{-1}[Y_i, \bar{Y}_j]_\zeta$ means the coefficient of the ζ part of $[Y_i, \bar{Y}_j]$ according to the above C^∞ vector bundle decomposition. By using this Y_1, Y_2, \dots, Y_{n-1} , we put an L^2 -norm on

$$\Gamma(M \cap U_\varepsilon(p), \Lambda^p({}^\circ T'')^*).$$

Namely for u in $\Gamma(M \cap U_\varepsilon(p), \Lambda^p({}^\circ T'')^*)$, we have C^∞ functions u_I by

$$u_I = u(Y_{i_1}, \dots, Y_{i_p}), \quad I = (i_1, \dots, i_p).$$

By using these u_I , we set

$$\|u\|_{M \cap U_\varepsilon(p)}^2 = \sum_I \int_{B_1(0)} |u_I \circ h|^2 dx_1 \cdots dx_{2n-1},$$

where I runs through all ordered indices of length p and h is a diffeomorphism map from $M \cap U_\varepsilon(p)$ to $B_1(0)$ defined as above. Furthermore we must introduce several notations. Namely $\bar{\partial}_1^*$ denotes the adjoint operator of $\bar{\partial}_1$ with respect to the above L^2 -norm. And we set

$$b = \sqrt{\sum_{i=1}^{n-1} |Y_i t|^2},$$

where $t = 2 \text{Re}\{1/2\sqrt{-1}\}z_n + z_n^2$. And we set the characteristic curve C by

$$C = \{(z_1, \dots, z_n), (z_1, \dots, z_n) \in M \cap U_\varepsilon(p), \\ Y_i t = 0, \quad 1 \leq i \leq n-1\}.$$

Then in [K], Kuranishi obtained

$$\|(1/b)v\|_{M \cap U_\varepsilon(p)}^2 \leq c\{\|\bar{\partial}_b v\|_{M \cap U_\varepsilon(p)}^2 + \|\bar{\partial}_b^* v\|_{M \cap U_\varepsilon(p)}^2\}$$

for v in $\Gamma(M \cap U_\varepsilon(p) - C, ({}^\circ T'')^*)$ satisfying:

$$v(Y^0) = 0 \quad \text{on } \{(z_1, \dots, z_n): (z_1, \dots, z_n) \in M \cap U_\varepsilon(p) - C, \\ t = \varepsilon\},$$

where

$$Y^0 = \sum_{i=1}^{n-1} (\bar{Y}_i t / b) Y_i,$$

if $\dim_{\mathbb{R}} M = 2n - 1 \geq 7$. Actually, Kuranishi obtained the estimate more precisely. However, in this paper, we discuss this estimate. Then, the L^2 -solution operator $\bar{\partial}_b^* N_b$ satisfies

$$\|(1/b)(\bar{\partial}_b^* N_b v)\|_{M \cap U_\varepsilon(p)} \leq c \|v\|_{M \cap U_\varepsilon(p)}$$

for v in $\Gamma(M \cap U_\varepsilon(p) - C, (\circ T''^*))$, which is of L^2 . We show that an explicit solution obtained by Henkin and Harvey-Polking satisfies the similar estimate.

3. Rigid hypersurfaces in C^n . In this paper, we study the $\bar{\partial}_b$ -operator over a special kind of real hypersurfaces in C^n . Namely let

$$M = \{(z_1, \dots, z_n) : \text{Im } z_n = k(z_i, \bar{z}_j), 1 \leq i, j \leq n-1\},$$

where k is a real valued C^∞ function which depends only on z_i, \bar{z}_j , and not on z_n, \bar{z}_n satisfying:

$$k(0, 0) = 0 \quad \text{and} \quad dk(0, 0) = 0.$$

We call M satisfying these relations a rigid hypersurface. Let M be a rigid hypersurface. And let M be strongly pseudo convex near the origin. Then by a change of coordinates, the defining equation of M becomes

$$\text{Im } z_n'' = \sum_{i=1}^{n-1} |z_i''|^2 + \text{terms of higher order in } z_j'', \bar{z}_j'',$$

where $1 \leq j \leq n-1$.

