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A COMBINATORIAL MATRIX IN 3-MANIFOLD THEORY

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In this paper we study a combinatorial matrix considered by W. B. R. Lickorish. We prove a conjecture by Lickorish that completes his topological and combinatorial proof of the existence of the Witten-Reshetikhin-Turaev 3-manifold invariants. We derive a recursive formula for the determinant of the matrix and discover some interesting numerical relations.

In this paper we study the matrix $A(n)$ which was defined by W. B. R. Lickorish [3]. We prove a result required by Lickorish which completes his topological and combinatorial approach to the 3-manifold invariants of Witten-Reshetikhin-Turaev [4], [5]. This matrix arises from a pairing on a set of geometric configurations. These are the configurations of n nonintersecting arcs in the disk with $2n$ specified boundary points. There are C_n such configurations where C_n is the n th Catalan number so the matrix increases in size very rapidly. The Catalan numbers were discovered by Euler who considered the ways to partition a polygon into triangles [1]. These two counting problems correspond naturally by considering "restricted sequences".

The matrix has entries in $\mathbf{Z}[\delta]$. Lickorish needed that $\det A(n) = 0$ if $\delta = \pm 2 \cos \frac{\pi}{n+1}$. We find a recursive formula for $\det A(n)$ and show that all the roots are of the form $2 \cos \frac{k\pi}{m+1}$ for $1 \leq m \leq n$ and $1 \leq k \leq m$ and verify the result. Using this formula, we derive a simple rule that allows one to recursively compute $\det A(n)$ by generating all of its factors.

There have been three approaches to study polynomial invariants of classical links: the topological and combinatorial approach considered by Kauffman, Lickorish and many other topologists; the study of quantized Yang-Baxter equations and related Lie algebras by Reshetikhin and Turaev; and the study of subfactors and traces of von Neumann and Hecke algebras by Jones. We took a topological and combinatorial viewpoint. The authors have been informed that the essential result needed by Lickorish could have been obtained by pursuing the two other approaches.

1. Combinatorial manipulation. Let D_n be the set of configurations of n non-intersecting arcs on a disk joining $2n$ points on the boundary of the disk. We draw these configurations by taking S^1 to be $[0, 1]/0 \sim 1$ as in Fig. 1.

The cardinality of D_n is equal to $(2n)!/n!(n+1)!$, known as the Catalan number, denoted here by C_n . It satisfies the recursive relation:

$$C_n = C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-2} C_1 + C_{n-1}.$$

We can inductively represent the elements of D_n by sequences of n integers (a_1, a_2, \dots, a_n) where $1 \leq a_i \leq n - i + 1$. The first entry a_1 means that there is an innermost arc in the configuration joining the a_1 th point and the $(a_1 + 1)$ th point on the interval. One then deletes that arc and has an element of D_{n-1} remaining. The sequence (a_2, a_3, \dots, a_n) then represents this element of D_{n-1} . See Fig. 2 for an example.

Note that every configuration in D_n must contain an innermost arc between adjacent points among the first $n + 1$ points. Thus this representation captures all possible configurations but with repetitions. For example the configuration in Fig. 3 has 12 distinct associated sequences.

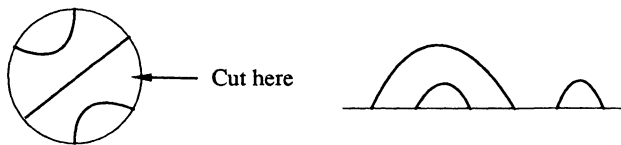


FIGURE 1

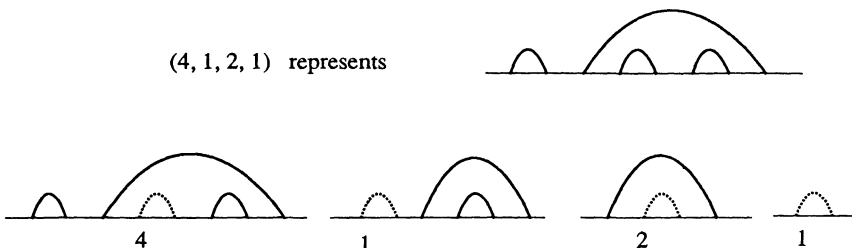


FIGURE 2



FIGURE 3

Given such a sequence (a_1, a_2, \dots, a_n) one may construct the unique configuration inductively. Into the configuration (a_2, a_3, \dots, a_n) one can insert two additional points between $(a_1 - 1)$ st and a_1 th points then joining these two new points by an innermost arc. Thus two distinct configurations cannot have the same sequence. To a given configuration in D_n , one can associate the unique sequence (a_1, a_2, \dots, a_n) in which a_1 indicates the initial position of the first occurring innermost arc and a_2 does the same for the configuration without the previous innermost arc and so on. Such a sequence is said to be *restricted*.

PROPOSITION 1.1. *A sequence (a_1, a_2, \dots, a_n) of a configuration is restricted if and only if $a_{i-1} - 1 \leq a_i$ for all $i = 2, \dots, n$.*

Proof. For a restricted sequence (a_1, a_2, \dots, a_n) , it is enough to prove $a_1 - 1 \leq a_2$ since (a_2, \dots, a_n) is also a restricted sequence. After removing the first innermost arc, either the second innermost arc or the arc joining the $(a_1 - 1)$ th and the $(a_1 + 2)$ th point in the original configuration will become the first innermost arc in the remaining configuration. Thus $a_1 - 1 \leq a_2$.

Conversely if $a_{i-1} - 1 \leq a_i$, then the newly inserted innermost arc into the configuration of (a_{i-1}, \dots, a_n) becomes the first innermost arc in the configuration of (a_i, \dots, a_n) . □

REMARK. The number of ways to divide an $(n + 2)$ gon into triangles or the number of ways to interpret the product $x_1 x_2 \cdots x_{n+1}$ in a non-associative algebra is equal to the Catalan number C_n . Restricted sequences are useful to see the correspondence between these and configurations defined earlier. Label the vertex of the $(n + 2)$ gon counterclockwise 1 through n except fixed adjacent vertices. A triangle in a partition is said to be outermost if it has a vertex contained in no other triangle. To a partition of the $(n + 2)$ gon we assign the sequence (a_1, a_2, \dots, a_n) where a_1 is the vertex that is solely contained in the first occurring outermost triangle. Then the sequence (a_2, \dots, a_n) inductively represents the partition of the $(n + 1)$ gon obtained by deleting the vertex a_1 and its adjacent sides. See Fig. 4 for an example.

