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**BOUNDED HANKEL FORMS WITH WEIGHTED NORMS AND  
LIFTING THEOREMS**

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## BOUNDED HANKEL FORMS WITH WEIGHTED NORMS AND LIFTING THEOREMS

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**Bounded Hankel forms with respect to weighted norms are studied. The Nehari's theorem about the norms of the classical Hankel forms is generalized. This is essentially a lifting theorem due to Cotlar and Sadosky. Moreover a theorem about the essential norms of Hankel forms is proved. This relates with a theorem of Adamjan, Arov and Krein in the special case and gives a new lifting theorem which has applications to weighted norm inequalities, and the F. and M. Riesz theorem.**

### 1. Introduction. Let

$$A[a, b] = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} A_{ij} a_i b_j$$

where  $a$  and  $b$  are finite sequences. Then  $A[a, b]$  is called a sesquilinear form in the variables  $a$  and  $b$ .

Let  $\mathcal{P}$  be the set of all trigonometric polynomials and  $m$  the normalized Lebesgue measure on the unit circle  $T$ . If we put  $u = \sum_{j=-n}^n a_j z^j$  for  $a = (\dots, 0, a_{-n}, \dots, a_0, a_1, \dots, a_n, 0, \dots)$  then  $u$  belongs to  $\mathcal{P}$  and  $\int |u|^2 dm = \sum_{j=-n}^n |a_j|^2$ . Let

$$A(u, v) = A[a, b]$$

where  $u = \sum_{j=-n}^n a_j z^j$  and  $v = \sum_{j=-m}^m \bar{b}_j \bar{z}^j$ . Then we say that  $A(u, v)$  is a sesquilinear form on  $\mathcal{P} \times \mathcal{P}$ . It is clear that

$$A(\beta_1 u_1 + \beta_2 u_2, v) = \beta_1 A(u_1, v) + \beta_2 A(u_2, v)$$

and

$$A(u, \alpha_1 v_1 + \alpha_2 v_2) = \bar{\alpha}_1 A(u, v_1) + \bar{\alpha}_2 A(u, v_2).$$

If  $A_{ij} = \alpha(i + j)$  then  $A(u, v)$  is called a Hankel form on  $\mathcal{P} \times \mathcal{P}$  and we will write those forms  $\varphi(u, v)$ ,  $\psi(u, v)$  or etc.

Let  $\mathcal{P}_+ = \{f \in \mathcal{P} : \hat{f}(j) = 0 \text{ if } j < 0\}$  and  $\mathcal{P}_- = \{f \in \mathcal{P} : \hat{f}(j) = 0 \text{ if } j \geq 0\}$ . If  $A$  is restricted to  $\mathcal{P}_+ \times \mathcal{P}_-$  then the restriction of  $A$  is called a sesquilinear form on  $\mathcal{P}_+ \times \mathcal{P}_-$ . If  $\varphi$  is a Hankel form on  $\mathcal{P} \times \mathcal{P}$  then we will write

$$H_\varphi = \text{the restriction of } \varphi \text{ to } \mathcal{P}_+ \times \mathcal{P}_-$$

and  $\varphi$  is called a symbol of  $H_\varphi$ .

A sesquilinear form  $A$  on  $\mathcal{P} \times \mathcal{P}$  is said to be bounded if there exists a positive constant  $\gamma$  such that  $|A(u, v)| \leq \gamma$  if  $\int |u|^2 dm \leq 1$  and  $\int |v|^2 dm \leq 1$ . We will generalize this definition. Let  $\mu$  and  $\nu$  be finite positive Borel measures on  $T$ . A sesquilinear form  $A$  on  $\mathcal{P} \times \mathcal{P}$  is said to be bounded w.r.t.  $(\mu, \nu)$  if there exists a positive constant  $\gamma$  such that

$$|A(u, v)|^2 \leq \gamma^2 \int |u|^2 d\mu \int |v|^2 d\nu \quad (u, v \in \mathcal{P}).$$

The smallest number  $\gamma$  for which the inequality above is referred to as the norm of the form  $A$  and we will write  $\gamma = |||A|||$ , where the pair of measures is fixed. Similarly for the norm  $\gamma$  of the form  $A$  on  $\mathcal{P}_+ \times \mathcal{P}_-$  we will write  $\gamma = \|A\|$ . When the form  $A(u, v)$  is bounded on  $\mathcal{P} \times \mathcal{P}$  w.r.t.  $(\mu, \nu)$ , it can be extended to a form on (the  $L^2(\mu)$ -closure of  $\mathcal{P}$ )  $\times$  (the  $L^2(\nu)$ -closure of  $\mathcal{P}$ ). Then we will still write  $A(u', v')$  for  $u'$  and  $v'$  in the closures. It is the same for the case of  $\mathcal{P}_+ \times \mathcal{P}_-$ .

For  $0 < p \leq \infty$   $H^p = H^p(m)$  denotes the usual Hardy space, that is, the  $L^p = L^p(m)$ -closure of  $\mathcal{P}_+$ .  $C$  denotes the set of all continuous functions on  $T$ . Then  $H^\infty + C$  is the closure of  $\bigcup_{n=1}^{\infty} \bar{z}^n H^\infty$  [9, Theorem 2].

Our program is as follows. In §2 we will give representations of bounded Hankel forms on  $\mathcal{P} \times \mathcal{P}$ . In §3 generalizing Nehari's theorem ([13], [15, p. 6]) we will calculate the norms of bounded Hankel forms on  $\mathcal{P}_+ \times \mathcal{P}_-$ . This is, in fact, the lifting theorem of Cotlar and Sadosky [4] that appears as a corollary in §6. In §4 we will determine compact bounded Hankel forms on  $\mathcal{P}_+ \times \mathcal{P}_-$ . This relates with Hartman's theorem [8] in a special case. In §5 we will give the distance between a given Hankel form and the set of all compact sesquilinear forms. In §6 as a result of the previous sections we will obtain a new lifting theorem which contains one due to Cotlar and Sadosky [4]. In §7 we will apply results in the previous sections to problems in weighted norm inequalities as in [3] and to get a quantitative F. and M. Riesz theorem [16].

