ON THE ROMANOV KERNEL AND KURANISHI'S $L^2$-ESTIMATE FOR $\bar{\partial}_b$ OVER A BALL IN THE STRONGLY PSEUDO CONVEX BOUNDARY

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As is proved by Kerzman-Stein, over a compact strongly pseudo 
convex boundary in $C^n$, Szegö projection $S$ is the operator defined 
by Henkin-Ramirez modulo compact operators. While, over a special 
ball, $U_\varepsilon$, in the strongly pseudo convex boundary, in order to obtain 
a local embedding theorem of CR-structures, Kuranishi constructed 
the Neumann type operator $N_b$ for $\bar{\partial}_b$ and so we have a local Szegö 
operator by 

$$S_{U_\varepsilon} = \text{id} - \overline{\partial}_b^* N_b \overline{\partial}_b \quad \text{on } U_\varepsilon,$$

where $\overline{\partial}_b^*$ means the adjoint operator of $\overline{\partial}_b$. There might be a rela-
tion between $S_{U_\varepsilon}$ and the Romanov kernel like the case of the Szegö 
operator and the Henkin-Ramirez kernel. We study this problem and 
show some estimates for the Romanov kernel.

0. Introduction. Let $(M, \circ T'')$ be an abstract strongly pseudo con-
 vex CR-manifold. Then as is well known, if $\dim_R M = 2n - 1 \geq 7$, $(M, \circ T'')$ is locally embeddable in a complex euclidean space 
$C^n((Ak3), (K))$. In the proof of this local embedding theorem, it is 
shown that: over a special ball in the strongly pseudo convex bound-
 ary, an $L^2$-estimate for $\overline{\partial}_b$, which is stronger than the standard $L^2$-
estimate, is established and so the $L^2$-solution operator for $\overline{\partial}_b$ is 
obtained. This operator plays an essential role in our local embed-
ding theorem. Therefore it must be important to study this solution 
operator for $\overline{\partial}_b$ precisely.

In order to get a solution operator, there exists another method. 
By using an integral formula, a local solution operator for $\overline{\partial}_b$ is con-
structed explicitly by Henkin and Harvey-Polking. Obviously, these 
solution operators are different. And it seems quite interesting to study 
the relation between the $L^2$-solution for $\overline{\partial}_b$ and the explicit solution, 
obtained by using an integral formula. We recall the $\partial$-case over a 
strongly pseudo convex domain in $C^n$. In this case, the explicit solu-
tion, constructed by Lieb and Range, is a certain kind of the essential 
part of the Kohn's $L^2$-solution. Therefore we could hope for a sim-
lar result in the $\overline{\partial}_b$ case over a special ball in the strongly pseudo
convex boundary. As mentioned already, our $L^2$-a priori estimate is different from the standard $L^2$-estimate. Therefore in the above sense, it seems to be natural to consider that the explicit solution operator would satisfy the similar $L^2$-estimate. In this paper, we discuss this point over rigid hypersurfaces in $\mathbb{C}^n$ (for the definition, see §3 in this paper). And we prove our a priori estimate (Main Theorem in §5 in this paper) for the explicit solution operator.

1. CR-structure and $\bar{\partial}_b$-operator. Let $M$ be a real hypersurface in $\mathbb{C}^n$. Let $p$ be a reference point of $M$. We assume that $p$ is a smooth point, namely let $\rho$ be a defining function of $M$ in a neighborhood of $p$ in $\mathbb{C}^n$, i.e., there is a neighborhood $V(p)$ of $p$ satisfying:

$$M \cap V(p) = \{q : q \in V(p), \rho(q) = 0\}$$

and

$$d\rho \neq 0 \quad \text{over } M \cap V(p).$$

Then over $M \cap V(p)$, we can introduce an CR-structure induced from $\mathbb{C}^n$. Namely, let

$$\mathcal{O}_\theta'' = \mathcal{T}\mathbb{C}^n \cap C \otimes TM \quad \text{over } M \cap V(p).$$

Then this $\mathcal{O}_\theta''$ satisfies

(1-1) $\mathcal{O}_\theta'' \cap \mathcal{O}_\theta'' = 0, \quad f\text{-dim}_C(C \otimes TM/(\mathcal{O}_\theta'' + \mathcal{O}_\theta'')) = 1,$

(1-2) $[\Gamma(M \cap V(p), \mathcal{O}_\theta''), \Gamma(M \cap V(p), \mathcal{O}_\theta'') \subset \Gamma(M \cap V(p), \mathcal{O}_\theta'').$

This pair $(M \cap V(p), \mathcal{O}_\theta'')$ is called a CR-structure, or a CR-manifold.