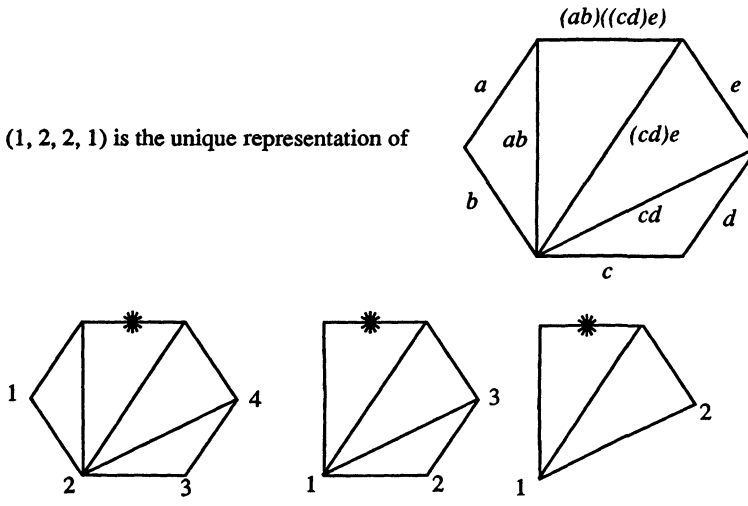


FIGURE 4

We give the lexicographic order to the set of all the sequences of configurations, i.e., $(a_1, a_2, \dots, a_n) < (b_1, b_2, \dots, b_n)$ if there is an index k such that $a_1 = b_1, \dots, a_{k-1} = b_{k-1}$ and $a_k < b_k$. If two distinct sequences α and β represent the same configuration and α is restricted, then clearly $\alpha < \beta$.

Let $B(n, k)$ be the set of restricted sequences of length n with initial entry k and let $b(n, k)$ be the cardinality $|B(n, k)|$ of the set $B(n, k)$. Since D_n can be identified with the set of all restricted sequences of length n , $C_n = \sum_{k=1}^n b(n, k)$. It is convenient to set $b(n, k) = 0$ for $k = 0$ or $k > n$.

PROPOSITION 1.2. $b(n, k) = \sum_{i=k-1}^{n-1} b(n-1, i)$ for $k = 1, \dots, n$.

Proof. Immediately follows from Proposition 1.1. □

It is interesting that $b(n, 1) = b(n, 2) = C_{n-1}$, $b(n, n-1) = n-1$, and $b(n, n) = 1$. The only element in $B(n, n)$ is $(n, n-1, \dots, 2, 1)$



FIGURE 5

which represents the configuration in Fig. 5. In fact we have:

COROLLARY 1.3. *Using the binomial coefficients,*

$$b(n, k) = \frac{k}{n} \binom{2n - k - 1}{n - 1}.$$

Proof. By Proposition 1.2, $b(n+1, k) - b(n+1, k+1) = b(n, k-1)$. And this recursive formula together with initial conditions $b(2, 1) = b(n, n) = 1$ for all n generates all $b(n, k)$'s. But a computation shows that

$$\frac{k}{n+1} \binom{2n - k + 1}{n} - \frac{k+1}{n+1} \binom{2n - k}{n} = \frac{k-1}{n} \binom{2n - k}{n-1}. \quad \square$$

Let $\mathfrak{B}(n, k)$ be the set of sequences with initial entry k and the remaining terms forming a restricted sequence of length $n - 1$. We will sometimes write (k, α) with α restricted for such a sequence. Note that $|\mathfrak{B}(n, k)| = C_{n-1}$.

Let V be the free $\mathbf{Z}[\delta]$ module generated by D_n where δ is a variable. We define a bilinear form on $V \times V$. If α, β are two configurations in D_n , we can form the union of their respective disks along the boundary to obtain a configuration of circles in the 2-sphere. We denote this configuration in S^2 by $\alpha \cup \beta$. Let c be the number of circles in $\alpha \cup \beta$; then $\langle \alpha, \beta \rangle = \delta^c$. Then we linearly extend this pairing to all elements in the free module. Lickorish first considered this symmetric bilinear form to give a more geometric and combinatorial proof of the existence of the 3-manifold invariant developed by Witten and Reshetikhin-Turaev. See [2], [3], [4] and [5]. So we call it *Lickorish's bilinear form*. We can also consider this a pairing of restricted sequences or of sequences since they correspond to configurations.

LEMMA 1.4. *For $\alpha, \beta \in D_n$, $\langle \alpha, \beta \rangle = \delta^n$ if and only if $\alpha = \beta$.*

Proof. If $\alpha = \beta$ then each component of $\alpha \cup \beta$ consists of one arc of α and one of β so $\alpha \cup \beta$ has n components. If $\langle \alpha, \beta \rangle = \delta^n$ then each arc of α is in a separate component of $\alpha \cup \beta$. But if $\alpha \neq \beta$ then some arc of β joins endpoints of two distinct arcs of α and these arcs are in the same component of $\alpha \cup \beta$. □

THEOREM 1.5 (*Properties of Lickorish's bilinear form*). (1) *Let S be any subset of D_n . Then $\langle \ , \ \rangle$ is nondegenerate over the free $\mathbf{Z}[\delta]$ module generated by S .*