**2. Bounded Hankel forms on  $\mathcal{P} \times \mathcal{P}$ .** For some pair  $\mu$  and  $\nu$  of finite positive Borel measures on  $T$ , there exist nonzero bounded sesquilinear forms w.r.t.  $(\mu, \nu)$  but in Corollary 1 it is shown that no nonzero Hankel forms can exist.

**PROPOSITION 1.** *If  $\phi$  is a bounded Hankel form on  $\mathcal{P} \times \mathcal{P}$  w.r.t.  $(\mu, \nu)$  and  $|||\phi||| = \gamma$  then the following are valid.*

(1) *There exists a finite Borel measure  $\lambda$  on  $T$  such that*

$$\varphi(u, v) = \int u\bar{v} d\lambda \quad (u, v \in \mathcal{P})$$

and

$$|\lambda(E)| \leq \gamma |\mu(E)| |\nu(E)|$$

for any Borel set  $E$  in  $T$ .

(2) *If  $\mu = \mu_a + \mu_s$  and  $\nu = \nu_a + \nu_s$  are Lebesgue decompositions w.r.t.  $\lambda$  then  $\varphi$  can be assumed to be a bounded Hankel form on  $\mathcal{P} \times \mathcal{P}$  with respect to  $(\mu_a, \nu_a)$ .*

*Proof.* There exists a bounded linear operator  $\Phi$  from  $L^2(\mu)$  to  $L^2(\nu)$  such that  $\varphi(u, v) = \int (\Phi u)\bar{v} d\nu$ . Since  $\varphi(z^i, \bar{z}^j) = \varphi(1, z^{i+j})$ ,

$$\varphi(u, v) = \int u\bar{v}k d\nu \quad (u, v \in \mathcal{P})$$

where  $k = \Phi 1 \in L^2(\nu)$ . Set  $d\lambda = k d\nu$ ; then

$$\left| \int u\bar{v} d\lambda \right|^2 \leq \gamma^2 \int |u|^2 d\mu \int |v|^2 d\nu$$

for any  $u \in L^2(\mu)$  and  $v \in L^2(\nu)$ , and hence (1) follows. There is a Borel set  $E_a$  in  $T$  with  $\mu_s(E_a) = \nu_s(E_a) = 0$  on which  $\lambda$  is concentrated. Then  $\chi_{E_a} \in L^2(\mu) \cap L^2(\nu)$  and so

$$\left| \int u\bar{v} d\lambda \right|^2 \leq \gamma^2 \int |u|^2 d\mu_a \int |v|^2 d\nu_a$$

for any  $u \in L^2(\mu_a) = \chi_{E_a} L^2(\mu)$  and  $v \in L^2(\nu_a) = \chi_{E_a} L^2(\nu)$ . This implies (2).

**COROLLARY 1.** *If  $\varphi$  is a bounded Hankel form on  $\mathcal{P} \times \mathcal{P}$  w.r.t.  $(\mu, \nu)$ , and  $\mu$  and  $\nu$  are mutually singular, then  $\varphi \equiv 0$ .*

**COROLLARY 2.** *If  $\varphi$  is a bounded Hankel form on  $\mathcal{P} \times \mathcal{P}$  w.r.t.  $(w_1 dm, w_2 dm)$ , then for some  $k$  in  $L^\infty$*

$$\varphi(u, v) = \int u\bar{v}k\sqrt{w_1 w_2} dm \quad (u, v \in \mathcal{P}).$$

Conversely such  $\varphi$  is bounded w.r.t.  $(w_1 dm, w_2 dm)$ .

**3. Bounded Hankel forms on  $\mathcal{P}_+ \times \mathcal{P}_-$ .** In this section we will give a generalization of Nehari's theorem (see [13], [15, p. 6]) which was proved in the case of  $\mu = \nu = m$ . For any Hankel form  $\varphi$  on  $\mathcal{P} \times \mathcal{P}$ , if  $H_\varphi$  is bounded on  $\mathcal{P}_+ \times \mathcal{P}_-$  w.r.t.  $(\mu, \nu)$  then there exists a finite Borel measure  $\lambda$  on  $T$  such that

$$\varphi(u, v) = \int u\bar{v} d\lambda \quad (u \in \mathcal{P}_+, v \in \mathcal{P}_-).$$

The proof is similar to the proof of Proposition 1. Let  $\lambda = \lambda_a + \lambda_s$ ,  $\mu = \mu_a + \mu_s$  and  $\nu = \nu_a + \nu_s$  be Lebesgue decompositions with respect to  $m$ . Put

$$\varphi_a(u, v) = \int u\bar{v} d\lambda_a \quad \text{and} \quad \varphi_s(u, v) = \int u\bar{v} d\lambda_s$$

for any  $u, v$  in  $\mathcal{P}$ . Then  $H_{\varphi_a}$  and  $H_{\varphi_s}$  are bounded Hankel forms on  $\mathcal{P}_+ \times \mathcal{P}_-$  w.r.t.  $(\mu_a, \nu_a)$  and  $(\mu_s, \nu_s)$ , respectively. Moreover  $\max(\|H_{\varphi_a}\|, \|H_{\varphi_s}\|) = \|H_\varphi\|$ .

For set

$$H^2(\mu) = \text{the } L^2(\mu)\text{-closure of } \mathcal{P}_+.$$

Then  $\overline{\mathcal{P}_-}^2(\mu)$  is the  $L^2(\mu)$ -closure of  $\mathcal{P}_-$ . Suppose  $E_s$  is a Borel set with  $m(E_s) = 0$  where  $\mu_s$  and  $\nu_s$  are concentrated on  $E_s$ , and  $E_a$  is a Borel set with  $m(E_a) = 1$  where  $\mu_a$  and  $\nu_a$  are concentrated on  $E_a$ .  $E_a$  can be chosen to be the complement of  $E_s$  in  $T$ . Then both the characteristic functions  $\chi_{E_a}$  and  $\chi_{E_s}$  belong to  $H^2(\mu) \cap \overline{\mathcal{P}_-}^2(\nu)$ . Moreover  $H^2(\mu) = \chi_{E_a} H^2(\mu) \oplus \chi_{E_s} H^2(\mu)$ , and  $\chi_{E_a} H^2(\mu) = H^2(\mu_a)$  and  $\chi_{E_s} H^2(\mu) = H^2(\mu_s) = L^2(\mu_s)$ . This implies the above statement about  $H_{\varphi_a}$  and  $H_{\varphi_s}$ .

