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**AN INTRINSIC CHARACTERIZATION OF A CLASS OF
MINIMAL SURFACES IN CONSTANT CURVATURE
MANIFOLDS**

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Let X be an N -manifold of constant sectional curvature. A class of minimal surfaces in X , called exceptional minimal surfaces, will be defined in terms of the structure of their normal bundles. It will be shown that these surfaces can be characterized intrinsically in a way that generalizes the Ricci condition for minimal surfaces in Euclidean 3-space. It will also be shown that these surfaces are rigid when N is even and belong to 1-parameter families of isometric surfaces when N is odd.

0. Introduction. Let $X^N(c)$ denote an N -dimensional manifold of constant sectional curvature c , and suppose that M is a minimal surface in $X^N(c)$ with Riemannian metric ds^2 and Gauss curvature K . The classical theorem of Ricci, as extended by Lawson [3], says that when $N = 3$ minimal surfaces of $X^3(c)$ are characterized by the conditions that $K \leq c$ and at points where $K < c$ the metric $d\hat{s}^2 = \sqrt{c - K} ds^2$ is flat. Moreover, for each minimal surface M in $X^3(c)$, there is a 1-parameter family of isometric minimal surfaces M_τ , $0 \leq \tau < 2\pi$, such that M is congruent to one of the members of this family.

This paper will describe a class of minimal surfaces in $X^N(c)$, called exceptional minimal surfaces, and a sequence of functions A_1^c, A_2^c, \dots on each surface such that when $N = 2n + 1$, these surfaces are characterized by the conditions that $A_r^c \geq 0$, $1 \leq r \leq n$, and at points where each $A_r^c > 0$, the metric $d\hat{s}^2 = (A_n^c)^{1/(n+1)} ds^2$ is flat. This reduces to the Ricci-Lawson condition when $n = 1$, in that $A_1^c = c - K$. The exceptional minimal surfaces in $X^{2n+1}(c)$ will be seen to belong to 1-parameter families of isometric surfaces, just as happens in $X^3(c)$.

In $X^{2n+2}(c)$, the exceptional minimal surfaces will be characterized by the conditions that $A_r^c \geq 0$, $1 \leq r \leq n$, and $A_{n+1}^c \equiv 0$. Additionally, in $X^{2n+2}(c)$ the exceptional minimal surfaces will be rigid. These results given here for the case where $N = 2n + 2$ are actually implicit in [2], although they are stated there in terms of minimal immersions of the 2-sphere S^2 into $X^{2n+2}(c)$.

Sections 1 and 2 summarize the structure equations for minimal surfaces and some results from [2] that are needed here. Section 3 contains the statements of the theorems, which are proved in §§4 and 5. Section 6 contains two corollaries on isometric minimal surfaces.

1. Structure equations of surfaces. Suppose M is a Riemannian surface with Gauss curvature K . Let e_1, e_2 be a local orthonormal frame field on M , and let θ_1, θ_2 be the coframe dual to e_1, e_2 . Then the structure equations of M are

$$(1) \quad d\theta_1 = \omega_{12} \wedge \theta_2, \quad d\theta_2 = -\omega_{12} \wedge \theta_1, \quad \text{and} \quad d\omega_{12} = -K\theta_1 \wedge \theta_2,$$

where ω_{12} ($= -\omega_{21}$) is the connection form on M , and K is the Gauss curvature of M .

If $f: M \rightarrow \mathbb{R}$ is a smooth function, let f_1, f_2 be given by

$$df = f_1\theta_1 + f_2\theta_2.$$

Taking the exterior derivative of this expression and applying Cartan's Lemma gives f_{11}, f_{12}, f_{21} , and f_{22} , with $f_{12} = f_{21}$, such that

$$\begin{aligned} df_1 - f_2\omega_{12} &= f_{11}\theta_1 + f_{12}\theta_2, \\ df_2 + f_1\omega_{12} &= f_{21}\theta_1 + f_{22}\theta_2. \end{aligned}$$

If Δ is the Laplace-Beltrami operator of M , then $\Delta f = f_{11} + f_{22}$. Let $\bar{\partial}f = (f_1 + if_2)/2$ and $\varphi = \theta_1 + i\theta_2$. Then

$$(2) \quad \Delta f \varphi \wedge \bar{\varphi} = 4(d\bar{\partial}f + i\bar{\partial}f\omega_{12}) \wedge \bar{\varphi}.$$

Note that this is not the usual $\bar{\partial}$ -operator. If $z = x + iy$ gives local isothermal coordinates on M , then there is a positive function λ such that $ds^2 = \lambda^2|dz|^2$. The $\bar{\partial}$ defined here is λ times the usual $\bar{\partial}$ -operator.