Let $(M \cap V(p), \mathcal{O}_\theta'')$ be a CR-manifold. We introduce a $C^\infty$ vector bundle decomposition

(1-3) $C \otimes TM = \mathcal{O}_\theta'' + \bar{\partial}_\theta'' + C\zeta,$

where

(1-3-1) $\zeta$ is a real vector field,

(1-3-2) $\zeta_q \notin \mathcal{O}_\theta'' + \bar{\partial}_\theta''$ for $q$ in $M \cap V(p)$.

By using this decomposition, we have a Levi form

$$L(X, Y) = \sqrt{-1}[X, \bar{Y}]_\zeta \quad \text{for } X, Y \text{ in } \Gamma(M \cap V(p), \mathcal{O}_\theta''),$$

where $[X, \bar{Y}]_\zeta$ means the $\zeta$-part of $[X, \bar{Y}]$ according to (1-3). As is well known, this map $L$ makes sense for elements $X, Y$ in $\mathcal{O}_\theta''$. And if this Levi form is positive or negative definite, $(M \cap V(p), \mathcal{O}_\theta'')$
is called a strongly pseudo convex real hypersurface. Next we briefly explain $\overline{\partial}_b$-complex. For $u$ in $\Gamma(M \cap V(p), C)$, we set
\[ \overline{\partial}_b u(x) = Xu \quad \text{for } X \text{ in } \mathcal{O}T^m, \]
where $\Gamma(M \cap V(p), c)$ means the spacing consisting of $C^\infty$ functions over $M \cap V(p)$. Namely we have a first order differential operator
\[ \overline{\partial}_b : \Gamma(M \cap V(p), c) \rightarrow \Gamma(M \cap V(p), (\mathcal{O}T^m)^*). \]
By the same way as for usual differential forms, we have
\[ \overline{\partial}_b^{(p+1)} \circ \overline{\partial}_b^{(p)} = 0. \]

2. Kuranishi's $L^2$-estimate. Let $(M, \mathcal{O}T^m)$ be a strongly pseudo convex CR manifold, embedded as a real hypersurface in $C^n$. Let $p$ be a reference point of $M$. Then by a change of coordinates, we can assume that there is a neighborhood $W(p)$ of $p$ in $C^n$, satisfying:
\[ M \cap W(p) = \{(z_1, \ldots, z_n): (z_1, \ldots, z_n) \in W(p), \]
\[ \text{Im} z_n = h(z_1, \ldots, z_{n-1}, \text{Re} z_n)\}, \]
where $z_i(p) = 0$, $1 \leq i \leq n - 1$, and $h$ is a real valued $C^\infty$ function, and
\[ (\partial^2 h/\partial z_i \partial \overline{z}_j)(0) = \delta_{ij}, \quad 1 \leq i, j \leq n - 1, \]
\[ (\partial^2 h/\partial z_i \partial z_j)(0) = \delta_{ij}, \quad 1 \leq i, j \leq n - 1, \]
\[ dh(0, \ldots, 0) = 0. \]
In this set up, we introduce a neighborhood $M \cap U_\varepsilon(p)$ of $p$ as follows:
\[ M \cap U_\varepsilon(p) = \{(z_1, \ldots, z_n): (z_1, \ldots, z_n) \in W(p), \]
\[ \text{Im} z_n = h(z_1, \ldots, z_{n-1}, \text{Re} z_n), \]
\[ 2 \text{Re}\{(1/2\sqrt{-1})z_n + z_n^2\} < \varepsilon\}. \]
Now we briefly sketch Kuranishi's $L^2$-estimate over $M \cap U_\varepsilon(p)$. Obviously by the above assumption, our $M \cap U_\varepsilon(p)$ is diffeomorphic to the real $2n - 1$ dimensional ball. We denote this diffeomorphism map by $h$ and we fix this. If $\varepsilon$ is chosen sufficiently small, there is a system of bases $Y_1', Y_2', \ldots, Y_{n-1}'$ of $\mathcal{O}T^m$ over $M \cap U_\varepsilon(p)$, where $\mathcal{O}T^m$ means the CR structure over $M \cap U_\varepsilon(p)$ induced from $C^n$. In our case, we can define a real vector field $\zeta$, dual to
\[ \sqrt{-1}\partial \rho, \]
where \( \rho = \text{Im} z_n - h(z_1, \ldots, z_{n-1}, \text{Re} z_n) \). And by using this \( \zeta \), we have a \( C^\infty \) vector bundle decomposition and so we have the Levi form. By the Schmidt orthogonal process, form \( Y'_1, Y'_2, \ldots, Y'_{n-1} \), we have a system of bases \( Y_1, Y_2, \ldots, Y_{n-1} \) of \( ^oT'' \) satisfying