To prove the generalized Nehari's theorem, we need the following lemma which will be used in later sections, too.

**LEMMA 1.** *Let  $A$  be a bounded sesquilinear form on  $\mathcal{P}_+ \times \mathcal{P}_-$  w.r.t.  $(w_1 dm, w_2 dm)$  and  $w_j = |h_j|^2$  for  $j = 1, 2$  where both  $h_1$  and  $h_2$  are outer functions in  $H^2$ . If we put*

$$B(f, g) = A(h_1^{-1} f, \bar{h}_2^{-1} g) \quad (f \in \mathcal{P}_+, g \in \mathcal{P}_-)$$

*then  $B$  is a bounded sesquilinear form w.r.t.  $(m, m)$  and  $\|B\| = \|A\|$ .*

*Proof.* Let  $\gamma = \|A\|$ ; then

$$|A(f, g)|^2 \leq \gamma^2 \int |f|^2 |h_1|^2 dm \int |g|^2 |h_2|^2 dm$$

for any  $f \in \mathcal{P}_+$  and  $g \in \mathcal{P}_-$ . For any  $f \in \mathcal{P}_+$  and  $g \in \mathcal{P}_-$ , set  $F = h_1 f$  and  $G = \bar{h}_2 g$ . Then  $F \in H^2$  and  $G \in \bar{z}\bar{H}^2$ . Hence

$$|A(h_1^{-1}F, h_2^{-1}G)|^2 \leq \gamma^2 \int |F|^2 dm \int |G|^2 dm.$$

Since both  $h_1$  and  $h_2$  are outer functions, we get the lemma.

The following theorem is a generalization of Nehari's theorem (cf. [15, Theorem 1.3]) but this is the lifting theorem of Cotlar and Sadosky in [4], with other notation. A new proof is given here (cf. [17]).

**THEOREM 2.** *Let  $\varphi$  be a Hankel form on  $\mathcal{P} \times \mathcal{P}$ . If  $H_\varphi$  is bounded w.r.t.  $(\mu, \nu)$  then there exists a Hankel form  $\psi$  bounded w.r.t.  $(\mu, \nu)$  on  $\mathcal{P} \times \mathcal{P}$  such that*

$$H_\psi = H_\varphi \quad \text{and} \quad |||\psi||| = \|H_\varphi\|.$$

*Proof.* Let  $\gamma = \|H_\varphi\|$ . By the remark above Lemma 1

$$|\varphi_s(f, g)|^2 \leq \gamma^2 \int |f|^2 d\mu_s \int |g|^2 d\nu_s$$

for all  $f \in \mathcal{P}_+$  and  $g \in \mathcal{P}_-$ . Since  $H^2(\mu_s) = L^2(\mu_s)$ , this implies that  $|||\varphi_s||| \leq \gamma$ . Now we will prove that there exists a bounded Hankel form  $\varphi_a$  with respect to  $(\mu_a, \nu_a)$  such that

$$H_{\psi_a} = H_{\varphi_a} \quad \text{and} \quad |||\psi_a||| = \|H_{\varphi_a}\|.$$

Then setting  $\psi = \psi_a + \varphi_s$ , the theorem follows because  $\varphi = \varphi_a + \varphi_s$  and  $\max(\|H_{\varphi_a}\|, \|H_{\varphi_s}\|) = \|H_\varphi\|$ . Let  $d\mu_a = w_1 dm$  and  $d\nu_a = w_2 dm$ .

*Case I.*  $\log w_1 \notin L^1$  or  $\log w_2 \notin L^1$ . We may assume that  $\log w_1 \notin L^1$ . By the remark above Lemma 1,

$$|\varphi_a(f, g)|^2 \leq \gamma^2 \int |f|^2 w_1 dm \int |g|^2 w_2 dm \quad (f \in \mathcal{P}_+, g \in \mathcal{P}_-).$$

Since  $\log w_1 \notin L^1$ ,  $H^2(w_1 dm) = L^2(w_1 dm)$  and hence for any  $u \in \mathcal{P}$  and  $g \in \mathcal{P}_-$

$$|\varphi_a(u, g)|^2 \leq \gamma^2 \int |u|^2 w_1 dm \int |g|^2 w_2 dm.$$

Fix any  $n \in \mathbb{Z}_+$ . For any  $u_1 \in \mathcal{P}$  and  $g_1 \in z^n \mathcal{P}_-$ , there exists  $u \in \mathcal{P}$  and  $g \in \mathcal{P}_-$  such that  $u_1 = z^n u$  and  $g_1 = z^n g$ . Hence

$$\begin{aligned} |\varphi_a(u_1, g_1)|^2 &= |\varphi_a(z^n u, z^n g)|^2 = |\varphi_a(u, g)|^2 \\ &\leq \gamma^2 \int |u_1|^2 w_1 dm \int |g_1|^2 w_2 dm. \end{aligned}$$

By the same argument for any  $u, v \in \mathcal{P}$

$$|\varphi_a(u, v)|^2 \leq \gamma^2 \int |u|^2 w_1 dm \int |v|^2 w_2 dm.$$

This implies that  $\|\varphi_a\| \leq \gamma$ . Put  $\psi_a = \varphi_a$ .