2. Structure equations and normal planes. Suppose M is a minimal surface in $X^N(c)$. When clear from context, this latter manifold will be denoted simply as X . Assume that M lies fully in X , i.e., does not lie in a totally geodesic submanifold of X . Let the integer n be given by $N = 2n + 1$ or $2n + 2$, and let indices have the following ranges unless otherwise indicated:

$$\begin{aligned} 1 \leq j, k \leq 2, \quad 3 \leq \alpha, \beta \leq N, \quad 1 \leq A, B, C \leq N, \\ 1 \leq p, q, r \leq n. \end{aligned}$$

(The symbol i will be reserved for $\sqrt{-1}$.)

Let \tilde{e}_A be a local orthonormal frame field on X , and let $\tilde{\theta}_A$ be the coframe dual to \tilde{e}_A . Then the structure equations of X are

$$(3) \quad d\tilde{\theta}_A = \sum_B \tilde{\omega}_{AB} \wedge \tilde{\theta}_B \quad \text{and} \quad d\tilde{\omega}_{AB} = \sum_C \tilde{\omega}_{AC} \wedge \tilde{\omega}_{CB} - c\tilde{\theta}_A \wedge \tilde{\theta}_B$$

where the $\tilde{\omega}_{AB}$ ($= -\tilde{\omega}_{BA}$) are the connection forms on X . If $\langle \cdot, \cdot \rangle$ is the Riemannian metric on X , then $\tilde{\omega}_{AB} = \langle d\tilde{e}_A, \tilde{e}_B \rangle$.

Suppose that e_1, e_2 is a frame on M as described in the previous section and that the frame \tilde{e}_A is chosen so that on M , $e_j = \tilde{e}_j$, and the \tilde{e}_α are normal to M . Then differential forms on M can be identified with differential forms on X restricted to M :

$$\theta_j = \tilde{\theta}_j|_M \quad \text{and} \quad \omega_{12} = \tilde{\omega}_{12}|_M.$$

To simplify the notation, when forms and vectors on X are restricted to M , let them be denoted by the same symbol without tilde:

$$\begin{aligned} \theta_A & \text{ will denote } \tilde{\theta}_A|_M, \\ \omega_{AB} & \text{ will denote } \tilde{\omega}_{AB}|_M, \quad \text{and} \\ e_A & \text{ will denote } \tilde{e}_A|_M. \end{aligned}$$

Then $\theta_\alpha = 0$ on M since the e_α are normal to M . When the relation $d\theta_\alpha = 0$ is expanded using the structure equations (3), Cartan's Lemma can be applied to show that there are functions $h_{\alpha jk}$ such that

$$(4) \quad \omega_{\alpha j} = \sum_k h_{\alpha jk} \theta_k, \quad h_{\alpha jk} = h_{\alpha kj}.$$

The $h_{\alpha jk}$ are the coefficients of the second fundamental form. The assumption that M is a minimal surface is equivalent to assuming that the second fundamental form has zero trace:

$$(5) \quad h_{\alpha 11} + h_{\alpha 22} = 0.$$

Let $T_x M$ and $T_x X$ denote the tangent space to M and X , respectively, at a point x . Curves on M through x have their first derivatives at x in $T_x M$, but higher order derivatives will have components normal to M . The space spanned by the derivatives of order up to r is called the *rth osculating space of M at x* , denoted $T_x^{(r)} M$. If for all r , $T_x^{(r)} M \subsetneq T_x X$ at all x in a neighborhood of M , then that neighborhood would lie in a totally geodesic submanifold of X ([4], p. 241). The assumption that M lies fully in X means that for some r , $T_x^{(r)} M = T_x X$ at generic points of M .

The r th normal space of M at x , denoted $\text{Nor}_x^{(r)}M$, is the orthogonal complement of $T_x^{(r)}M$ in $T_x^{(r+1)}M$, so

$$T_x^{(r+1)}M = T_x^{(r)}M \oplus \text{Nor}_x^{(r)}M.$$

The results in §2 and §3 of [2] show that at generic points of M , the dimension of $\text{Nor}_x^{(r)}M$ is 2 when $1 \leq r \leq n - 1$, and the dimension of $\text{Nor}_x^{(n)}M$ is 1 or 2, depending on whether N is odd or even. Those normal spaces that have dimension 2 will be called the *normal planes* of M . Let β_N denote the number of normal planes possessed by M at a generic point:

$$\beta_N = \begin{cases} n, & \text{if } N = 2n + 2, \\ n - 1, & \text{if } N = 2n + 1. \end{cases}$$