\[
-\sqrt{-1}[Y_i, \overline{Y}_j]_\zeta = \delta_{ij},
\]

where \( -\sqrt{-1}[Y_i, \overline{Y}_j]_\zeta \) means the coefficient of the \( \zeta \) part of \([Y_i, \overline{Y}_j]\) according to the above \( C^\infty \) vector bundle decomposition. By using this \( Y_1, Y_2, \ldots, Y_{n-1} \), we put an \( L^2 \)-norm on

\[
\Gamma(M \cap U_\varepsilon(p), \Lambda^p(^oT'')^*).\]

Namely for \( u \) in \( \Gamma(M \cap U_\varepsilon(p), \Lambda^p(^oT'')^*) \), we have \( C^\infty \) functions \( u_I \) by

\[
u_I = u(Y_{i_1}, \ldots, Y_{i_p}), \quad I = (i_1, \ldots, i_p).
\]

By using these \( u_I \), we set

\[
\|u\|_{\mathcal{M} \cap U_\varepsilon(p)}^2 = \sum_I \int_{B_1(0)} |u_I \circ h|^2 dx_1 \cdots dx_{2n-1},
\]

where \( I \) runs through all ordered indices of length \( p \) and \( h \) is a diffeomorphism map from \( M \cap U_\varepsilon(p) \) to \( B_1(0) \) defined as above. Furthermore we must introduce several notations. Namely \( \overline{\partial}_1^* \) denotes the adjoint operator of \( \overline{\partial}_1 \) with respect to the above \( L^2 \)-norm. And we set

\[
b = \sqrt{\sum_{i=1}^{n-1} |Y_i t|^2},
\]

where \( t = 2 \text{Re}\{1/2\sqrt{-1})z_n + z_n^2\} \). And we set the characteristic curve \( C \) by

\[
C = \{(z_1, \ldots, z_n), (z_1, \ldots, z_n) \in M \cap U_\varepsilon(p), \quad Y_i t = 0, \quad 1 \leq i \leq n-1\}.
\]

Then in [K], Kuranishi obtained

\[
\|(1/b)v\|^2_{\mathcal{M} \cap U_\varepsilon(p)} \leq c\{\|\overline{\partial}_b v\|^2_{\mathcal{M} \cap U_\varepsilon(p)} + \|\overline{\partial}_b^* v\|^2_{\mathcal{M} \cap U_\varepsilon(p)}\}
\]

for \( v \) in \( \Gamma(M \cap U_\varepsilon(p) - C, (^oT'')^*) \) satisfying:

\[
u(Y^0) = 0 \quad \text{on} \quad \{(z_1, \ldots, z_n): (z_1, \ldots, z_n) \in M \cap U_\varepsilon(p) - C, \quad t = \varepsilon\},
\]
where

\[ Y^0 = \sum_{i=1}^{n-1} (\overline{Y}_i t/b) Y_i, \]

if \( \dim_R M = 2n - 1 \geq 7 \). Actually, Kuranishi obtained the estimate more precisely. However, in this paper, we discuss this estimate. Then, the \( L^2 \)-solution operator \( \overline{\partial}_b N_b \) satisfies

\[
\|(1/b)(\overline{\partial}_b N_b v)\|_{M \cap U_\varepsilon(p)} \leq c\|v\|_{M \cap U_\varepsilon(p)}
\]

for \( v \) in \( \Gamma(M \cap U_\varepsilon(p) - C, (\mathcal{O} T^n)^*) \), which is of \( L^2 \). We show that an explicit solution obtained by Henkin and Harvey-Polking satisfies the similar estimate.