*Case II.*  $\log w_1 \in L^1$  and  $\log w_2 \in L^1$ . There exist outer functions  $h_1$  and  $h_2$  in  $H^2$  such that  $w_1 = |h_1|^2$  and  $w_2 = |h_2|^2$  (cf. [6, p. 53]). Let  $d\lambda_a = w_3 dm$ . By Lemma 1

$$\begin{aligned} & \left| \int f \bar{g} (h_1 h_2)^{-1} w_3 dm \right|^2 \\ & \leq \gamma^2 \int |f|^2 dm \int |g|^2 dm \quad (f \in \mathcal{P}_+, g \in \mathcal{P}_-). \end{aligned}$$

Let  $s = w_3 (h_1 h_2)^{-1}$ ; then by a duality argument there exists  $l \in H^\infty$  such that  $\|s + l\|_\infty \leq \gamma$ . By Schwarz's lemma, this implies that

$$\left| \int (s + l) u_1 \bar{u}_2 dm \right|^2 \leq \gamma^2 \int |u_1|^2 dm \int |u_2|^2 dm \quad (u_1, u_2 \in \mathcal{P}).$$

Let  $v_1 = h_1^{-1} u_1$  and  $v_2 = \bar{h}_2^{-1} u_2$  for any  $u_1, u_2 \in \mathcal{P}$ . Then  $v_1 \in L^2(w_1 dm)$  and  $v_2 \in L^2(w_2 dm)$ . Hence

$$\begin{aligned} & \left| \int v_1 v_2 w_3 dm + \int v_1 \bar{v}_2 (l h_1 h_2) dm \right|^2 \\ & \leq \gamma^2 \int |v_1|^2 w_1 dm \int |v_2|^2 w_2 dm. \end{aligned}$$

Since  $h_1^{-1} \mathcal{P}$  and  $h_2^{-1} \mathcal{P}$  are dense in  $L^2(w_1 dm)$  and  $L^2(w_2 dm)$ , respectively, if we put

$$\varphi_0(u, v) = \int (l h_1 h_2) u \bar{v} dm \quad (u, v \in \mathcal{P})$$

then  $\varphi_0$  is a bounded Hankel form on  $\mathcal{P} \times \mathcal{P}$  w.r.t.  $(w_1 dm, w_2 dm)$ ,  $H_{\varphi_0} \equiv 0$  and  $\|\varphi_a + \varphi_0\| \leq \gamma$ . Put  $\psi_a = \varphi_a + \varphi_0$ .

Theorem 2 implies that  $\|H_\varphi\| = \inf\{\|\varphi + \varphi_0\| : H_{\varphi_0} \equiv 0\}$ .

In Theorem 2 if  $d\mu = d\nu = dm$  then Nehari's theorem follows and if  $d\mu = d\nu = w dm$  then the scalar version of a theorem of Page [9] follows.

**4. Compact bounded Hankel forms on  $\mathcal{P}_+ \times \mathcal{P}_-$ .** The ideas of this section are closely related to those of [2]. In particular, the concept of compact form and Theorem 3 are in Theorem 1a in [2]. Let  $A$  be a

bounded sesquilinear form on  $\mathcal{P}_+ \times \mathcal{P}_-$  w.r.t.  $(\mu, \nu)$ . We say that  $A$  is compact if there exists a null decreasing sequence  $\{\gamma_n\}$  such that

$$|A(z^n f, g)|^2 \leq \gamma_n^2 \int |f|^2 d\mu \int |g|^2 d\nu \quad (f \in \mathcal{P}_+, g \in \mathcal{P}_-)$$

and

$$|A(f, \bar{z}^n g)|^2 \leq \gamma_n^2 \int |f|^2 d\mu \int |g|^2 d\nu \quad (f \in \mathcal{P}_+, g \in \mathcal{P}_-)$$

for  $n = 1, 2, \dots$ . When  $\gamma_n = 0$  and  $\gamma_{n-1} \neq 0$  for some  $n$ ,  $A$  is called finite  $n$ . In this section we will give a generalization of Hartman's theorem [8] which was proved in the case of  $\mu = \nu = m$  and describes compact Hankel forms. However Theorem 4 does not show Hartman's theorem (see Remark).

**LEMMA 2.** *If  $A$  is a nonzero compact (finite  $n \neq 0$ , resp.) sesquilinear form w.r.t.  $(\mu, \nu)$  associated with  $\{\gamma_n\}$ , then it is a nonzero compact (finite  $n \neq 0$ , resp.) sesquilinear form w.r.t.  $(w_1 dm, w_2 dm)$  associated with  $\{\gamma_n\}$  where  $d\mu/dm = w_1$  and  $d\nu/dm = w_2$ . Moreover both  $\log w_1$  and  $\log w_2$  are integrable.*

*Proof.* Let  $E_a$  and  $E_s$  be Borel sets as in the remark before Lemma 1. Then  $\chi_{E_a}$  and  $\chi_{E_s}$  belong to  $H^2(\mu) \cap \bar{z}H^2(\nu)$ . Hence for  $n = 1, 2, \dots$

$$|A(\chi_{E_s} z^n f, g)|^2 \leq \gamma_n^2 \int |f|^2 d\mu_s \int |g|^2 d\nu \quad (f \in \mathcal{P}_+, g \in \mathcal{P}_-)$$

and

$$|A(f, \chi_{E_s} \bar{z}^n g)|^2 \leq \gamma_n^2 \int |f|^2 d\mu \int |g|^2 d\nu_s \quad (f \in \mathcal{P}_+, g \in \mathcal{P}_-).$$

Since  $H^2(\mu_s) = L^2(\mu_s)$  and  $H^2(\nu_s) = L^2(\nu_s)$ , for  $n = 1, 2, \dots$

$$|A(\chi_{E_s} u, g)|^2 \leq \gamma_n^2 \int |u|^2 d\mu_s \int |g|^2 d\nu \quad (u \in \mathcal{P}, g \in \mathcal{P}_-)$$

and

$$|A(f, \chi_{E_s} v)|^2 \leq \gamma_n^2 \int |f|^2 d\mu \int |v|^2 d\nu_s \quad (f \in \mathcal{P}_+, v \in \mathcal{P}_-).$$

As  $n \rightarrow \infty$ , it follows that  $A(\chi_{E_s} f, g) = A(f, \chi_{E_s} g) = 0$  for all  $f \in \mathcal{P}_+$  and  $g \in \mathcal{P}_-$ . Hence  $A(z^n f, g) = A(\chi_{E_a} z^n f, \chi_{E_a} g)$  and  $A(f, \bar{z}^n g) = A(\chi_{E_a} f, \chi_{E_a} \bar{z}^n g)$ . This implies that  $A$  is a nonzero

compact (finite  $n \neq 0$ , resp.) sesquilinear form w.r.t.  $(w_1 dm, w_2 dm)$  associated with  $\{\gamma_n\}$ . If  $\log w_1 \notin L^1$  or  $\log w_2 \notin L^1$  then  $H^2(w_1 dm) = L^2(w_1 dm)$  or  $H^2(w_2 dm) = L^2(w_2 dm)$ . By the same argument to the above, we can show that  $A$  is a zero form. Thus the lemma follows.