Choose the normal vectors e_α so that $\text{Nor}_x^{(r)}M$ is spanned by $\{e_{2r+1}, e_{2r+2}\}$, $1 \leq r \leq \beta_N$. When $N = 2n + 1$, $\text{Nor}_x^{(n)}M$ will be spanned by $\{e_{2n+1}\}$. The derivatives of vector fields in $T_x^{(r)}M$ must lie in $T_x^{(r+1)}M$, so de_{2r-1} and de_{2r} cannot have any e_α components for $\alpha > 2r + 2$. Since $\omega_{AB} = \langle de_A, e_B \rangle$ and $\omega_{AB} = -\omega_{BA}$,

$$\omega_{2r-1,A} = \omega_{2r,A} = 0 \quad \text{when } A > 2r + 2 \text{ and when } A < 2r - 3.$$

When $r = 2$, these relations imply that

$$(6) \quad h_{\alpha jk} = 0 \quad \text{when } \alpha > 4.$$

Set $H_\alpha = h_{\alpha 11} + ih_{\alpha 12}$ for $\alpha = 3$ and 4. Then using (5), (6), and the form $\varphi = \theta_1 + i\theta_2$, equations (4) can be written

$$\omega_{\alpha 1} + i\omega_{\alpha 2} = H_\alpha \bar{\varphi} \quad \text{for } \alpha = 3, 4,$$

$$\omega_{\alpha j} = 0 \quad \text{for } \alpha > 4.$$

Now applying the structure equations (3) to the relation $\omega_{51} + i\omega_{52} = 0$ leads to

$$(H_3\omega_{53} + H_4\omega_{54}) \wedge \bar{\varphi} = 0.$$

So for some H_5 ,

$$H_3\omega_{53} + H_4\omega_{54} = H_5\bar{\varphi}.$$

Applying this argument inductively to the relations

$$\omega_{2r+1,2r-3} + i\omega_{2r+1,2r-2} = 0,$$

$$\omega_{2r+2,2r-3} + i\omega_{2r+2,2r-2} = 0,$$

for $r = 2, \dots, n$, produces H_α such that

$$(7) \quad H_{2r-1}\omega_{\alpha,2r-1} + H_{2r}\omega_{\alpha,2r} = H_\alpha \bar{\varphi},$$

for $\alpha = 2r + 1$ and $2r + 2$. This relation extends to the case where $r = 1$ by setting $H_1 = 1$, and $H_2 = i$.

The r th normal plane, $\text{Nor}_x^{(r)}M$, of M will be called *exceptional* if $H_{2r+2} = \pm iH_{2r+1}$. (Note that the sign can be reversed by reversing the orientation of $\text{Nor}_x^{(r)}M$. Note also that when $N = 2n + 1$, $\text{Nor}_x^{(n)}M$ is a line, not a plane, and the notion of exceptionality does not apply.) The minimal surface M will be called *exceptional* if all of its normal planes are exceptional. Minimal immersions of the 2-sphere S^2 into $X^{2n+2}(c)$ are always exceptional [2]. (These surfaces are called “superminimal” by Bryant [1].)

For the remainder of this paper, assume that M is an exceptional minimal surface. The orientations of the normal planes of such a surface can be chosen so that $H_{2r+2} = +iH_{2r+1}$ for $1 \leq r \leq \beta_N$. Then equations (7) become

$$(8) \quad \begin{aligned} H_{2r-1}(\omega_{2r+1, 2r-1} + i\omega_{2r+1, 2r}) &= H_{2r+1}\bar{\varphi}, \\ H_{2r-1}(\omega_{2r+2, 2r-1} + i\omega_{2r+2, 2r}) &= iH_{2r+1}\bar{\varphi}, \end{aligned}$$

for $r = 1, \dots, \beta_N$, together with

$$(9) \quad H_{2n-1}(\omega_{2n+1, 2n-1} + i\omega_{2n+1, 2n}) = H_{2n+1}\bar{\varphi}$$

when $N = 2n + 1$.

3. The theorems. For each real number c , the quantities A_p^c are defined as follows:

$$\begin{aligned} A_0^c &= 1/2 \text{ for all } c, \\ A_1^c &= c - K, \\ A_{p+1}^c &= \begin{cases} A_p^c[\Delta \log(A_p^c) + A_p^c/A_{p-1}^c - 2(p+1)K], & \text{if } A_p^c > 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

THEOREM A. *Suppose M is an exceptional minimal surface lying fully in $X^N(c)$. Let K denote the Gauss curvature of M , ds^2 its Riemannian metric, and let n be given by $N = 2n + 1$ or $2n + 2$. Then $A_p^c \geq 0$ for $1 \leq p \leq n$, with equality only at isolated points. If $N = 2n + 2$, then $A_{n+1}^c \equiv 0$, and if $N = 2n + 1$, then at points where each $A_p^c > 0$, the metric $d\hat{s}^2 = (A_n^c)^{1/(n+1)} ds^2$ is flat (has Gauss curvature $\hat{K} \equiv 0$).*