3. Rigid hypersurfaces in \( C^n \). In this paper, we study the \( \overline{\partial}_b \)-operator over a special kind of real hypersurfaces in \( C^n \). Namely let

\[ M = \{(z_1, \ldots, z_n): \text{Im} z_n = k(z_i, \overline{z}_j), \ 1 \leq i, j \leq n-1\}, \]

where \( k \) is a real valued \( C^\infty \) function which depends only on \( z_i, \overline{z}_j \), and not on \( z_n, \overline{z}_n \) satisfying:

\[ k(0, 0) = 0 \quad \text{and} \quad dk(0, 0) = 0. \]

We call \( M \) satisfying these relations a rigid hypersurface. Let \( M \) be a rigid hypersurface. And let \( M \) be strongly pseudo convex near the origin. Then by a change of coordinates, the defining equation of \( M \) becomes

\[ \text{Im} z''_n = \sum_{i=1}^{n-1} |z''_i|^2 + \text{terms of higher order in } z''_j, \overline{z''}_j, \]

where \( 1 \leq j \leq n - 1 \).

4. Integral formula for \( \overline{\partial}_b \) and the Romanov kernel. Let \( u, v \) be \( C^\infty \) functions from \( C^n \times C^n \) to \( C^n \),

\[
\begin{align*}
    u(\zeta, z) &= (u_1(\zeta, z), \ldots, u_n(\zeta, z)), \\
    v(\zeta, z) &= (v_1(\zeta, z), \ldots, v_n(\zeta, z)).
\end{align*}
\]
We use the following notations:
\[
\begin{align*}
    u(\zeta, z)(\zeta - z) &= \sum_{j=1}^{n} u_j(\zeta, z)(\zeta_j - z_j), \\
    u(\zeta, z) d(\zeta - z) &= \sum_{j=1}^{n} u_j(\zeta, z) d(\zeta_j - z_j), \\
    \overline{\delta} u(\zeta, z) d(\zeta - z) &= \sum_{j=1}^{n} \overline{\delta} u_j(\zeta, z) \wedge d(\zeta_j - z_j),
\end{align*}
\]
and we define the following kernels:
\[
\begin{align*}
    (4-1-1) \quad \Omega^u(\zeta, z) &= (2\pi i)^{-n} \left(\frac{u(\zeta, z) d(\zeta - z)}{u(\zeta, z)(\zeta - z)}\right) \\
    &\quad \wedge \left(\frac{(\overline{\delta} u(\zeta, z) d(\zeta - z))}{(u(\zeta, z)(\zeta - z))}\right)^{n-1}, \\
    (4-1-2) \quad \Omega^v(\zeta, z) &= (2\pi i)^{-n} \left(\frac{v(\zeta, z) d(\zeta - z)}{v(\zeta, z)(\zeta - z)}\right) \\
    &\quad \wedge \left(\frac{(\overline{\delta} v(\zeta, z) d(\zeta - z))}{(v(\zeta, z)(\zeta - z))}\right)^{n-1}, \\
    (4-1-3) \quad \Omega^{u,v}(\zeta, z) &= (2\pi i)^{-n} \left(\frac{(u(\zeta, z) d(\zeta - z))}{(u(\zeta, z)(\zeta - z))}\right) \\
    &\quad \wedge \sum_{j+k=n-2} \left(\frac{(\overline{\delta} u(\zeta, z) d(\zeta - z))}{(u(\zeta, z)(\zeta - z))}\right)^{j} \\
    &\quad \wedge \left(\frac{(\overline{\delta} v(\zeta, z) d(\zeta - z))}{(v(\zeta, z)(\zeta - z))}\right)^{k}.
\end{align*}
\]