**THEOREM 3.** *Let  $n$  be a nonnegative integer.*

(1)  $H_\varphi$  is finite  $n = 0$  if and only if there exists a function  $h$  in  $H^1$  such that  $\varphi(f, g) = \int f \bar{g} h dm$  ( $f \in \mathcal{P}_+, g \in \mathcal{P}_-$ ).

(2) When  $n \neq 0$ ,  $H_\varphi$  is finite  $n$  if and only if there exists a function  $h$  in  $\bar{z}^n H^1$  and out of  $H^1$  such that  $\varphi(f, g) = \int f \bar{g} h dm$  ( $f \in \mathcal{P}_+, g \in \mathcal{P}_-$ ).

*Proof.* (1) There exists a finite Borel measure  $\lambda$  such that  $\varphi(f, g) = \int f \bar{g} d\lambda$  ( $f \in \mathcal{P}_+, g \in \mathcal{P}_-$ ). If  $H_\varphi$  is zero, by the proof of Lemma 2  $\varphi(f, g) = \varphi(\chi_{E_a} f, \chi_{E_a} g)$  and hence  $\lambda$  is absolutely continuous w.r.t.  $dm$ . Let  $d\lambda = h dm$ ; then  $h dm$  annihilates  $z\mathcal{P}_+$  and so  $h \in H^1$ . The converse is clear.

(2) Let  $H_\varphi$  be finite,  $n \neq 0$ . By Corollary 2, Theorem 2 and Lemma 2, there exists a nonzero function  $h$  in  $L^1$  such that

$$\varphi(f, g) = \int f \bar{g} h dm \quad (f \in \mathcal{P}_+, g \in \mathcal{P}_-).$$

Since  $H_\varphi$  is finite,  $n \neq 0$ , by Lemma 2 there exist  $\gamma_1, \gamma_2, \dots, \gamma_n$  with  $\gamma_n = 0$  such that for  $1 \leq j \leq n$

$$\left| \int z^j f \bar{g} h dm \right|^2 \leq \gamma_j^2 \int |f|^2 w_1 dm \int |g|^2 w_2 dm \quad (f \in \mathcal{P}_+, g \in \mathcal{P}_-),$$

where  $w_1 = d\mu/dm$  and  $w_2 = d\nu/dm$ . Moreover there exist outer functions  $h_1$  and  $h_2$  such that  $|h_j|^2 = w_j$  for  $j = 1, 2$ . By Lemma 1, for  $1 \leq j \leq n$

$$\left| \int z^j f \bar{g} (h_1 h_2)^{-1} h dm \right|^2 \leq \gamma_j^2 \int |f|^2 dm \int |g|^2 dm \quad (f \in \mathcal{P}_+, g \in \mathcal{P}_-)$$

and hence  $\|z^j (h_1 h_2)^{-1} h + H^\infty\| \leq \gamma_j$ . Since  $\gamma_n = 0$ ,  $(h_1 h_2)^{-1} h \in \bar{z}^n H^\infty$  and hence  $h \in \bar{z}^n H^1$  and  $h \notin H^1$  because  $H_\varphi$  is rank  $n \neq 0$ . The converse is clear because for such  $h$ ,  $\int z^n f \bar{g} h dm = 0$  ( $f \in \mathcal{P}_+, g \in \mathcal{P}_-$ ).

In the proof of Theorem 3,  $h_1 h_2 \in H^1$  and  $h = (h_1 h_2)u$  where  $u \in \bar{z}^n H^\infty$ . The following theorem is the generalization of this result.

**THEOREM 4.**  $H_\varphi$  is nonzero and compact w.r.t.  $(\mu, \nu)$  if and only if there exists a function  $h = h_0 \times u$  in  $H^1 \times (H^\infty + C)$  and out of  $H^1$  such that

$$\varphi(f, g) = \int f \bar{g} h \, d\mu \quad (f \in \mathcal{P}_+, g \in \mathcal{P}_-)$$

and  $h_0 = h_1 h_2$  where  $h_j$  is an outer function in  $H^2$ ,  $w_j = |h_j|$ ,  $d\mu/dm = w_1$  and  $d\nu/dm = w_2$ .

*Proof.* Let  $H_\varphi$  be nonzero and compact. By Lemma 2, we may assume that  $d\mu = w_1 \, dm$  and  $d\nu = w_2 \, dm$ , and there exists an outer function  $h_j$  in  $H^2$  with  $w_j = |h_j|^2$ . By the proof of Theorem 3,  $\|z^j (h_1 h_2)^{-1} h + H^\infty\| \leq \gamma_j$  and  $\gamma_j \rightarrow 0$  as  $j \rightarrow \infty$ . Thus  $(h_1 h_2)^{-1} h \in H^\infty + C$  and hence  $h = (h_1 h_2)u \in H^1 \times (H^\infty + C)$  and out of  $H^1$ . For the converse, put  $\|z^j u + H^\infty\| = \gamma_j$ ; then  $\gamma_j \rightarrow 0$  as  $j \rightarrow \infty$  and for each  $j$  there exists  $g_j \in H^\infty$  such that

$$|z^j h + h_1 h_2 g_j| \leq \gamma_j |h_1 h_2|.$$

Hence for each  $j$

$$\begin{aligned} |\varphi(z^j f, g)|^2 &= \left| \int z^j f \bar{g} h \, d\mu \right|^2 \leq \gamma_j^2 \int |f \bar{g}| |h_1 h_2| \, d\mu \\ &\leq \gamma_j^2 \int |f|^2 w_1 \, dm \int |g|^2 w_2 \, dm \end{aligned}$$

for all  $f \in \mathcal{P}_+$  and  $g \in \mathcal{P}_-$ . This implies that  $H_\varphi$  is nonzero and compact w.r.t.  $(\mu, \nu)$ .