THEOREM B. *Suppose M is a smooth Riemannian surface with Gauss curvature K and Riemannian metric ds^2 . Suppose $A_p^c > 0$ for $1 \leq p \leq n$ in some neighborhood of $x_0 \in M$. If $A_{n+1}^c \equiv 0$, set*

$N = 2n + 2$. If the metric $d\hat{s}^2 = (A_n^c)^{1/(n+1)} ds^2$ is flat, set $N = 2n + 1$. Then there is a neighborhood U of x_0 and an isometric immersion $f: U \rightarrow X^N(c)$ such that $f(U)$ is an exceptional minimal surface lying fully in $X^N(c)$.

The theorems will be proved in §4 and §5. Both proofs will use the following:

LEMMA. *The condition that $d\hat{s}^2 = (A_n^c)^{1/(n+1)} ds^2$ should be flat is equivalent to*

$$(10) \quad \Delta \log(A_n^c) - 2(n + 1)K \equiv 0.$$

Proof. Recall that if $ds^2 = \lambda^2 |dz|^2$ for isothermal coordinates $z = x + iy$, then the Gauss curvature is given by $K = -\Delta \log \lambda$. If $d\hat{s}^2 = (A_n^c)^{1/(n+1)} ds^2 = \lambda^2 (A_n^c)^{1/(n+1)} |dz|^2$, then $\hat{K} = K - [\Delta \log(A_n^c)]/2(n+1)$ so $\hat{K} \equiv 0$ if and only if (10) holds. \square

4. Proof of Theorem A.

LEMMA 4.1. *If $M \subset X^N(c)$ is an exceptional minimal surface, then*

$$(11) \quad d\omega_{2r-1, 2r} = 2[|H_{2r+1}|^2/|H_{2r-1}|^2 - |H_{2r-1}|^2/|H_{2r-3}|^2]\theta_1 \wedge \theta_2$$

for $r = 2, \dots, \beta_N$, whenever the denominators are not zero.

Proof. From the first equation in (8),

$$\begin{aligned} H_{2r-1}(\omega_{2r+1, 2r-1} + i\omega_{2r+1, 2r}) \wedge \overline{H}_{2r-1}(\omega_{2r+1, 2r-1} - i\omega_{2r+1, 2r}) \\ = -H_{2r+1}\overline{H}_{2r+1}\varphi \wedge \overline{\varphi}. \end{aligned}$$

Since $\varphi \wedge \overline{\varphi} = -2i\theta_1 \wedge \theta_2$, this implies that

$$(12) \quad \omega_{2r-1, 2r+1} \wedge \omega_{2r+1, 2r} = [|H_{2r+1}|^2/|H_{2r-1}|^2]\theta_1 \wedge \theta_2.$$

Similarly,

$$(13) \quad \omega_{2r-1, 2r+2} \wedge \omega_{2r+2, 2r} = [|H_{2r+1}|^2/|H_{2r-1}|^2]\theta_1 \wedge \theta_2.$$

From (8) with r replaced by $r - 1$,

$$\begin{aligned} \overline{H}_{2r-3}(\omega_{2r-1, 2r-3} - i\omega_{2r-1, 2r-2}) \wedge H_{2r-3}(\omega_{2r, 2r-3} + i\omega_{2r, 2r-2}) \\ = \overline{H}_{2r-1}\varphi \wedge iH_{2r-1}\overline{\varphi}. \end{aligned}$$

The real part of this expression simplifies to

$$(14) \quad \begin{aligned} \omega_{2r-1, 2r-3} \wedge \omega_{2r-3, 2r} + \omega_{2r-1, 2r-2} \wedge \omega_{2r-2, 2r} \\ = -2[|H_{2r-1}|^2/|H_{2r-3}|^2]\theta_1 \wedge \theta_2. \end{aligned}$$

Now (11) follows from the structure equation (3) for $d\omega_{2r-1, 2r}$ and from (12), (13), and (14). \square

LEMMA 4.2. For each $r = 1, \dots, \beta_N + 1$, there is a G_{2r-1} such that

$$(15) \quad dH_{2r-1} + iH_{2r-1}(r\omega_{12} - \omega_{2r-1,2r}) = G_{2r-1}\bar{\varphi}.$$