Then as is well known, in [B] and [BS], we have
\[
\overline{\delta} \Omega^{u,v}(\zeta, z) = \Omega^v(\zeta, z) - \Omega^u(\zeta, z),
\]
\[
\overline{\delta} \Omega^v(\zeta, z) = 0.
\]
Let \( M \) be as in §1 in this paper. Then we can define formally
\[
R_M(u, v)(\phi)(z) := \left\{ \int_{\zeta \in M} \Omega^{u,v}(\zeta, z) \wedge \phi(\zeta) \right\}_{T_M},
\]
\[
L(u)(\phi)(z) := \int_{\zeta \in M} \Omega^u(\zeta, z) \wedge \phi(\zeta),
\]
for \( \phi \in \mathcal{D}^{0,1}(M \cap U) \), where \( \{ \}_{T_M} \) means the tangential part of \( \{ \} \).
Of course without any assumption for \( u, v \) and \( M \), the operators \( R_M, L \) do not make sense. However if we assume that \( u \) is a local support function for \((M, D)\) at a point \( p \) (for the definition, see 2.4 Definition in [BS]), then \( R_M(u, v)(\phi), L(u)(\phi) \) make sense. And
furthermore, the boundary value of $L(u)(\phi)$ from $D^-$ and $D^+$ exists respectively, where $D$ means $U$ and

$$
D^+ = \{ z : z \in \mathbb{C}^n, \rho(z) > 0 \},
$$

$$
D^- = \{ z : z \in \mathbb{C}^n, \rho(z) > 0 \}.
$$

And for $\phi \in \mathcal{D}^{0,1}(M \cap U)$,

$$
\phi = - (\overline{\partial}_b R_M(u, v)(\phi) + R_M(u, v)\overline{\partial}_b \phi) + L_M^+(v)(\phi) - L_M^-(u)(\phi) \quad \text{on } M \cap U.
$$

Note from this equality, the terms $L_M^+(v)(\phi)$ and $L_M^-(u)(\phi)$ are obstructions to solving the equations $\overline{\partial}_b g = \phi$. If we set

$$
u_j(\zeta, z) = \partial \rho / \partial \zeta_j(\zeta), \quad v_j(\zeta, z) = -\partial \rho / \partial z_j(z), \quad 1 \leq j \leq n,
$$
then $u(\zeta, z) = (u_1(\zeta, z), \ldots, u_n(\zeta, z))$ and $v(\zeta, z) = (v_1(\zeta, z), \ldots, v_n(\zeta, z))$ are local support functions for $(M, D^-)$ and $(M, D^+)$ respectively. And in the case,

$$L_M^-(u)(\phi) = 0 \quad \text{unless } \phi \in \mathcal{D}^{p,0}(M \cap U),
$$

$$L_M^+(v)(\phi) = 0 \quad \text{unless } \phi \in \mathcal{D}^{p,n-1}(M \cap U).
$$

And so we have: for $\phi \in \mathcal{D}^{p,1}(M \cap U)$,

$$\phi = -\{\overline{\partial}_b R_M(u, v)(\phi) + R_M(u, v)(\overline{\partial}_b \phi)\},
$$

if $n \geq 3$.

Henceforth, we abbreviate $R$ for $R_M(u, v)$, where $u$ and $v$ are defined as above, and $R\phi$ stands for $R_M(u, v)(\phi)(z)$.

5. Kuranishi's $L^2$-estimate for the Romanov kernel. In §4, we see that the Romanov kernel $R$ is a certain kind of the solution operator for $\overline{\partial}_b$. Concerning this $R$ kernel, in this section, we show an $L^2$-estimate which the $L^2$ solution satisfies. Namely, we show

**MAIN THEOREM.** *For any $\phi$ in $\Gamma(M \cap U_\varepsilon(p) - C, (\mathcal{O}''T'')^*)$, which is of $L^2$, and for any $\delta < 1$, we have:*

$$
\| (1/b^\delta) R\phi \|_{M \cap U_\varepsilon(p)} \leq C_\delta \| \phi \|_{M \cap U_\varepsilon(p)},
$$