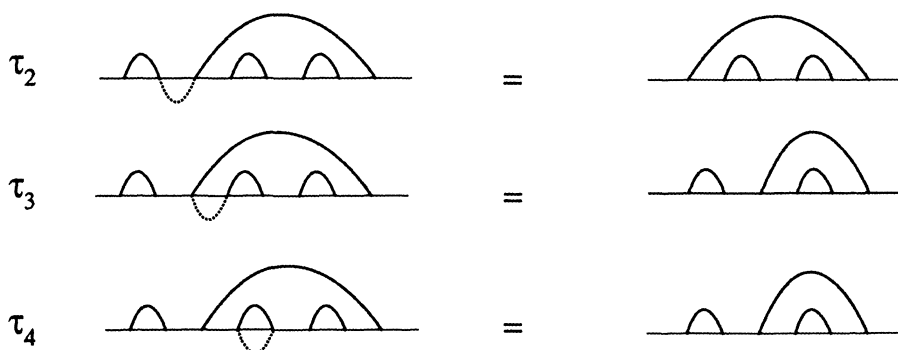


FIGURE 6

(2) Suppose α is any configuration in D_n . Then for any $b \in \{1, 2, \dots, n\}$ with $\alpha \notin \mathfrak{B}(n, b)$, there is a $\beta \in \mathfrak{B}(n, b)$ such that $\delta\langle\alpha, \gamma\rangle = \langle\beta, \gamma\rangle$ for all $\gamma \in \mathfrak{B}(n, b)$.

(3) $\delta\langle a, \varepsilon\rangle = \langle(a, \alpha), (a, \varepsilon)\rangle = \delta\langle(a \pm 1, \alpha), (a, \varepsilon)\rangle$ for all sequences ε, α whenever $a \pm 1$ makes sense.

(4) Suppose $(a, \alpha), (b, \beta)$ are restricted sequences of length n and there is an $\eta \in \mathbf{Z}[\delta]$ such that $\langle(a, \alpha), \gamma\rangle = \eta\langle(b, \beta), \gamma\rangle$ for all $\gamma \in \mathfrak{B}(n, a)$ with $\gamma \leq (a, \alpha)$.

- (i) If $b = a, a \pm 1$ then $\alpha \leq \beta$.
- (ii) If $b \neq a, a \pm 1$ then $\alpha < \beta$.

Before we begin the proof, we first define a set of maps

$$\tau_a: D_n \rightarrow D_{n-1} \quad \text{for } a = 1, 2, \dots, n.$$

These mappings eliminate the a th and the $(a + 1)$ st points in D_n by an inverse of a “finger move” as in Fig. 6.

Note that $\tau_a((a, \alpha)) = \alpha$ for any sequence α .

Proof of Theorem 1.5. (1) Suppose $\sum_{\alpha \in S} q_\alpha \alpha$ is an arbitrary element in the free $\mathbf{Z}[\delta]$ module generated by S . From among the q_α , pick a β so that the degree of q_β is maximal. Then by Lemma 1.4, the degree of $\langle\beta, \beta\rangle$ is strictly greater than the degree $\langle\alpha, \beta\rangle$ for all $\alpha \neq \beta$. Therefore $\langle\sum_{\alpha \in S} q_\alpha \alpha, \beta\rangle$ has a nonvanishing term of degree $(n + \deg q_\beta)$.

(2) If $\alpha \notin \mathfrak{B}(n, b)$, then

$$\delta\langle\alpha, \gamma\rangle = \langle(b, \tau_b(\alpha)), \gamma\rangle \quad \text{for } \gamma \in \mathfrak{B}(n, b)$$

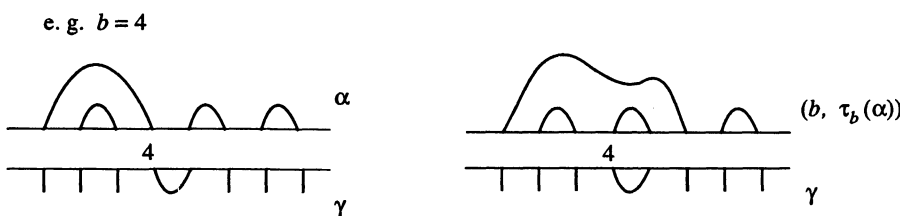


FIGURE 7

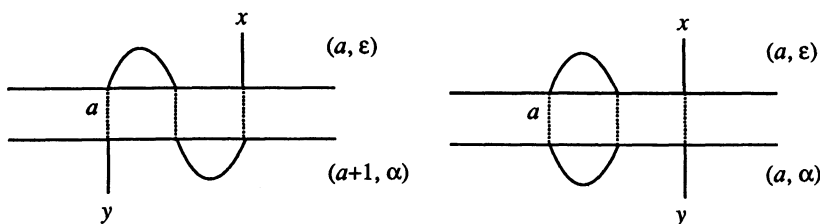


FIGURE 8

since the innermost arc at b performs τ_b when joined to α . See Fig. 7.

(3) $\tau_a(a, \alpha) = \tau_a(a + 1, \alpha) = \tau_a(a - 1, \alpha) = \alpha$. See Fig. 8.

(4) It follows from Lemma 1.4 that

$$\eta = \langle (a, \alpha), (a, \alpha) \rangle / \langle (b, \beta), (a, \alpha) \rangle = \delta^k$$

for some $k \geq 0$. First suppose $b = a$ and so $(b, \beta) \in \mathfrak{B}(n, a)$. Let $S = \{\varepsilon \in D_{n-1} | \varepsilon \leq \alpha\}$. If $(b, \beta) < (a, \alpha)$, i.e., $\beta < \alpha$ then $\delta \langle \alpha - \delta^k \beta, \varepsilon \rangle = \langle (a, \alpha) - \delta^k (b, \beta), (a, \varepsilon) \rangle = 0$ for all $\varepsilon \in S$. This contradicts property (1). Thus $(b, \beta) \geq (a, \alpha)$.

Suppose that $b = a \pm 1$. If $\beta < \alpha$ then this together with property (3) contradicts property (1). Thus $\beta \geq \alpha$.