4. Integral formula for $\bar{\partial}_b$ and the Romanov kernel. Let u, v be C^∞ functions from $C^n \times C^n$ to C^n ,

$$\begin{aligned} u(\zeta, z) &= (u_1(\zeta, z), \dots, u_n(\zeta, z)), \\ v(\zeta, z) &= (v_1(\zeta, z), \dots, v_n(\zeta, z)). \end{aligned}$$

We use the following notations:

$$\begin{aligned} u(\zeta, z)(\zeta - z) &= \sum_{j=1}^n u_j(\zeta, z)(\zeta_j - z_j), \\ u(\zeta, z) d(\zeta - z) &= \sum_{j=1}^n u_j(\zeta, z) d(\zeta_j - z_j), \\ \bar{\partial}u(\zeta, z) d(\zeta - z) &= \sum_{j=1}^n \bar{\partial}u_j(\zeta, z) \wedge d(\zeta_j - z_j), \end{aligned}$$

and we define the following kernels:

$$(4-1-1) \quad \Omega^u(\zeta, z) = (2\pi i)^{-n} ((u(\zeta, z) d(\zeta - z)) / (u(\zeta, z)(\zeta - z))) \wedge ((\bar{\partial}u(\zeta, z) d(\zeta - z)) / (u(\zeta, z)(\zeta - z)))^{n-1},$$

$$(4-1-2) \quad \Omega^v(\zeta, z) = (2\pi i)^{-n} ((v(\zeta, z) d(\zeta - z)) / (v(\zeta, z)(\zeta - z))) \wedge ((\bar{\partial}v(\zeta, z) d(\zeta - z)) / (v(\zeta, z)(\zeta - z)))^{n-1},$$

$$(4-1-3) \quad \begin{aligned} \Omega^{u,v}(\zeta, z) &= (2\pi i)^{-n} ((u(\zeta, z) d(\zeta, z)) / (u(\zeta, z)(\zeta - z))) \\ &\quad \wedge ((\bar{\partial}v(\zeta, z) d(\zeta - z)) / (v(\zeta, z)(\zeta - z))) \\ &\quad \wedge \sum_{j+k=n-2} ((\bar{\partial}u(\zeta, z) d(\zeta - z)) / (u(\zeta, z)(\zeta - z)))^j \\ &\quad \wedge ((\bar{\partial}v(\zeta, z) d(\zeta - z)) / (v(\zeta, z)(\zeta - z)))^k. \end{aligned}$$

Then as is well known, in [B] and [BS], we have

$$\begin{aligned} \bar{\partial}\Omega^{u,v}(\zeta, z) &= \Omega^v(\zeta, z) - \Omega^u(\zeta, z), \\ \bar{\partial}\Omega^v(\zeta, z) &= 0. \end{aligned}$$

Let M be as in §1 in this paper. Then we can define *formally*

$$\begin{aligned} R_M(u, v)(\phi)(z) &:= \left\{ \int_{\zeta \in M} \Omega^{u,v}(\zeta, z) \wedge \phi(\zeta) \right\}_{T_M}, \\ L(u)(\phi)(z) &:= \int_{\zeta \in M} \Omega^u(\zeta, z) \wedge \phi(\zeta), \end{aligned}$$

for $\phi \in \mathcal{D}^{0,1}(M \cap U)$, where $\{ \}_{T_M}$ means the tangential part of $\{ \}^{\bar{\partial}}$. Of course without any assumption for u, v and M , the operators R_M, L do not make sense. However if we assume that u is a local support function for (M, D) at a point p (for the definition, see 2.4 Definition in [BS]), then $R_M(u, v)(\phi), L(u)(\phi)$ make sense. And

furthermore, the boundary value of $L(u)(\phi)$ from D^- and D^+ exists respectively, where D means U and

$$\begin{aligned} D^+ &= \{z: z \in C^n, \rho(z) > 0\}, \\ D^- &= \{z: z \in C^n, \rho(z) < 0\}. \end{aligned}$$