If  $h = h_0 \times u$  is in  $H^1 \times (H^\infty + C)$  and  $\varphi_1(f, g) = \int f \bar{g} h \, d\mu$  ( $f \in \mathcal{P}_+, g \in \mathcal{P}_-$ ) then  $H_{\varphi_1}$  is compact w.r.t.  $(\mu_1, \nu_1)$  where  $d\mu_1 = d\mu$  and  $d\nu_1 = |h_0|^2 \, dm$ .

If  $\mu$  is a complex finite Borel measure on  $T$  and  $\hat{\mu}(n) = \int e^{-in\theta} \, d\mu = 0$  for any negative integer  $n$ , then  $d\mu = h \, dm$  for some  $h$  in  $H^1$ . This is the famous F. and M. Riesz theorem (cf. [11, p. 47]) and a corollary of the following corollary which follows from Theorem 3 and 4. That is, it is just the case of  $\varepsilon_0 = 0$ .

**COROLLARY 4.** Let  $\mu$  be a complex finite Borel measure on  $T$  and

$$\varepsilon_n = \sup \left\{ \left| \int z^n F \, d\mu \right| ; F \in \mathcal{P}_+, \int |F| \, d|\mu| \leq 1 \right\}.$$

If  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$  then  $\mu = h dm$  and  $h$  is in  $H^1 \times (H^\infty + C)$ . If  $\varepsilon_n = 0$  for some  $n \geq 0$  then  $h$  belongs to  $\bar{z}^n H^1$ .

*Proof.* By Schwarz's lemma,

$$\sup \left\{ \left| \int z^n f \bar{g} d\mu \right| ; f \in \mathcal{P}_+, g \in \mathcal{P}_-, \int |f|^2 d|\mu| \leq 1 \right. \\ \left. \text{and } \int |g|^2 d|\mu| \leq 1 \right\} \leq \varepsilon_n.$$

Now apply Theorems 3 and 4 for  $\varphi(z^n f, g) = \int z^n f \bar{g} d\mu$ .

### 5. Distance between $H_\varphi$ and the set of all compact sesquilinear forms.

**THEOREM 5.** *Let  $H_\varphi$  be a bounded Hankel form and  $A$  a compact (finite  $n$ , resp.) sesquilinear form on  $\mathcal{P}_+ \times \mathcal{P}_-$  w.r.t.  $(\mu, \nu)$ . If  $\|H_\varphi + A\| \leq \gamma$  then there exists a symbol  $\psi$  such that  $H_\psi$  is a compact (finite  $n$ , resp.) Hankel form w.r.t.  $(\mu, \nu)$  and  $\|\varphi + \psi\| \leq \gamma$ .*

*Proof.* By the remark preceding Lemma 1, we can decompose  $\varphi = \varphi_a + \varphi_s$  where  $H_{\varphi_a}$  is bounded w.r.t.  $(\mu_a, \nu_a)$  and  $H_{\varphi_s}$  is bounded w.r.t.  $(\mu_s, \nu_s)$ . If  $\|H_\varphi + A\| \leq \gamma$  then by Lemma 2 and the proof of Theorem 2  $\|\varphi_s\| \leq \gamma$  and  $\|H_{\varphi_a} + A\| \leq \gamma$ . Hence we may assume that  $\varphi = \varphi_a$ ,  $\mu = \mu_a = w_1 dm$  and  $\nu = \nu_a = w_2 dm$ . If  $\log w_1 \notin L^2(m)$  or  $\log w_2 \notin L^1(m)$ , by Lemma 2  $A(f, g) = 0$  ( $f \in \mathcal{P}_+, g \in \mathcal{P}_-$ ) and hence Theorem 2 implies the theorem. By Lemma 1

$$|\varphi(h_1^{-1} f, \bar{h}_2^{-1} g) + A(h_1^{-1} f, h_2^{-1} g)|^2 \\ \leq \gamma^2 \int |f|^2 dm \int |g|^2 dm \quad (f \in \mathcal{P}_+, g \in \mathcal{P}_-)$$

and there exists a null decreasing sequence  $\{\gamma_n\}$  such that

$$|A(h_1^{-1} z^n f, h_2^{-1} g)|^2 \\ \leq \gamma_n^2 \int |z^n f|^2 dm \int |g|^2 dm \quad (f \in \mathcal{P}_+, g \in \mathcal{P}_-).$$

Hence there exist bounded linear operators  $H_l$  and  $\mathcal{A}$  from  $H^2(m)$  to  $\bar{z}H^2(m)$  such that

$$(H_l f, g) = (lf, g) = \varphi(h_1^{-1} f, h_2^{-1} g)$$

and

$$(\mathcal{A} f, g) = A(h_1^{-1} f, h_2^{-1} g)$$

where  $l \in L^\infty(m)$  and  $(\cdot, \cdot)$  denotes the usual inner product with respect to  $m$ . Let  $U$  be a unilateral shift on  $H^2$ ; then  $\|\mathcal{A}U^n\| \rightarrow 0$  because  $\gamma_n \rightarrow 0$ . By the same argument as in [10, p. 6], there exists a function  $k \in H^\infty + C$  such that  $\|l + k\|_\infty < 1$ . Similarly to the proof of Theorem 2 put

$$\psi(u, v) = \int (kh_1h_2)u\bar{v} dm \quad (u, v \in \mathcal{P}).$$

Then  $\psi$  is a bounded Hankel form w.r.t.  $(w_1 dm, w_2 dm)$  and by Theorem 4  $H_\psi$  is compact. Thus  $\|\|\varphi + \psi\|\| \leq \gamma$ .