Now suppose that $b \neq a, a \pm 1$. If $(b, \beta) \in \mathfrak{B}(n, a)$ then $b < a$ because (b, β) is a restricted sequence. So $(b, \beta) < (a, \alpha)$, which again contradicts property (1). Thus $(b, \beta) \notin \mathfrak{B}(n, a)$. We then have as in Fig. 9,

$$\langle (b, \beta), \gamma \rangle = \langle \tau_a(b, \beta), \tau_a \gamma \rangle \quad \text{for all } \gamma \in \mathfrak{B}(n, a) \text{ with } \gamma \leq (a, \alpha).$$

Then $\delta \langle \alpha, \tau_a \gamma \rangle = \langle (a, \alpha), \gamma \rangle = \delta^k \langle (b, \beta), \gamma \rangle = \delta^k \langle \tau_a(b, \beta), \tau_a \gamma \rangle$. Thus

$$\delta \langle \alpha, \varepsilon \rangle = \delta^k \langle \tau_a(b, \beta), \varepsilon \rangle \quad \text{for all } \varepsilon \in D_{n-1} \text{ with } \varepsilon \leq \alpha.$$

Thus we have $\tau_a(b, \beta) \geq \alpha$ by property (1). Let α_1, β_1 , and β'_1 be the first entry of restricted sequences α, β , and $\tau_a(b, \beta)$ respectively.

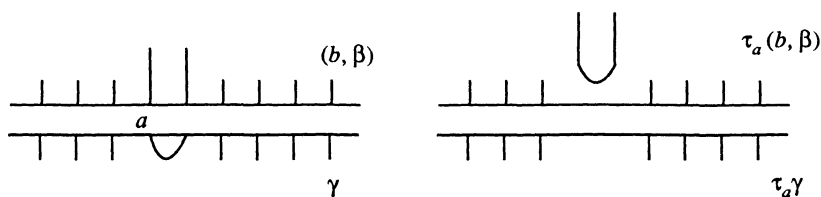


FIGURE 9

If $b < a - 1$ then $\beta'_1 \leq b < a - 1 \leq \alpha_1$ because (a, α) is a restricted sequence. So $\tau_a(b, \beta) < \alpha$ and this is a contradiction. Thus $b > a + 1$. Since (b, β) has the first occurring innermost arc at b , $\beta'_1 = b - 2 < \beta_1$ and so $\tau_a(b, \beta) < \beta$. Therefore $\alpha < \beta$. \square

2. Matrix manipulation. Let T_n be the $(n \times n)$ tridiagonal matrix with δ in each diagonal element and 1 in each upper and lower superdiagonal. For example

$$T_5 = \begin{pmatrix} \delta & 1 & 0 & 0 & 0 \\ 1 & \delta & 1 & 0 & 0 \\ 0 & 1 & \delta & 1 & 0 \\ 0 & 0 & 1 & \delta & 1 \\ 0 & 0 & 0 & 1 & \delta \end{pmatrix}.$$

Let $\Delta_n = \det T_n$, then it is a polynomial in δ for $n \geq 1$.

- PROPOSITION 2.1.** (1) $\Delta_n = \delta\Delta_{n-1} - \Delta_{n-2}$ for $n \geq 3$.
 (2) $\Delta_n = \prod_{k=1}^n (\delta - 2 \cos \frac{k\pi}{n+1})$.

Proof. (1) Compute Δ_n by expanding along the first row.

(2) Note that Δ_n is of degree n and the coefficient of δ^n is 1 so that we must find the roots of Δ_n . Since $\Delta_n = n + 1$ when $\delta = 2$ and $\Delta_n = (-1)^n(n + 1)$ when $\delta = -2$, $\delta = \pm 2$ are not roots. We solve the recursion formula by a standard method. Let

$$\alpha = \frac{\delta + \sqrt{\delta^2 - 4}}{2} \quad \text{and} \quad \beta = \frac{\delta - \sqrt{\delta^2 - 4}}{2}$$

so that $\alpha\beta = 1$ and $\alpha + \beta = \delta$. From the recursion we get $\Delta_n - \alpha\Delta_{n-1} = \beta(\Delta_{n-1} - \alpha\Delta_{n-2}) = \beta^n$. Similarly $\Delta_n - \beta\Delta_{n-1} = \alpha^n$. Then $(\alpha - \beta)\Delta_n = \alpha^{n+1} - \beta^{n+1}$. Thus $\Delta_n = 0$ exactly when $\alpha \neq \beta$ and $\alpha^{n+1} = \beta^{n+1}$. Since $\beta < 1 < \alpha$ when $\delta > 2$ and $\beta < -1 < \alpha$ when $\delta < -2$, δ cannot be a root for $|\delta| > 2$. Thus we may assume $|\delta| < 2$ so $\delta = 2 \operatorname{Re} \alpha = 2 \operatorname{Re} \beta$. Also $\alpha^{n+1} = \beta^{n+1}$ is equivalent to $\alpha^{2n+2} = 1$. If we take α to be one among the first n of $(2n + 2)$ th roots of unity, then α is not equal to β which is now the conjugate

of α . Thus $\delta = 2 \cos(k\pi/(n + 1))$ for $k = 1, \dots, n$. Since they are all distinct, we found all of the roots of $\Delta_n = 0$. \square

LEMMA 2.2. *Let A be a symmetric matrix over a ring and A' be obtained by deleting the last row and column. If $\det A' \neq 0$, then a series of row operations and the corresponding column operations within the ring convert A into $\begin{pmatrix} A' & 0 \\ 0 & \det A' \det A \end{pmatrix}$.*

Proof. Let

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

be the last column of A . Let y be the solution of the system of equations:

$$A'x = \det A' \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \end{pmatrix}.$$

Define

$$E = \begin{pmatrix} I & -y \\ 0 & \det A' \end{pmatrix}.$$

Then

$$E^{\text{tr}}AE = \begin{pmatrix} A' & 0 \\ 0 & \det A' \det A \end{pmatrix}. \quad \square$$

REMARK. Applying row operations, one gets

$$E^{\text{tr}}A = \begin{pmatrix} & v_1 \\ A' & \vdots \\ & v_{n-1} \\ 0 & \det A \end{pmatrix}.$$

Let $A(n)$ be the matrix representation of Lickorish's bilinear form \langle , \rangle over the basis D_n ordered by restricted sequences. $A(n)$ consists of n^2 blocks of matrices M_{ij} such that M_{ij} represents \langle , \rangle on $B(n, i) \times B(n, j)$. So M_{ij} is a $b(n, i) \times b(n, j)$ matrix. Let $A(n, k)$ be the submatrix $(M_{ij})_{i, j=1, \dots, k}$ of $A(n)$. Thus $A(n, n) = A(n)$ and $A(n, 1) = \delta A(n - 1)$ in this notation. By Theorem 1.5(3),

$$A(n, 2) = \begin{pmatrix} \delta A(n - 1) & A(n - 1) \\ A(n - 1) & \delta A(n - 1) \end{pmatrix}.$$

Thus we have the following proposition.