And for $\phi \in \mathcal{D}^{0,1}(M \cap U)$,

$$\begin{aligned} \phi &= -(\bar{\partial}_b R_M(u, v)(\phi) + R_M(u, v)\bar{\partial}_b \phi) \\ &\quad + L_M^+(v)(\phi) - L_M^-(u)(\phi) \quad \text{on } M \cap U. \end{aligned}$$

Note from this equality, the terms $L_M^+(v)(\phi)$ and $L_M^-(u)(\phi)$ are obstructions to solving the equations $\bar{\partial}_b g = \phi$. If we set

$$u_j(\zeta, z) = \partial \rho / \partial \zeta_j(\zeta), \quad v_j(\zeta, z) = -\partial \rho / \partial z_j(z), \quad 1 \leq j \leq n,$$

then $u(\zeta, z) = (u_1(\zeta, z), \dots, u_n(\zeta, z))$ and $v(\zeta, z) = (v_1(\zeta, z), \dots, v_n(\zeta, z))$ are local support functions for (M, D^-) and (M, D^+) respectively. And in the case,

$$\begin{aligned} L_M^-(u)(\phi) &= 0 \quad \text{unless } \phi \in \mathcal{D}^{p,0}(M \cap U), \\ L_M^+(v)(\phi) &= 0 \quad \text{unless } \phi \in \mathcal{D}^{p,n-1}(M \cap U). \end{aligned}$$

And so we have: for $\phi \in \mathcal{D}^{p,1}(M \cap U)$,

$$\phi = -\{\bar{\partial}_b R_M(u, v)(\phi) + R_M(u, v)(\bar{\partial}_b \phi)\},$$

if $n \geq 3$.

Henceforth, we abbreviate R for $R_M(u, v)$, where u and v are defined as above, and $R\phi$ stands for $R_M(u, v)(\phi)(z)$.

5. Kuranishi's L^2 -estimate for the Romanov kernel. In §4, we see that the Romanov kernel R is a certain kind of the solution operator for $\bar{\partial}_b$. Concerning this R kernel, in this section, we show an L^2 -estimate which the L^2 solution satisfies. Namely, we show

MAIN THEOREM. *For any ϕ in $\Gamma(M \cap U_\varepsilon(p) - C, (\circ T^n)^*)$, which is of L^2 , and for any $\delta < 1$, we have:*

$$\|(1/b^\delta)R\phi\|_{M \cap U_\varepsilon(p)} \leq C_\delta \|\phi\|_{M \cap U_\varepsilon(p)},$$

where C_δ depends only on δ .

In order to prove the main theorem, we first show

LEMMA 5.1.

$$C_1 \sqrt{\sum_{i=1}^{n-1} |z_i''|^2} \leq b \leq C_2 \sqrt{\sum_{i=1}^{n-1} |z_i''|^2},$$

where C_1, C_2 are positive constants, and b is defined by

$$b = \sqrt{\sum_{i=1}^{n-1} |Y_i'' t|^2},$$

where $\{Y_i''\}_{1 \leq i \leq n-1}$ is obtained from $\{Y_i\}_{1 \leq i \leq n-1}$, by the Schmidt orthogonal process, and

$$Y_i = \partial / \partial \bar{z}_i'' - (\rho_i^- / \rho_n^-) \partial / \partial \bar{z}_n'', \quad 1 \leq i \leq n-1,$$

$$\rho = \text{Im } z_n'' - \sum_{i=1}^{n-1} |z_i''|^2 - Q(z_j'', \bar{z}_j''),$$

where $\{z_i''\}_{1 \leq i \leq n}$ means the coordinate obtained in §3 in this paper.