Proof. When $r = 1$, $H_1 = 1$, so $G_1 = 0$. Suppose (15) holds when $r = p$. Set $r = p$ in the two equations in (8) and take their exterior derivatives using the structure equations (3). (Note that $d\bar{\varphi} = i\omega_{12} \wedge \bar{\varphi}$.) In both cases, the result is

$$\{dH_{2p+1} + iH_{2p+1}[(p+1)\omega_{12} - \omega_{2p+1,2p+2}]\} \wedge \bar{\varphi} = 0,$$

so for some G_{2p+1} ,

$$dH_{2p+1} + iH_{2p+1}[(p+1)\omega_{12} - \omega_{2p+1,2p+2}] = G_{2p+1}\bar{\varphi}$$

and the lemma follows by induction. □

COROLLARY. If H_{2r-1} is not identically zero, then its zeros are isolated.

Proof. By (15),

$$d\bar{H}_{2r-1} - i\bar{H}_{2r-1}(r\omega_{12} - \omega_{2r-1,2r}) \equiv 0 \pmod{\varphi},$$

and the corollary follows from the theorem in §4 in [2]. □

LEMMA 4.3. If $M \subset X^N(c)$ is an exceptional minimal surface, then

$$(16) \quad A_r^c = 2^{2r-1}|H_{2r+1}|^2, \quad r = 0, \dots, \beta_N,$$

and the zeros of each A_r^c are isolated.

Proof. The preceding corollary shows that the zeros of A_r^c are isolated whenever (16) holds. Clearly (16) holds when $r = 0$. To show (16) when $r = 1$, note that the structure equations (1) and (3) give two ways of computing $d\omega_{12}$:

$$d\omega_{12} = -K\theta_1 \wedge \theta_2 = \sum_{\alpha=3}^4 \omega_{1\alpha} \wedge \omega_{\alpha 2} - c\theta_1 \wedge \theta_2.$$

Expanding the summation using equations (8) with $r = 1$ and then extracting the coefficients of $\theta_1 \wedge \theta_2$ yields

$$A_1^c = c - K = 2|H_3|^2.$$

Now proceed inductively: suppose that for some p , (16) holds for $r < p$. If $A_{p-1}^c = 0$, then $A_p^c = 0$ by definition. Also, $H_{2p-1} = 0$ by (16), so $H_{2p+1} = 0$ by (8), showing that (16) holds for $r = p$. Since

the zeros of the A 's propagate along the sequence, the only other case to consider is $A_{p-2}^c \neq 0$ and $A_{p-1}^c \neq 0$. By Lemma 4.1,

$$(17) \quad d\omega_{2p-1, 2p} = \frac{1}{2}[2^{2p-1}|H_{2p+1}|^2/A_{p-1}^c - A_{p-1}^c/A_{p-2}^c]\theta_1 \wedge \theta_2.$$

Using this, the exterior derivative of (15) when $r = p$ is

$$(18) \quad \{dG_{2p-1} + iG_{2p-1}[(p+1)\omega_{12} - \omega_{2p-1, 2p}]\} \wedge \bar{\varphi} \\ = -\frac{i}{2}H_{2p-1}[2^{2p-1}|H_{2p+1}|^2/A_{p-1}^c \\ - A_{p-1}^c/A_{p-2}^c + 2pK]\theta_1 \wedge \theta_2.$$

By Lemma 4.2, the exterior derivative of $A_{p-1}^c = 2^{2p-3}H_{2p-1}\bar{H}_{2p-1}$ is

$$(A_{p-1}^c)_1\theta_1 + (A_{p-1}^c)_2\theta_2 = 2^{2p-3}(H_{2p-1}\bar{G}_{2p-1}\varphi + \bar{H}_{2p-1}G_{2p-1}\bar{\varphi}).$$

Wedging this with φ and comparing coefficients of $i\theta_1 \wedge \theta_2$ yields

$$(19) \quad \bar{\partial}A_{p-1}^c = 2^{2p-3}\bar{H}_{2p-1}G_{2p-1}.$$

By (2), (15), and (19),

$$(20) \quad \Delta A_{p-1}^c \varphi \wedge \bar{\varphi} = 2^{2p-1}\bar{H}_{2p-1} \\ \times \{dG_{2p-1} + iG_{2p-1}[(p+1)\omega_{12} - \omega_{2p-1, 2p}]\} \\ \wedge \bar{\varphi} + 2^{2p-1}|G_{2p-1}|^2\varphi \wedge \bar{\varphi}.$$

Note that by (19) and the inductive hypothesis,

$$|G_{2p-1}|^2 = |\bar{\partial}A_{p-1}^c|^2/2^{4p-6}|H_{2p-1}|^2 = \|dA_{p-1}^c\|^2/2^{2p-1}A_{p-1}^c.$$