PROPOSITION 2.3. (1) $\det A(n, 1) = \Delta_1^{C_{n-1}} \det A(n - 1)$.

(2) $\det A(n, 2) = \Delta_2^{C_{n-1}} (\det A(n - 1))^2$.

Proof. Just calculate. □

LEMMA 2.4. $A(n)$ and all of its principal minors have nonzero determinants.

Proof. By Lemma 1.4, only the diagonal entries of $A(n)$ have the highest degree n . Thus the term δ^{nC_n} in the determinant of $A(n)$ has the coefficient 1. And the same argument applies to all principal minors. □

Given a matrix M , $M^{(p)}$ denotes the matrix obtained from M by deleting the last p rows and columns. And $A(n, k)^{(\overline{p})}$ denotes the matrix $(M_{ij}^{(p)})_{i,j=1,\dots,k}$.

LEMMA 2.5. (1) For $0 \leq p \leq b(n, k) - 1$, we have the following recursion formula:

$$\det A(n, k)^{(\overline{p})} = \Delta_k \left(\frac{\det A(n-1)^{(p)}}{\det A(n-1)^{(p+1)}} \right)^k \det A(n, k)^{(\overline{p+1})}.$$

(2) For $2 \leq j \leq k - 1$ and $b(n, j + 1) \leq p \leq b(n, j) - 1$, we have the following recursion formula:

$$\det A(n, k)^{(\overline{p})} = \Delta_j \left(\frac{\det A(n-1)^{(p)}}{\det A(n-1)^{(p+1)}} \right)^j \det A(n, k)^{(\overline{p+1})}.$$

In order to help the understanding of the proof given below, we will describe some of the properties of $A(n)$ that reflect the properties of Lickorish’s bilinear form in Theorem 1.5. Let \mathfrak{M}_{ij} be the matrix representing $\langle \ , \ \rangle$ on $\mathfrak{B}(n, i) \times \mathfrak{B}(n, j)$. Then property (2) in Theorem 1.5 means that each column of \mathfrak{M}_{ij} is equal to either one of columns of \mathfrak{M}_{ii} or δ^{-1} times one of the columns. Property (3) implies that $\mathfrak{M}_{ii} = \delta A(n - 1)$ and $\mathfrak{M}_{i(i\pm 1)} = A(n - 1)$. Furthermore the last column of \mathfrak{M}_{ii} is independent of every column in the blocks \mathfrak{M}_{ij} for $j \neq i, i \pm 1$. This can be seen through property (4) since unrestricted sequences (i.e., repeated configurations) always appear first in the sets $\mathfrak{B}(n, k)$ for $k \geq 3$. Then row operations as in Lemma 2.2 with $A = A(n - 1)$ convert the last row of \mathfrak{M}_{ii} into $(0, \dots, 0, \delta \det A(n - 1))$, $M_{i(i\pm 1)}$ into $(0, \dots, 0, \det A(n - 1))$, and \mathfrak{M}_{ij} for $j \neq i, i \pm 1$ into $(0, \dots, 0, 0)$.

The matrix $(\mathfrak{M}_{ij})_{i,j=1,\dots,n}$ of the blocks has repeated rows due to the presence of unrestricted sequences. $A(n)$ then is obtained from this matrix by deleting repeated rows and corresponding columns. From \mathfrak{M}_{ij} one would delete the first $\sum_{k=1}^{i-2} b(n-1, k)$ rows and $\sum_{k=1}^{j-2} b(n-1, k)$ columns so an undeleted column in \mathfrak{M}_{ij} does not change its position when counted from the rear. Let $\mathfrak{B}(n, i)^{(p)}$ and $B(n, i)^{(p)}$ denote $\mathfrak{B}(n, i)$ and $B(n, i)$ with the last p configurations deleted. Consider the matrix given by \langle , \rangle on $\mathfrak{B}(n, i)^{(p)} \times \bigcup_{j=1}^n B(n, j)^{(p)}$. Any multiple of the column corresponding to the last element of $B(n, i)^{(p)}$ appears only at the spots corresponding to the last elements of $B(n, i \pm 1)^{(p)}$. By property (4) any other multiples were eliminated in the p deletions since they occur nearer the rear of their respective $\mathfrak{B}(n, i)^{(p)} \times B(n, j)^{(p)}$ block.

In the matrix $A(n)$ there is still a minor which is \mathfrak{M}_{ij} ; however it does not appear as a solid block since some of its configurations have innermost arcs which occur before the i th spot. However one can perform the desired row operations by borrowing the missing rows from the blocks above. One may do similar operations on $A(n)^{(\overline{p})}$.

Proof of Lemma 2.5. (1) Let E be the matrix as in the proof of Lemma 2.2 such that

$$E^{\text{tr}} A(n-1)^{(p)} E = \begin{pmatrix} A(n-1)^{(p+1)} & 0 \\ 0 & \det A(n-1)^{(p+1)} \det A(n-1)^{(p)} \end{pmatrix}.$$