Proof of Lemma 5.1. By the construction of Y_i'' , Y_i'' is a linear combination of Y_j , $1 \leq j \leq n$, satisfying:

$$Y_i'' := \sum_{j=1}^{n-1} a_{ji} Y_j,$$

where a_{ji} is a C^∞ function over $M \cap U_\varepsilon(p)$ and $a_{ji}(p) = 0$. So

$$Y_i'' t = Y_i t + \sum_{j=1}^{n-1} a_{ji} Y_j t.$$

While

$$\begin{aligned} Y_j t &= (\partial / \partial \bar{z}_j'' - (\rho_j^- / \rho_n^-) \partial / \partial \bar{z}_n'') 2 \text{Re}\{(1/2\sqrt{-1})z_n'' + z_n''^2\} \\ &= z_j''(1 + 4\sqrt{-1}z_n''). \end{aligned}$$

Therefore we have our lemma. □

And we have

LEMMA 5.2. *There is a constant c satisfying:*

$$\int_{\zeta \in M \cap U_\varepsilon(p)} (1/b^\delta) |\Omega^{u,v}(\zeta, z)| dV_\zeta \leq c \quad \text{for } z \text{ in } U_\varepsilon(p).$$

This lemma is proved in [HP]. So we briefly sketch the proof. For a system of coordinates of $M \cap U_\varepsilon(p)$, we can adopt $(z_1'', \dots, z_{n-1}'', t)_{\frac{3}{2}}$ which we constructed in §3 in this paper, where $t = \text{Re } z_n''$. Then over $M \cap U_\varepsilon(p)$,

$$c_1 \left(|t| + \sum_{i=1}^{n-1} |z_i''|^2 \right) \leq |z_n''| \leq c_2 \left(|t| + \sum_{i=1}^{n-1} |z_i''|^2 \right),$$

where c_1, c_2 are positive constants. So over $M \cap U_\varepsilon(p)$,

$$c_3 \left(|t| + \sum_{i=1}^{n-1} |z''_i|^2 \right) \leq |u(\zeta - z'')| \leq c_4 \left(|t| + \sum_{i=1}^{n-1} |z''_i|^2 \right),$$

where c_3, c_4 are positive constants. And

$$c_5 \left(|t| + \sum_{i=1}^{n-1} |z''_i|^2 \right) \leq |v(\zeta - z'')| \leq c_6 \left(|t| + \sum_{i=1}^{n-1} |z''_i|^2 \right),$$

where c_5, c_6 are positive constants. And

$$u d(\zeta - z) \wedge v d(\zeta - z) = Q(|\zeta - z|).$$

So each coefficient of $(1/b^\delta)R$ is dominated by

$$\left(\sum_{i=1}^{n-1} |z''_i|^2 \right)^{-(\delta/2)} \left(|t| + \sqrt{\sum_{i=1}^{n-1} |z''_i|^2} \right) \left(|t| + \sum_{i=1}^{n-1} |z''_i|^2 \right)^{-n}.$$

And this is locally integrable on $C^{n-1} \times R$ if $\delta < 1$. In fact, by using polar coordinates, we compute the following integral. We set

$$\begin{aligned} x_1 &= r \cos \theta_1 \cos \theta_2 \cdots \cos \theta_{2n-3} \cos \theta_{2n-2}, \\ y_1 &= r \cos \theta_1 \cos \theta_2 \cdots \cos \theta_{2n-3} \sin \theta_{2n-2}, \\ x_2 &= r \cos \theta_1 \cdots \cos \theta_{2n-4} \sin \theta_{2n-3}, \\ y_2 &= r \cos \theta_1 \cdots \sin \theta_{2n-4}, \\ &\dots \\ x_{n-1} &= r \cos \theta_1 \sin \theta_2, \\ y_{n-1} &= r \sin \theta_1, \end{aligned}$$

where $z''_j = x_j + \sqrt{-1}y_j$, $1 \leq j \leq n-1$. Then

$$\begin{aligned} &\left(\sum_{i=1}^{n-1} |z''_i|^2 \right)^{-(\delta/2)} \left(|t| + \sqrt{\sum_{i=1}^{n-1} |z''_i|^2} \right) \left(|t| + \sum_{i=1}^{n-1} |z''_i|^2 \right)^{-n} \\ &= r^{-\delta} (t+r)(t+r^2)^{-n}. \end{aligned}$$