Combining this with (20) and (18) shows that

$$\Delta A_{p-1}^c = A_{p-1}^c[2^{2p-1}|H_{2p+1}|^2/A_{p-1}^c - A_{p-1}^c/A_{p-2}^c + 2pK] \\ + \|dA_{p-1}^c\|^2/2A_{p-1}^c.$$

Thus,

$$2^{2p-1}|H_{2p+1}|^2 = A_{p-1}^c[\Delta \log(A_{p-1}^c) + A_{p-1}^c/A_{p-2}^c - 2pK] \\ = A_p^c,$$

completing the induction. \square

To finish the proof of Theorem A, consider first the case $N = 2n+2$. Calculations analogous to those in the proof of Lemma 4.1 show that

$$d\omega_{2n+1, 2n+2} = -2[|H_{2n+1}|^2/|H_{2n-1}|^2]\theta_1 \wedge \theta_2 = -[A_n^c/2A_{n-1}^c]\theta_1 \wedge \theta_2.$$

Using this in the proof of Lemma 4.3 with $p = n + 1$ shows that

$$A_{n+1}^c = A_n^c[\Delta \log(A_n^c) + A_n^c/A_{n-1}^c - 2(n + 1)K] \equiv 0.$$

The proof of Theorem A when $N = 2n + 1$ follows by modifying the arguments in the proofs of the lemmas to apply to (9) instead of (8). Using (9) in the proof of Lemma 4.1 when $r = n$ yields

$$d\omega_{2n-1,2n} = 2[|H_{2n+1}|^2/2|H_{2n-1}|^2 - |H_{2n-1}|^2/|H_{2n-3}|^2]\theta_1 \wedge \theta_2.$$

The proof of Lemma 4.2 applied to (9) implies that for some G_{2n+1} ,

$$dH_{2n+1} + iH_{2n+1}(n + 1)\omega_{12} = G_{2n+1}\bar{\varphi}.$$

The proof of Lemma 4.3 when $p = n$ shows that

$$A_n^c = 2^{2n-2}|H_{2n+1}|^2$$

so that

$$d\omega_{2n-1,2n} = \frac{1}{2}[A_n^c/A_{n-1}^c - A_{n-1}^c/A_{n-2}^c]\theta_1 \wedge \theta_2.$$

Using this, the proof of Lemma 4.3 can be repeated again with appropriate modifications when $p = n + 1$ to show that $\Delta \log(A_n^c) - 2(n + 1)K = 0$. By the lemma in §3, $d\hat{s}^2 = (A_n^c)^{1/(n+1)}ds^2$ is flat. \square

5. Proof of Theorem B. Let $F(M)$ and $F(X)$ denote the bundles of orthonormal frames on M and X , respectively. Consider the manifold

$$P = F(M) \times F(X) \times \mathbb{C}^{2n}$$

where \mathbb{C}^{2n} has coordinates $(H_3, \dots, H_{2n+1}, G_3, \dots, G_{2n+1})$. Use the projections $\pi_1: P \rightarrow F(M)$ and $\pi_2: P \rightarrow F(X)$ to pull the forms θ_j , ω_{12} , $\tilde{\theta}_A$, and $\tilde{\omega}_{AB}$ back to P , and let the pulled-back forms be denoted by the same symbols. For example, the pull-back $\pi_1^*(\theta_j)$ will be denoted simply θ_j . Let I denote the ideal of differential forms on P generated by the following 1-forms:

$$\begin{aligned} &\tilde{\theta}_j - \theta_j, \quad \tilde{\theta}_\alpha, \quad \tilde{\omega}_{12} - \omega_{12}, \\ &\tilde{\omega}_{2r-1,A} \quad \text{and} \quad \tilde{\omega}_{2r,A} \quad \text{for } 1 \leq r \leq \beta_N, \\ &\quad \text{and } A > 2r + 2 \text{ or } A < 2r - 3, \\ &H_{2r-1}(\tilde{\omega}_{2r+1,2r-1} + i\tilde{\omega}_{2r+1,2r}) - H_{2r+1}\bar{\varphi}, \quad 1 \leq r \leq \beta_N, \\ &H_{2r-1}(\tilde{\omega}_{2r+2,2r-1} + i\tilde{\omega}_{2r+2,2r}) - iH_{2r+1}\bar{\varphi}, \quad 1 \leq r \leq \beta_N, \\ &dH_{2r+1} + iH_{2r+1}[(r + 1)\omega_{12} - \tilde{\omega}_{2r+1,2r+2}] - G_{2r+1}\bar{\varphi}, \\ &\quad 0 \leq r \leq \beta_N, \end{aligned}$$