We may assume that the entries of E are indexed by the first C_{n-1-p} elements in D_{n-1} that is ordered by the restricted sequences. Consider the set \mathfrak{S} of sequences (i, α) for $i = 1, \dots, k$ and the first C_{n-1-p} restricted sequences α in D_{n-1} . There is an obvious equivalence relation in which two sequences are equivalent if they represent the same configuration. Mod out \mathfrak{S} by this relation and we obtain a subset S of D_n . For $i = 1, \dots, k$ define a matrix E_i whose entries are indexed by S . The $([(i, \alpha)], [(i, \beta)])$ th entry of E_i is equal to the (α, β) th entry of E for all elements $[(i, \alpha)], [(i, \beta)]$ of S . All other diagonal entries of E_i are 1 and all other off-diagonal entries are 0. Hence E_1 is the identity except in the upper $(C_{n-1-p}) \times (C_{n-1-p})$ corner where it is E . And E_i is obtained from E_1 by permuting rows and corresponding columns. Perform row operations E_1^{tr} to $A(n, k)^{(\overline{p})}$ and denote the blocks of $E_1^{\text{tr}} A(n, k)^{(\overline{p})}$ by (G_{ij}) . Then the last row of G_{12} consists of zeros except the last entry because the $(1, 2)$ th block of $A(n, k)^{(\overline{p})}$ is exactly equal to $A(n-1)^{(p)}$. And by Theorem 1.5(3), $G_{11} = \delta G_{12}$. Theorem 1.5(2) and (4) say

that every column of the $(1, 3)$ th, \dots , $(1, k)$ th blocks of $A(n, k)^{\langle \bar{p} \rangle}$ is equal to one of the columns of the $(1, 2)$ th block of $A(n, k)^{\langle \bar{p} \rangle}$ which is not the last. Thus the last rows of G_{13}, \dots, G_{1k} are all zero. We now perform additional row operations $E_2^{\text{tr}}, \dots, E_k^{\text{tr}}$ and all the corresponding column operations. Then the resulting matrix $E_k^{\text{tr}} \dots E_1^{\text{tr}} A(n, k)^{\langle \bar{p} \rangle} E_1 \dots E_k$ looks like

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ M_{11}^{(p+1)} & \vdots & M_{12}^{(p+1)} & \vdots & M_{13}^{(p+1)} & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 \dots 0 & \delta\xi & 0 \dots 0 & \xi & 0 \dots 0 & 0 & 0 \dots 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ M_{21}^{(p+1)} & \vdots & M_{22}^{(p+1)} & \vdots & M_{23}^{(p+1)} & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 \dots 0 & \xi & 0 \dots 0 & \delta\xi & 0 \dots 0 & \xi & 0 \dots 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ M_{31}^{(p+1)} & \vdots & M_{32}^{(p+1)} & \vdots & M_{33}^{(p+1)} & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 \dots 0 & 0 & 0 \dots 0 & \xi & 0 \dots 0 & \delta\xi & \dots 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 \dots 0 & 0 & 0 \dots 0 & 0 & 0 \dots 0 & 0 & 0 \dots \delta\xi \end{pmatrix}$$

and $\xi = \det A(n - 1)^{\langle p+1 \rangle} \det A(n - 1)^{\langle p \rangle}$. By permuting rows and corresponding columns, the matrix becomes

$$\begin{pmatrix} \xi T_k & 0 \\ 0 & A(n, k)^{\langle \overline{p+1} \rangle} \end{pmatrix}.$$

But

$$\det E_i = \det A(n - 1)^{\langle p+1 \rangle}$$

and

$$\det(\xi T_k) = \Delta_k (\det A(n - 1)^{\langle p+1 \rangle} \det A(n - 1)^{\langle p \rangle})^k.$$

(2) The proof is similar. The only difference is that $A(n, k)^{\langle \bar{p} \rangle}$ now has j^2 blocks so we try to factor the tridiagonal matrix T_j out from it. □

LEMMA 2.6. For $3 \leq k \leq n$, we have the following recursion formulae:

$$\det A(n - 1) = \Delta_k^{b(n, k)} \left(\frac{\det A(n - 1)}{\det A(n - 1, k - 2)} \right)^k \det A(n - 1)^{\langle \overline{b(n, k)} \rangle},$$

and when $2 \leq j \leq k - 1$,

$$\begin{aligned} \det A(n-1)^{\langle \overline{b(n, j+1)} \rangle} \\ = \Delta_j^{b(n, j) - b(n, j+1)} \left(\frac{\det A(n-1, j-1)}{\det A(n-1, j-2)} \right)^j \det A(n-1)^{\langle \overline{b(n, j)} \rangle}. \end{aligned}$$

Proof. We successively apply the recursion formula (1) in Lemma 2.5. Then

$$\begin{aligned} \det A(n, k) &= \Delta_k \left(\frac{\det A(n-1)}{\det A(n-1)^{\langle 1 \rangle}} \right)^k \det A(n-1)^{\langle \bar{1} \rangle} \\ &= \Delta_k^2 \left(\frac{\det A(n-1)}{\det A(n-1)^{\langle 1 \rangle}} \right)^k \left(\frac{\det A(n-1)^{\langle 1 \rangle}}{\det A(n-1)^{\langle 2 \rangle}} \right)^k \det A(n, k)^{\langle \bar{2} \rangle} \\ &\quad \dots \\ &= \Delta_k^{b(n, k)} \left(\frac{\det A(n-1)}{\det A(n-1)^{\langle 1 \rangle}} \right)^k \left(\frac{\det A(n-1)^{\langle 1 \rangle}}{\det A(n-1)^{\langle 2 \rangle}} \right)^k \\ &\quad \dots \left(\frac{\det A(n-1)^{\langle b(n, k)-1 \rangle}}{\det A(n-1)^{\langle b(n, k) \rangle}} \right)^k \det A(n, k)^{\langle \overline{b(n, k)} \rangle} \\ &= \Delta_k^{b(n, k)} \left(\frac{\det A(n-1)}{\det A(n-1)^{\langle b(n, k) \rangle}} \right)^k \det A(n, k)^{\langle \overline{b(n, k)} \rangle}. \end{aligned}$$

But $A(n-1)^{\langle b(n, k) \rangle} = A(n-1, k-2)$ because

$$b(n, k) = \sum_{i=k-1}^{n-1} b(n-1, i).$$

The other formula can be shown by using the formula (2) in Lemma 2.5. □

THEOREM 2.7. *For $3 \leq k \leq n$, we have the following recursion formula:*

$$\begin{aligned} \det A(n, k) \\ = \frac{\Delta_k^{b(n, k)} \Delta_{k-1}^{b(n, k-1) - b(n, k)} \dots \Delta_2^{b(n, 2) - b(n, 3)} (\det A(n-1))^k}{(\det A(n-1, k-2)) (\det A(n-1, k-3)) \dots (\det A(n-1, 1))}. \end{aligned}$$