So

$$\begin{aligned}
& \int_{M \cap U_\varepsilon(p)} \left(\sum_{i=1}^{n-1} |z_i''|^2 \right)^{-(\delta/2)} \left(|t| + \sqrt{\sum_{i=1}^{n-1} |z_i''|^2} \right) \\
& \quad \times \left(|t| + \sum_{i=1}^{n-1} |z_i''|^2 \right) dV_{z,t} \\
& \leq \int_0^\varepsilon \int_0^\infty r^{-\delta} (t+r)(t+r^2)^{-n} r^{2n-3} dt dr \\
& = \int_0^\varepsilon \int_0^\infty \left\{ (1/(t+r^2)^{n-1}) r^{2n-3-\delta} \right. \\
& \quad \left. + (1/(t+r^2)^n) ((r-r^2)/r^\delta) r^{2n-3} \right\} dt dr.
\end{aligned}$$

While

$$\begin{aligned}
& \int_0^\infty (1/(t+r^2)^{n-1}) r^{2n-3-\delta} dt \\
& = -(1/(n-2)) [(1/(t+r^2)^{n-2}) r^{2n-3-\delta}]_0^\infty \\
& = (1/(n-2)) r^{1-\delta},
\end{aligned}$$

$$\begin{aligned}
& \int_0^\infty (1/(t+r^2)^n) ((r-r^2)/r^\delta) r^{2n-3} dt \\
& = -(1/(n-1)) [(1/(t+r^2)^{n-1} (1-r)) r^{2n-2-\delta}]_0^\infty \\
& = (1/(n-1)) (1-r) r^{-\delta}.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \int_{M \cap U_\varepsilon(p)} \left(\sum_{i=1}^{n-1} |z_i''|^2 \right)^{-(\delta/2)} \left(|t| + \sqrt{\sum_{i=1}^{n-1} |z_i''|^2} \right) \left(|t| + \sum_{i=1}^{n-1} |z_i''|^2 \right) dV_{z,t} \\
& \leq \int_0^\varepsilon \left((1/(n-2)) r^{1-\delta} + (1/(n-1)) r^{-\delta} - (1/(n-1)) r^{1-\delta} \right) dr \\
& = (1/((n-2)(2-\delta))) \varepsilon^{1-(\delta/2)} \\
& \quad + (1/((n-1)(1-\delta))) \varepsilon^{(1/2)-(\delta/2)} \\
& \quad - (1/((n-1)(2-\delta))) \varepsilon^{1-(\delta/2)}.
\end{aligned}$$

Therefore we have our lemma. \square

Now we prove our main theorem.

$$\begin{aligned}
 & \int_{M \cap U_\varepsilon(p)} (1/b^{2\delta}) |R_M(u, v)(\phi)|^2 dV \\
 & \leq \int_{M \cap U_\varepsilon(p)} \left\{ (1/b^{2\delta}) \left(\int_{M \cap U_\varepsilon(p)} \Omega^{u, v}(\zeta, z) \phi(\zeta) dV_\zeta \right)^2 \right\} dV_z \\
 & \leq \int_{M \cap U_\varepsilon(p)} \left(\int_{M \cap U_\varepsilon(p)} |(1/b^\delta) \Omega^{u, v}(\zeta, z) \phi(\zeta)| dV_\zeta \right)^2 dV_z \\
 & \leq \int_{M \cap U_\varepsilon(p)} \left\{ \left(\int_{M \cap U_\varepsilon(p)} |(1/b^\delta) \Omega^{u, v}(\zeta, z)| dV_\zeta \right) \right. \\
 & \quad \left. \times \left(\int_{M \cap U_\varepsilon(p)} |(1/b^\delta) \Omega^{u, v}(\zeta, z)| |\phi(\zeta)|^2 dV_\zeta \right) \right\} dV_z \\
 & \hspace{15em} \text{(by the Schwarz lemma)} \\
 & \leq c^2 \int_{M \cap U_\varepsilon(p)} |\phi(\zeta)|^2 dV_\zeta \quad \text{(by Lemma 5.2 in this paper)} \\
 & \leq c^2 \|\phi\|_{M \cap U_\varepsilon(p)}^2.
 \end{aligned}$$

So we have our theorem. \square

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