together with the forms

$$H_{2n-1}(\tilde{\omega}_{2n+1,2n-1} + i\tilde{\omega}_{2n+1,2n}) - H_{2n+1}\bar{\varphi},$$

$$dH_{2n+1} + iH_{2n+1}(n+1)\omega_{12} - G_{2n+1}\bar{\varphi},$$

when $N = 2n + 1$. Let Q be the submanifold of P determined by the relations

$$(21) \quad A_r^c = 2^{2r-1}|H_{2r+1}|^2 \quad \text{and} \quad \bar{\partial}A_r^c = 2^{2r-1}\bar{H}_{2r+1}G_{2r+1}$$

for $1 \leq r \leq \beta_N$,

together with

$$(22) \quad A_n^c = 2^{2n-2}|H_{2n+1}|^2 \quad \text{and} \quad \bar{\partial}A_n^c = 2^{2n-2}\bar{H}_{2n+1}G_{2n+1}$$

when $N = 2n + 1$. Lengthy calculations (most of which are outlined in §2 and §4) show that if $N = 2n + 2$ and $A_{n+1}^c \equiv 0$, or if $N = 2n + 1$ and $\Delta \log(A_n^c) - 2(n + 1)K \equiv 0$, then I is a closed ideal in Q , i.e., if $\eta \in I$, then $d\eta \in I$. Choose an initial point of Q by choosing initial points $x_0 \in M$, $y_0 \in X$, initial frames e_i and \tilde{e}_A , and initial values of the H 's and G 's that satisfy (21) and (22) at x_0 . By the Frobenius Theorem, there is a submanifold $S \subset Q$ passing through this initial point such that all forms in I are zero on S .

When $N = 2n + 2$, $\dim Q = 2n^2 + 6n + 6$ and there are $2n^2 + 6n + 3$ independent 1-forms in I , so $\dim S = 3$. By a standard argument ([5], pp. 73–77) the restricted projection $\pi_1|_S$ is locally a diffeomorphism from S to $F(M)$. More precisely, there is a neighborhood $V \subset S$ and a neighborhood $W \subset F(M)$ containing (x_0, e_1, e_2) such that $\pi_1|_V: V \rightarrow W$ is a diffeomorphism. Define $f_*: W \rightarrow F(X)$ by

$$f_* = \pi_2 \circ (\pi_1|_V)^{-1}.$$

Let $f_*(x, e_1, e_2) = (y, \tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_{2n+2})$ for $(x, e_1, e_2) \in W$. If f_* is a bundle map, then it projects down to a map $f: U \rightarrow X$ where $U \subset M$ is a neighborhood of x_0 . The forms in I were chosen so that f would be an isometry ($\theta_i = f^*(\tilde{\theta}_i)$, where f^* is the pull-back induced by f) and so that $f(U)$ would be an exceptional minimal surface. Note that since $A_r^c > 0$, $1 \leq r \leq n$, each $H_{2r+1} \neq 0$ by (21), so $f(U)$ lies fully in $X^{2n+2}(c)$.

To show that f_* is a bundle map, let a be a complex number with absolute value 1 and consider the action on P given by

$$e_1 + ie_2 \rightarrow a(e_1 + ie_2),$$

$$\tilde{e}_{2r-1} + i\tilde{e}_{2r} \rightarrow a^r(\tilde{e}_{2r-1} + i\tilde{e}_{2r}), \quad 1 \leq r \leq n + 1,$$

$$G_{2r+1} \rightarrow aG_{2r+1}, \quad 1 \leq r \leq n,$$

with $x \in M$, $y \in X$, and the H 's unchanged. This induces the following action on forms:

$$\begin{aligned} \theta_1 + i\theta_2 &\rightarrow a(\theta_1 + i\theta_2), \\ \tilde{\theta}_{2r-1} + i\tilde{\theta}_{2r} &\rightarrow a^r(\tilde{\theta}_{2r-1} + i\tilde{\theta}_{2r}), \quad 1 \leq r \leq n+1, \\ \eta_{p,q} &\rightarrow a^{p+q}\eta_{p,q}, \quad 1 \leq p, q \leq n+1, \end{aligned}$$

where

$$\eta_{p,q} = \tilde{\omega}_{2p-1,2q-1} + i\tilde{\omega}_{2p-1,2q} + i(\tilde{\omega}_{2p,2q-1} + i\tilde{\omega}_{2p,2q}).$$

The forms ω_{12} and $\tilde{\omega}_{2r+1,2r+2}$ are unchanged. Also,

$$\bar{\partial} \rightarrow a\bar{\partial}.$$

The submanifold Q and the ideal I are invariant under this action, so the integral submanifolds of I are also invariant. It follows that f_* is a bundle map, which completes the proof when $N = 2n + 2$.