*where $C_\delta$ depends only on $\delta$.*

In order to prove the main theorem, we first show

**LEMMA 5.1.**

$$
C_1 \sqrt{\sum_{i=1}^{n-1} |z''_i|^2} \leq b \leq C_2 \sqrt{\sum_{i=1}^{n-1} |z''_i|^2},
$$

$$
C_1 \sqrt{\sum_{i=1}^{n-1} |z''_i|^2} \leq b \leq C_2 \sqrt{\sum_{i=1}^{n-1} |z''_i|^2},
$$
where \( C_1, C_2 \) are positive constants, and \( b \) is defined by

\[
b = \sqrt{\sum_{i=1}^{n-1} |Y''_i t|^2},
\]

where \( \{Y''_i\}_{1 \leq i \leq n-1} \) is obtained from \( \{Y_i\}_{1 \leq i \leq n-1} \), by the Schmidt orthogonal process, and

\[
Y_i = \partial / \partial \overline{z''_i} - (\rho_i^- / \rho_n^-) \partial / \partial \overline{z''_n}, \quad 1 \leq i \leq n - 1,
\]

\[
\rho = \text{Im} z''_n - \sum_{i=1}^{n-1} |z''_i|^2 - Q(z''_i, \overline{z''_j}),
\]

where \( \{z''_i\}_{1 \leq i \leq n} \) means the coordinate obtained in §3 in this paper.

**Proof of Lemma 5.1.** By the construction of \( Y''_i, Y''_j \), \( Y''_i \) is a linear combination of \( Y_j, 1 \leq j \leq n \), satisfying:

\[
Y''_i := \sum_{j=1}^{n-1} a_{ji} Y_j,
\]

where \( a_{ji} \) is a \( C^\infty \) function over \( M \cap U_\epsilon(p) \) and \( a_{ji}(p) = 0 \). So

\[
Y''_i t = Y_i t + \sum_{j=1}^{n-1} a_{ji} Y_j t.
\]

While

\[
Y_j t = (\partial / \partial \overline{z''_j} - (\rho_i^- / \rho_n^-) \partial / \partial \overline{z''_n}) 2 \text{Re}\{(1/2\sqrt{-1})z''_n + z''_n^2\}
\]

\[
= z''_j (1 + 4\sqrt{-1} z''_n).
\]

Therefore we have our lemma. \( \square \)

And we have

**Lemma 5.2.** There is a constant \( c \) satisfying:

\[
\int_{\zeta \in M \cap U_\epsilon(p)} (1/b^\delta)|\Omega^{\mu, \nu}(\zeta, z)|dV_\zeta \leq c \quad \text{for } z \text{ in } U_\epsilon(p).
\]

This lemma is proved in [HP]. So we briefly sketch the proof. For a system of coordinates of \( M \cap U_\epsilon(p) \), we can adopt \( (z''_1, \ldots, z''_{n-1}, t)_\zeta \), which we constructed in §3 in this paper, where \( t = \text{Re } z''_n \). Then over \( M \cap U_\epsilon(p) \),

\[
c_1 \left( |t| + \sum_{i=1}^{n-1} |z''_i|^2 \right) \leq |z''_n| \leq c_2 \left( |t| + \sum_{i=1}^{n-1} |z''_i|^2 \right),
\]
where \( c_1, c_2 \) are positive constants. So over \( M \cap U_\varepsilon(p) \),
\[
c_3 \left( |t| + \sum_{i=1}^{n-1} |z_i''|^2 \right) \leq |u(\zeta - z'')| \leq c_4 \left( |t| + \sum_{i=1}^{n-1} |z_i''|^2 \right),
\]
where \( c_3, c_4 \) are positive constants. And
\[
c_5 \left( |t| + \sum_{i=1}^{n-1} |z_i''|^2 \right) \leq |v(\zeta - z'')| \leq c_6 \left( |t| + \sum_{i=1}^{n-1} |z_i''|^2 \right),
\]
where \( c_5, c_6 \) are positive constants. And
\[
u d(\zeta - z) \wedge v d(\zeta - z) = \mathcal{O}(\nu \zeta - z|).
\]
So each coefficient of \((1/b^\delta)R\) is dominated by
\[
\left( \sum_{i=1}^{n-1} |z_i''|^2 \right)^{-(\delta/2)} \left( |t| + \sqrt{\sum_{i=1}^{n-1} |z_i''|^2} \right) \left( |t| + \sum_{i=1}^{n-1} |z_i''|^2 \right)^{-n}.
\]
And this is locally integrable on \( C^{n-1} \times R \) if \( \delta < 1 \). In fact, by using polar coordinates, we compute the following integral. We set
\[
x_1 = r \cos \theta_1 \cos \theta_2 \cdots \cos \theta_{2n-3} \cos \theta_{2n-2},
\]
\[
y_1 = r \cos \theta_1 \cos \theta_2 \cdots \cos \theta_{2n-3} \sin \theta_{2n-2},
\]
\[
x_2 = r \cos \theta_1 \cdots \cos \theta_{2n-4} \sin \theta_{2n-3},
\]
\[
y_2 = r \cos \theta_1 \cdots \sin \theta_{2n-4},
\]
\[
\vdots
\]
\[
x_{n-1} = r \cos \theta_1 \sin \theta_2,
\]
\[
y_{n-1} = r \sin \theta_1,
\]
where \( z_j'' = x_j + \sqrt{-1}y_j, \ 1 \leq j \leq n - 1 \). Then
\[
\left( \sum_{i=1}^{n-1} |z_i''|^2 \right)^{-(\delta/2)} \left( |t| + \sqrt{\sum_{i=1}^{n-1} |z_i''|^2} \right) \left( |t| + \sum_{i=1}^{n-1} |z_i''|^2 \right)^{-n}
\]
\[= r^{-\delta} (t + r)(t + r^2)^{-n}.
\]
So