Theorem 5 implies that  $\inf\{\|H_\varphi + A\|: A \text{ ranges over all compact sesquilinear forms}\} = \inf\{\|\|\varphi + \psi\|\|: H_\psi \text{ ranges over all compact Hankel forms}\}$ . When  $d\mu = d\nu = dm$ , this relates a theorem of Adamjan, Arov and Krein (cf. [1], [15, p. 6]). However the former does not imply the latter (see Remark).

**6. Lifting theorem.** In this section we obtain a new lifting theorem which contains one due to Cotlar and Sadosky [2]. Let  $A_{ij}$  ( $i, j = 1, 2$ ) be bilinear forms on  $\mathcal{P} \times \mathcal{P}$  and suppose

$$A_{11}(u, u) \geq 0, \quad A_{22}(u, u) \geq 0 \quad \text{and} \quad A_{12}(u, v) = \overline{A_{21}(u, v)}.$$

Set

$$\mathbf{A}(\mathbf{u}, \mathbf{u}) = \sum_{i,j=1}^2 A_{ij}(u_i, u_j)$$

where  $\mathbf{u} = (u_1, u_2)$  and  $u_i \in \mathcal{P}$  for  $i = 1, 2$ . We write  $\mathbf{A} = [A_{ij}]$ . If  $\rho_{ij}$  ( $i, j = 1, 2$ ) are finite Borel measures on  $T$  and

$$A_{ij}(u, v) = \int u\bar{v} d\rho_{ij} \quad (u \in \mathcal{P}_+, v \in \mathcal{P}_-),$$

then  $A_{ij}$  ( $i, j = 1, 2$ ) are bounded Hankel forms on  $\mathcal{P} \times \mathcal{P}$  w.r.t.  $(|\rho_{ij}|, |\rho_{ij}|)$ . By the hypothesis on  $[A_{ij}]$

$$\rho_{11} \geq 0, \quad \rho_{22} \geq 0 \quad \text{and} \quad \rho_{12} = \bar{\rho}_{21}.$$

We write  $\mathbf{A} = [A_{ij}] = [\rho_{ij}] = \rho$  and we call  $\rho$  a matrix of measures.  $\mathbf{A} > 0$  w.r.t.  $\Gamma$  means that  $\mathbf{A}$  is positive w.r.t.  $\Gamma$ :

$$\mathbf{A}(\mathbf{u}, \mathbf{u}) = \sum_{i,j=1}^2 A_{ij}(u_i, u_j) \geq 0 \quad (\mathbf{u} \in \Gamma)$$

where  $\Gamma$  denotes  $\mathcal{P} \times \mathcal{P}$  or  $\mathcal{P}_+ \times \mathcal{P}_-$ .

We say that  $\mathbf{A}$  is compact (finite  $n$ , resp.) w.r.t.  $\rho$  if  $A_{11} = A_{22} = 0$  and  $A_{12}$  is compact (finite  $n$ ) w.r.t.  $(\rho_{11}, \rho_{22})$ .

**THEOREM 6.** *Let  $\rho$  be a matrix of measures. If*

$$\rho + \mathbf{A} \succ 0 \quad \text{w.r.t. } \mathcal{P}_+ \times \mathcal{P}_-$$

*where  $\mathbf{A}$  is compact (finite  $n$ , resp.) w.r.t.  $\rho$ , then there exists a compact (finite  $n$ , resp.) matrix  $\tau$  of measures w.r.t.  $\rho$  such that*

$$\rho + \tau \succ 0 \quad \text{w.r.t. } \mathcal{P} \times \mathcal{P}.$$

*Proof.* Let

$$\varphi_{12}(f, g) = \int f \bar{g} d\rho_{12} \quad (f \in \mathcal{P}_+, g \in \mathcal{P}_-).$$

Then  $\varphi_{12} + A_{12}$  is a bounded bilinear form on  $\mathcal{P}_+ \times \mathcal{P}_-$  w.r.t.  $(\rho_{11}, \rho_{22})$  because  $\rho + \mathbf{A} \succ 0$ . Let  $\|\varphi_{12} + A_{12}\| \leq \gamma$ . By Theorem 5, there exists a symbol  $\psi$  such that  $H_\psi$  is a compact (finite  $n$ , resp.) w.r.t.  $(\rho_{11}, \rho_{22})$  and  $\|\varphi_{12} + \psi\| \leq \gamma$ . By Theorems 3 and 4, there exists a function  $h$  in  $L^1$  such that

$$\psi(f, g) = \int f \bar{g} h dm \quad (f \in \mathcal{P}_+, g \in \mathcal{P}_-).$$

Then  $d\tau_{12} = h dm$  is the desired measure.

**COROLLARY 3** (Cotlar and Sadosky). *Let  $\rho$  be a matrix of measures. If*

$$\rho \succ 0 \quad \text{w.r.t. } \mathcal{P}_+ \times \mathcal{P}_-$$

*then there exists a finite  $n = 0$  matrix  $\tau$  of measures such that*

$$\rho + \tau \succ 0 \quad \text{w.r.t. } \mathcal{P} \times \mathcal{P}.$$

*By Theorems 3 and 4, we can describe compact (finite  $n$ , resp.) matrices of measures w.r.t.  $\rho$ .*

**7. Weighted norm inequalities.** In this section we show known results in the  $L^2$  weighted problem, using the theorems of §§3, 4 and 5. For any fixed nonnegative integer  $n$ , we want to find the positive measure  $\mu$  for which there is a nonzero positive measure  $\nu_n$  such that

$$\int |z^n f|^2 d\nu_n \leq \int |z^n f + g|^2 d\mu \quad (f \in \mathcal{P}_+, g \in \mathcal{P}_-).$$

The inequality above is equivalent to the following one:

$$\left| \int z^n f \bar{g} d\mu \right|^2 \leq \int |f|^2 d(\mu - \nu_n) \int |g|^2 d\mu \quad (f \in \mathcal{P}_+, g \in \mathcal{P}_-).$$

Hence the problem is related with prediction problems when such a measure  $\mu$  arises as the spectral density of a discrete weakly stationary Gaussian stochastic process. The following proposition is due to Arocena, Cotlar and Sadosky [3]. The Helson-Szegö theorem [10] and the Koosis theorem [12] follow from the first part in it.