When $N = 2n + 1$, $\dim Q = 2n^2 + 4n + 4$ and there are $2n^2 + 4n + 2$ independent 1-forms in I , so $\dim S = 2$. The initial conditions are such that $\theta_1 \wedge \theta_2 \neq 0$ at the initial point and therefore in a neighborhood of the initial point. Thus, there are neighborhoods $V \subset S$ and $U \subset M$ such that projection from S to M is a diffeomorphism from V to U , and f can be defined as the inverse of this projection followed by the projection from S to X . As in the previous case, f is an isometry and $f(U)$ is an exceptional minimal surface lying fully in $X^{2n+1}(c)$. □

6. Isometric exceptional minimal surfaces.

COROLLARY 6.1. *Suppose M_1 and M_2 are exceptional minimal surfaces lying fully in $X^{2n+2}(c)$. If M_1 and M_2 are isometric, then they are congruent.*

Proof. The surfaces M_1 and M_2 are real analytic, so it suffices to show that they are locally congruent. By Theorem A, $A_{n+1}^c \equiv 0$ and there are isometric neighborhoods U_1, U_2 in M_1, M_2 , respectively, on which $A_p^c > 0, 1 \leq p \leq n$. Let $g: U_1 \rightarrow U_2$ be the isometry and let \tilde{e}_A^j denote frame fields on X adapted to U_j such that $\tilde{e}_k^2 = g_*(\tilde{e}_k^1)$. These frames determine H_{2r+1}^j and G_{2r+1}^j as in §2 and §4, and thus determine submanifolds of P that are integral manifolds for the ideal I . Since that ideal satisfies the Frobenius condition, its integral manifolds are completely determined by initial conditions. Let $x_1 \in U_1$ and $x_2 = g(x_1) \in U_2$ be the initial points. The H 's and G 's

satisfy (21), so for each r there is a value of τ such that at the initial points, $H_{2r+1}^2 = e^{i\tau} H_{2r+1}^1$ and $G_{2r+1}^2 = e^{i\tau} G_{2r+1}^1$. By (8), rotating the vectors $\tilde{e}_{2r+1}^1, \tilde{e}_{2r+2}^2$ counterclockwise in $\text{Nor}_{x_1}^{(r)} M_1$ through an angle $-\tau$ changes H_{2r+1}^1 to $e^{-i\tau} H_{2r+1}^1$. By (15), this rotation changes G_{2r+1}^1 to $e^{-i\tau} G_{2r+1}^1$. It follows that the normal vectors \tilde{e}_α^1 can be chosen so that $H_{2r+1}^1 = H_{2r+1}^2$ and $G_{2r+1}^1 = G_{2r+1}^2$ at the initial points. With this choice of the frame \tilde{e}_A^1 , an isometry of X that takes x_1 to x_2 and \tilde{e}_A^1 and \tilde{e}_A^2 will take U_1 to U_2 and therefore M_1 to M_2 . \square

COROLLARY 6.2. *Suppose M is an exceptional minimal surface lying fully in $X^{2n+1}(c)$. Then there is a 1-parameter family M_τ , $0 \leq \tau < 2\pi$, of exceptional minimal surfaces in $X^{2n+1}(c)$ such that every exceptional minimal surface in $X^{2n+1}(c)$ that is isometric to M is congruent to some M_τ .*

Proof. As in the proof of the previous corollary, different exceptional minimal surfaces in $X^{2n+1}(c)$ that are isometric to M can only arise through a choice of different initial conditions, and choosing different initial values for $H_3, H_5, \dots, H_{2n-1}, G_3, G_5, \dots, G_{2n-1}$ is equivalent to choosing different initial normal vectors $\tilde{e}_3, \dots, \tilde{e}_{2n}$. As for H_{2n+1} and G_{2n+1} , they can be replaced by $e^{i\tau} H_{2n+1}$ and $e^{i\tau} G_{2n+1}$ in (22), but this is not equivalent to a rotation in $\text{Nor}_x^{(n)} M$, which is 1-dimensional. Thus, an integral manifold for the ideal I is determined by initial points in M and $X^{2n+1}(c)$, initial frames e_j and \tilde{e}_A , and one value of τ to determine the initial values $e^{i\tau} H_{2n+1}$ and $e^{i\tau} G_{2n+1}$. Varying τ will produce a 1-parameter family M_τ of minimal surfaces and every exceptional minimal surface in $X^{2n+1}(c)$ that is isometric to M will be congruent to a member of this family. \square

Added in proof. When N is even, my exceptional minimal surfaces are also known as *isotopic minimal surfaces*. See [6] for an alternate definition in this case.

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