\[
\int_{M \cap U_i(p)} \left( \sum_{i=1}^{n-1} |z''_i|^2 \right)^{-\delta/2} \left( |t| + \sqrt{\sum_{i=1}^{n-1} |z''_i|^2} \right) dV_z, t
\times \left( |t| + \sum_{i=1}^{n-1} |z''_i|^2 \right) dV_z, t
\leq \int_0^\epsilon \int_0^\infty r^{-\delta} (t + r)(t + r^2)^{-n} r^{2n-3} dt dr
\]

\[
= \int_0^\epsilon \int_0^\infty \left\{ \frac{1}{(t + r^2)^{n-1}} r^{2n-3-\delta} + \frac{1}{(t + r^2)^n} r^{2n-3} \right\} dt dr.
\]

While

\[
\int_0^\infty \frac{1}{(t + r^2)^{n-1}} r^{2n-3-\delta} dt
\]

\[
= - \frac{1}{(n - 2)[(1/((n - 2))^2 - (1 - r))r^{2n-2-\delta}]}\bigg|_0^\infty
\]

\[
= \frac{1}{(n - 1)}(1 - r)^{-\delta}.
\]

Therefore

\[
\int_{M \cap U_i(p)} \left( \sum_{i=1}^{n-1} |z''_i|^2 \right)^{-\delta/2} \left( |t| + \sqrt{\sum_{i=1}^{n-1} |z''_i|^2} \right) dV_z, t
\leq \int_0^\epsilon \left( \frac{1}{(n - 2)(2 - \delta)} \right)^{1-\delta/2} + \frac{1}{(n - 1)(1 - \delta)} \right)^{1/2} - (\delta/2)
\]

\[
= (1/((n - 2)(2 - \delta)))^{1-\delta/2} + (1/((n - 1)(1 - \delta)))^{1/2} - (\delta/2)
\]

Therefore we have our lemma.
Now we prove our main theorem.

\[
\int_{M \cap U_\epsilon(p)} (1/b^{2\delta}) |R_M(u,v)(\phi)|^2 dV \\
\leq \int_{M \cap U_\epsilon(p)} \left\{ (1/b^{2\delta}) \left( \int_{M \cap U_\epsilon(p)} \Omega^{u,v}(\zeta, z) \phi(\zeta) dV_\zeta \right)^2 \right\} dV_z \\
\leq \int_{M \cap U_\epsilon(p)} \left( \int_{M \cap U_\epsilon(p)} |(1/b^\delta)\Omega^{u,v}(\zeta, z)\phi(\zeta)| dV_\zeta \right)^2 dV_z \\
\leq \int_{M \cap U_\epsilon(p)} \left\{ \left( \int_{M \cap U_\epsilon(p)} |(1/b^\delta)\Omega^{u,v}(\zeta, z)| dV_\zeta \right) \times \left( \int_{M \cap U_\epsilon(p)} |\phi(\zeta)|^2 dV_\zeta \right) \right\} dV_z \\
\leq c^2 \int_{M \cap U_\epsilon(p)} |\phi(\zeta)|^2 dV_\zeta \quad \text{(by Lemma 5.2 in this paper)} \\
\leq c^2 \|\phi\|_{M \cap U_\epsilon(p)}^2.
\]

So we have our theorem. \(\square\)

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