Proof. We recursively use the formulae in Lemma 2.6.

$$\begin{aligned}
 & \det A(n, k) \\
 &= \Delta_k^{b(n,k)} \left(\frac{\det A(n-1)}{\det A(n-1, k-2)} \right)^k \det A(n, k)^{\langle \overline{b(n,k)} \rangle} \\
 &= \Delta_k^{b(n,k)} \Delta_{k-1}^{b(n,k-1)-b(n,k)} \left(\frac{\det A(n-1)}{\det A(n-1, k-2)} \right)^k \\
 &\quad \cdot \left(\frac{\det A(n-1, k-2)}{\det A(n-1, k-3)} \right)^{k-1} \det A(n, k)^{\langle \overline{b(n,k-1)} \rangle} \\
 &\quad \dots \\
 &= \Delta_k^{b(n,k)} \Delta_{k-1}^{b(n,k-1)-b(n,k)} \dots \Delta_3^{b(n,3)-b(n,4)} \left(\frac{\det A(n-1)}{\det A(n-1, k-2)} \right)^k \\
 &\quad \cdot \left(\frac{\det A(n-1, k-2)}{\det A(n-1, k-3)} \right)^{k-1} \\
 &\quad \dots \left(\frac{\det A(n-1, 2)}{\det A(n-1, 1)} \right)^3 \det A(n, k)^{\langle \overline{b(n,3)} \rangle} \\
 &= \frac{\Delta_k^{b(n,k)} \Delta_{k-1}^{b(n,k-1)-b(n,k)} \dots \Delta_3^{b(n,3)-b(n,4)} (\det A(n-1))^k}{(\det A(n-1, k-2))(\det A(n-1, k-3)) \dots (\det A(n-1, 2))} \\
 &\quad \cdot \frac{\det A(n, k)^{\langle \overline{b(n,3)} \rangle}}{(\det A(n-1, 1))^3}.
 \end{aligned}$$

But

$$\begin{aligned}
 A(n, k)^{\langle \overline{b(n,3)} \rangle} &= A(n, 2)^{\langle \overline{b(n,3)} \rangle} \\
 &= \begin{pmatrix} \delta A(n-1)^{\langle b(n,3) \rangle} & A(n-1)^{\langle b(n,3) \rangle} \\ A(n-1)^{\langle b(n,3) \rangle} & \delta A(n-1)^{\langle b(n,3) \rangle} \end{pmatrix}.
 \end{aligned}$$

Since $A(n-1)^{\langle b(n,3) \rangle} = A(n-1, 1)$,

$$\begin{aligned}
 \det A(n, k)^{\langle \overline{b(n,3)} \rangle} &= \Delta_2^{b(n-1,1)} (\det A(n-1, 1))^2 \\
 &= \Delta_2^{b(n,2)-b(n,3)} (\det A(n-1, 1))^2. \quad \square
 \end{aligned}$$

REMARK. By inserting the factor $\Delta_1^{b(n,1)-b(n,2)}$, which is 1, into the formula in Theorem 2.7, we obtain a recursion formula that works for all $k = 1, \dots, n$. See Proposition 2.3.

COROLLARY 2.8. *The $\det A(n)$ vanishes at twice the real part of any primitive $2(n+1)$ st root of unity and $\det A(m, k)$ for $1 \leq m \leq n-1$ and $1 \leq k \leq n-1$ never vanishes at these values.*

Proof. The recursion formula of Theorem 2.7 shows that the determinants $\det A(m, k)$ for $1 \leq m \leq n$ and $1 \leq k \leq m$ can be written as a product of positive or negative powers of $\Delta_1, \dots, \Delta_n$. It also shows that $\det A(n)$ contains the factor Δ_n exactly once and all the other determinants of lower indexes do not contain the factor Δ_n . Therefore $2 \cos \frac{k\pi}{n+1}$ must be a root of $\det A(n)$ if k is relatively prime to $n + 1$. \square

COROLLARY 2.9. *After setting δ to be twice the real part of any primitive $2(n+1)$ st root of unity, Lickorish's pairing can be considered as a symmetric bilinear form over the real (or complex) vector space with a basis D_n . Then the basis element $\alpha = (n, n - 1, \dots, 2, 1)$ has the property that there is a linear combination $\sum_{\beta \neq \alpha} q_\beta \beta$ of basis elements other than α such that $\langle \alpha, \gamma \rangle = \langle \sum_{\beta \neq \alpha} q_\beta \beta, \gamma \rangle$ for all γ in the vector space.*

Proof. The last row of $A(n)$ corresponds to α and $A(n)^{(1)} = A(n, n - 1)$. By Corollary 2.8, $A(n)$ is singular but $A(n, n - 1)$ is nonsingular. Thus it follows from Lemma 2.2. In fact, q_β 's are equal up to sign to the (α, β) th cofactor of $A(n)$ divided by $\det A(n, n - 1)$. \square

REMARK. In fact the last elements of each block $B(n, k)$ of D_n as well as the rotations of the configuration $(n, n - 1, \dots, 1)$ have the property of Corollary 2.9.

COROLLARY 2.10. *We have the following recursive formula:*

$$\det A(n) = \prod_{i=1}^{[(n+1)/2]} (\det A(n-i))^{(-1)^{i-1} \binom{n-i+1}{i}} \prod_{i=0}^{[(n-1)/2]} \left(\frac{\Delta_{n-i}}{\Delta_i} \right)^{b(n-i, n-2i)}$$

where $\Delta_0 = 0$.

Proof. One can derive this from the formula in Theorem 2.7 using the following identities:

$$\binom{n-i+1}{i} = \sum_{k_1=1}^{n-2i+2} \sum_{k_2=1}^{k_1} \dots \sum_{k_{i-1}=1}^{k_{i-2}} 1$$

and

$$b(n, k) - b(n, k + 1) = b(n - 1, k - 1). \quad \square$$

Let $d(n, j)$ denote the exponent of Δ_j in $\det A(n)$. It is not hard to see that $d(n, j)$ is well defined for $j \geq 1$.