**PROPOSITION 7.** *Let  $\mu$  be a positive measure. There is a nonzero positive measure  $\nu$  such that*

$$\int |f|^2 d\nu \leq \int |f + g|^2 d\mu \quad (f \in \mathcal{P}_+, g \in \mathcal{P}_-)$$

*if and only if  $d\nu = u dm$  and there is a nonzero  $k$  in  $H^1$  such that*

$$|w + k|^2 \leq (w - u)w$$

*where  $d\mu = w dm + d\mu_s$ . Then if  $\log(w - u)$  is in  $L^1$  then  $u \leq (1 - \gamma^{-1})w$  and  $\gamma > 1$ .*

We can prove Proposition 7 using the lifting theorem of Cotlar and Sadosky (Theorem 2 or Corollary 3) as that in [3]. The following theorem is closely related to results in [3]. We will give a proof using Theorems 3 and 4.

**THEOREM 8.** *Let  $\mu$  be a positive measure. For any fixed nonnegative integer  $n$ , let  $\nu_n$  be a nonzero positive measure such that*

$$\int |z^n f|^2 d\nu_n \leq \int |z^n f + g|^2 d\mu \quad (f \in \mathcal{P}_+, g \in \mathcal{P}_-).$$

*Suppose that there exists a positive measure  $\lambda$  and a decreasing sequence  $\{\varepsilon_n\}$  such that  $\nu_n = \mu - \varepsilon_n \lambda$  and  $0 \leq \varepsilon_n \leq 1$ .*

*(1)  $\varepsilon_n = 0$  for some  $n$  if and only if  $d\nu_n = d\mu = w dm$  and  $w = sh$  where  $h$  is an outer function with  $w = |h|$  and  $s$  is in  $\bar{z}^n H^\infty$ .*

*(2)  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$  if and only if  $d\nu_n = (w_1 - \varepsilon_n w_2) dm$ ,  $d\mu = w_1 dm$ ,  $d\lambda = w_2 dm + d\lambda_s$  and  $w_1 = sh_1 h_2$  where  $h_j$  is an outer function with  $w_j = |h_j|^2$  for  $j = 1, 2$  and  $s$  is in  $H^\infty + C$ .*

*Proof.* Set

$$\varphi(u, v) = \int u\bar{v} d\mu \quad (u, v \in \mathcal{P});$$

then by the remark before Theorem 7  $H_\varphi$  is finite  $n$  and compact w.r.t.  $(\lambda, \mu)$  for (1) and (2), respectively. (1) follows from (2) of Theorem 3. For if  $\varepsilon_n = 0$  for some  $n$  then  $w \in \bar{z}^n H^1$  and hence  $w = |h| = \bar{z}^n qh$  where  $q$  is in  $H^\infty$ . (2) follows from Theorem 4.

In Theorem 8, if  $\lambda = \mu$  this was proved by Helson and Sarason [10]. Theorem 8 is also a corollary of Theorem 6 which is a new lifting theorem.

REMARK. Hankel operators from  $H^2(\mu)$  to  $\overline{z}H^2(\nu)$ . Let  $\mu$  and  $\nu$  be finite positive Borel measures on  $T$ .  $M_z^\mu$  and  $M_z^\nu$  are multiplication operators by the coordinate function  $z$  on  $L^2(\mu)$  and  $L^2(\nu)$ , respectively. Let  $\Phi$  be a bounded linear operator from  $L^2(\mu)$  to  $L^2(\nu)$  and  $(\Phi u, v) = \int (\Phi u)\overline{v} d\nu$  for  $u, v$  in  $\mathcal{P}$ . Then  $\Phi M_z^\mu = M_z^\nu \Phi$  if and only if  $\varphi(u, v) = (\Phi u, v)$  is a bounded Hankel form on  $\mathcal{P} \times \mathcal{P}$  w.r.t.  $(\mu, \nu)$ . Let  $P$  and  $Q$  be the orthogonal projections from  $L^2(\mu)$  to  $H^2(\mu)$  and from  $L^2(\nu)$  to  $\overline{z}H^2(\nu)$ , respectively. Put  $H = Q\Phi P$ ; then  $(Hf, g) = H_\varphi(f, g)$  for  $f$  in  $\mathcal{P}_+$  and  $g$  in  $\mathcal{P}_-$ . Put  $S_z^\mu = PM_z^\mu|_{H^2(\mu)}$  and  $S_z^\nu = QM_z^\nu|_{\overline{z}H^2(\nu)}$ ; then  $HS_z^\mu = (S_z^\nu)^*H$ . Theorem 2 calculates the norm of  $H$ . In general, even if  $H$  is a compact linear operator,  $H_\varphi$  may not be a compact sesquilinear form.

When  $\mu = \nu = m$ ,  $\Phi$  is a multiplication operator  $M_\Phi$  by a function  $\Phi$  in  $L^\infty(m)$  and  $\|\Phi\| = \|\Phi\|_\infty = \|\varphi\|$ .  $H$  is called a Hankel operator and  $\|H\| = \|H_\varphi\|$ .  $H_\varphi$  is a compact Hankel form if and only if  $H$  is a compact Hankel operator. For by Theorem 4  $H_\varphi$  is compact if and only if  $\varphi(f, g) = \int f\overline{g}h dm$  ( $f \in \mathcal{P}_+$ ,  $g \in \mathcal{P}_-$ ) and  $h \in H^\infty + C$ . By Hartman's theorem (cf. [15, Theorem 1.4])  $H$  is compact if and only if  $\Phi \in H^\infty + C$ . Moreover the essential norm  $\|H\|_e$  of  $H$  coincides with  $\inf\{\|H_\varphi + A\|: A \text{ ranges over all compact sesquilinear forms}\}$ . For by a theorem of Adamjan, Arov and Krein [1],  $\|H\|_e = \|\Phi + H^\infty + C\|$ . While by Theorems 4 and 5  $\inf\|H_\varphi + A\| = \inf\{\|\varphi + \psi\|: H_\psi \text{ ranges over all compact Hankel forms}\} = \|h + H^\infty + C\|$  where  $\varphi(f, g) = \int f\overline{g}h dm$ .

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