COROLLARY 2.11. *For $j \geq 1$, we have that*

$$\sum_{i=0}^{\lfloor (n+1)/2 \rfloor} (-1)^i \binom{n-i+1}{i} d(n-i, j) = b(j, 2j-n)$$

where $b(n, k) = -b(n-k, -k)$ for $k < 0$ and $b(n, 0) = b(n, k) = 0$ for $k > n$.

Proof. Immediate from Corollary 2.10. □

REMARK. It is interesting to note that the Catalan numbers satisfy the similar formula:

$$\begin{aligned} \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} (-1)^i \binom{n-i+1}{i} C_{j-i+1} \\ = b(j+2, n+2) \quad \text{for } j \geq \left\lfloor \frac{n+1}{2} \right\rfloor. \end{aligned}$$

This formula can be proved by recalling that $b(n, k)$ is the number of configurations in D_n that the first innermost arc occurs at the k th point and by applying the inclusion-exclusion principle.

The following theorem shows that $\det A(n)$ is generated by a simple rule.

THEOREM 2.12. *For $j \geq 1$, we have that*

$$d(n, j) = d(n-1, j-1) + 2d(n-1, j) + d(n-1, j+1)$$

where $d(n, 0) = 2C_n - C_{n+1} = -\frac{4}{n+2} \binom{2n-1}{n+1}$.

Proof. By the remark following Corollary 2.11, the formula in Corollary 2.11 holds for $j = 0$ if we set $b(0, k) = -b(-k, -k) = -1$ for $k < 0$. Use an induction on (n, j) with lexicographic order. Since $d(1, 0) = 2C_1 - C_2 = 0$, $d(2, 1) = d(1, 0) + 2d(1, 1) + d(1, 2)$.

From the formula in Corollary 2.11,

$$d(n, j) = b(j, 2j - n) + \sum_{i=1}^{[(n+1)/2]} (-1)^{i-1} \binom{n-i+1}{i} d(n-i, j).$$

By the induction hypothesis and the identity $\binom{n-i+1}{i} = \binom{n-i}{i} + \binom{n-i}{i-1}$,

$$\begin{aligned} & \sum_{i=1}^{[(n+1)/2]} (-1)^{i-1} \binom{n-i+1}{i} d(n-i, j) \\ &= \sum_{i=1}^{[(n+1)/2]} (-1)^{i-1} \binom{n-i}{i} (d(n-i-1, j-1) + 2d(n-i-1, j) \\ & \qquad \qquad \qquad + d(n-i-1, j+1)) \\ & \quad + \sum_{i=1}^{[(n+1)/2]} (-1)^{i-1} \binom{n-i}{i-1} (d(n-i-1, j-1) + 2d(n-i-1, j) \\ & \qquad \qquad \qquad + d(n-i-1, j+1)) \\ &= \sum_{i=1}^{[n/2]} (-1)^{i-1} \binom{n-i}{i} (d(n-i-1, j-1) + 2d(n-i-1, j) \\ & \qquad \qquad \qquad + d(n-i-1, j+1)) \\ & \quad + \sum_{i=0}^{[(n-1)/2]} (-1)^i \binom{n-i-1}{i} \\ & \qquad \qquad \qquad \times (d(n-i-2, j-1) + 2d(n-i-2, j) \\ & \qquad \qquad \qquad + d(n-i-2, j+1)) \\ &= d(n-1, j-1) - b(j-1, 2j-n-1) \\ & \quad + 2d(n-1, j) - 2b(j, 2j-n+1) + d(n-1, j+1) \\ & \quad - b(j+1, 2j-n+3) + b(j-1, 2j-n) \\ & \quad + 2b(j, 2j-n+2) + b(j+1, 2j-n+4) \\ &= -b(j, 2j-n) + d(n-1, j-1) + 2d(n-1, j) \\ & \quad + d(n-1, j+1). \end{aligned}$$

The last equality is achieved by several uses of the identity

$$b(n, k) - b(n, k+1) = b(n-1, k-1) \qquad \text{for all integer } k \text{ and all } n \geq 2. \quad \square$$

One can now easily generate $\det A(n)$ by using the rule in Theorem 2.12 as in the following table. The term $\Delta_0 = 1$ is inserted for a

computational purpose.

$$\begin{aligned}
 \det A(1) &= \Delta_0^0 \Delta_1 \\
 \det A(2) &= \Delta_0^{-1} \Delta_1^2 \Delta_2 \\
 \det A(3) &= \Delta_0^{-4} \Delta_1^4 \Delta_2^4 \Delta_3 \\
 \det A(4) &= \Delta_0^{-14} \Delta_1^8 \Delta_2^{13} \Delta_3^6 \Delta_4 \\
 \det A(5) &= \Delta_0^{-48} \Delta_1^{15} \Delta_2^{40} \Delta_3^{26} \Delta_4^8 \Delta_5 \\
 \det A(6) &= \Delta_0^{-165} \Delta_1^{22} \Delta_2^{121} \Delta_3^{100} \Delta_4^{43} \Delta_5^{10} \Delta_6 \\
 \det A(7) &= \Delta_0^{-572} \Delta_1^0 \Delta_2^{364} \Delta_3^{364} \Delta_4^{196} \Delta_5^{64} \Delta_6^{12} \Delta_7 \\
 \det A(8) &= \Delta_0^{-2002} \Delta_1^{-208} \Delta_2^{1092} \Delta_3^{1288} \Delta_4^{820} \Delta_5^{336} \Delta_6^{89} \Delta_7^{14} \Delta_8
 \end{aligned}$$

REMARK. Notice that the exponents of Δ_i may be negative; however $\det A(n)$ is a polynomial in δ . The negative exponents arise since the Δ_i 's are not relatively prime to each other. In fact the factor $\delta - 2 \cos \frac{k\pi}{i+1}$ of Δ_i is also a factor of Δ_j if $i+1$ divides $j+1$. Moreover, if k is relatively prime to $i+1$, then the converse holds. For example δ is a factor of Δ_{2i+1} for all i and $\delta^2 - 1$ is a factor of Δ_{3i+2} for all i . R. A. Litherland has shown that the exponent of δ in $\det A(n)$ is C_n and that the exponent of $\delta^2 - 1$ is $C_n - 1$.

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