ON COVERINGS OF FIGURE EIGHT KNOT SURGERIES

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We show that over half of the Dehn surgeries on $S^3$ along the figure eight knot $K$ yield manifolds having finite covers with positive first Betti number by explicitly constructing these covers and exhibiting their homology.

1. Introduction. Denote by $K$ the figure eight knot, pictured in Figure 1. In his celebrated Notes, [T], Thurston showed that all but finitely many Dehn surgeries along $K$ in $S^3$ yield hyperbolic non-Haken manifolds—the first such examples. It remains an open question whether or not these manifolds (or every closed, irreducible 3-manifold with infinite $\pi_1$) are finitely covered by Haken manifolds, or stronger still, by manifolds with positive first Betti number.

![Figure 1](image-url)

In this paper we will show that over half of the Dehn surgeries along $K$ yield manifolds having finite covers with positive first Betti number by explicitly constructing these covers and exhibiting their homology.

Section 2 is devoted to notation and preliminaries. Section 3 contains a statement of our results as well as a summary of previous results on the problem. The method of proof is outlined in §4. Proofs are given in §§5–7.

2. Preliminaries. Throughout this paper $K$ will denote the figure eight knot and $M$ the complement, in $S^3$, of an open regular neighborhood of $K$. We will use the fact that $M$ is a bundle over $S^1$ with fiber a once-punctured torus.

2.1. Let $T_0$ denote the torus with an open disk removed, pictured in Figure 2. Let $D_x$ denote the left-handed Dehn twist about the loop
$x$ and $D_y$ the right-handed Dehn twist about the loop $y$ in $T_0$. Then

$$M \cong T_0 \times [0, 1]/(g(s), 0) \sim (s, 1)$$

where $g = D_x \circ D_y$.

We fix a basepoint, $b$, in $\partial T_0$ and let $x$, $y$ be the elements of $\pi_1(T_0, b)$ represented by the loops $x$, $y$ in $T_0$ based at $b$ via the arc $\sigma$. Then $x$ and $y$ freely generate $\pi_1(T_0, b)$ and $D_x$, $D_y$ induce the isomorphisms:

$$(D_x)_#: x \rightarrow x, \quad y \rightarrow yx,$$

$$(D_y)_#: x \rightarrow xy, \quad y \rightarrow y.$$

The loop $\alpha = b \times [0, 1]/\sim$ is a meridian for $K$ and $\beta = \partial T_0$ is a longitude for $K$. Then

$$\pi_1(M) \cong \langle x, y, \alpha | \alpha^{-1}xx = xyx, \quad \alpha^{-1}yx = yx \rangle$$

which is easily seen to be isomorphic to the following Wirtinger presentation for $S^3 \setminus K$:

$$\pi_1(S^3 \setminus K) \cong \langle a, b | (a^{-1}bab^{-1})a(a^{-1}bab^{-1})^{-1}b^{-1} = \text{id} \rangle.$$ 

Indeed, first eliminate $y$ ($y = x^{-1}\alpha^{-1}xax^{-1}$) then set $\alpha = a^{-1}$ and $x = ba^{-1}$.

2.2. By Dehn filling on a 3-manifold $X$ with respect to a loop in a boundary torus, we mean attaching a solid torus to $\partial X$ so that this loop bounds a meridional disk in the solid torus.

We say that $X$ has a virtually $\mathbb{Z}$-representable fundamental group if $\pi_1(X)$ contains a finite index subgroup with non-trivial representation to $\mathbb{Z}$. If $X$ is compact, this is equivalent to the existence of a finite cover $\widetilde{X} \rightarrow X$ with $\beta_1(\widetilde{X}) \equiv \text{rank} H_1(\widetilde{X}) > 0$.

Given a surface $F$ and a homeomorphism $h: F \rightarrow F$, we define the corresponding bundle over $S^1$ by $F \times I/\sim = F \times [0, 1]/(h(s), 0) \sim (s, 1)$. Note that the back face $F \times \{1\}$ is attached to the front face $F \times \{0\}$ via $h$. 

![Figure 2](image-url)
Given $M_h = T_0 \times I/h$ with $h$ the identity on $\partial T_0$, define (as for $M$) the loops $\alpha_h = b \times I/\sim$, $\beta = \partial T_0$.

**Definitions.** (1) $M_h(\mu, \lambda)$ represents the manifold obtained by Dehn filling on $M_h$ with respect to the loop $\alpha^\mu_\lambda$.

(2) By $\mu/\lambda$ Dehn surgery along $K$ in $S^3$, we mean Dehn filling on $M$ with respect to $\alpha^\mu_\lambda$. Let $M(\mu, \lambda)$ denote the resulting manifold.

**Remarks.** (1) $M(\mu, \lambda) \cong M(\mu, -\lambda)$ since there exists an orientation reversing homeomorphism on $M$ sending $\alpha$ to $\alpha$ and $\beta$ to $\beta^{-1}$ (see [H2] or [T]).

(2) Since $M_h(\mu, \lambda) = M_h(-\mu, -\lambda)$ we will assume that $\mu \geq 1$.

3. Statement of results. $M(\mu, \lambda)$ is known to have a virtually $\mathbb{Z}$-representable fundamental group if:

(i) $\lambda \equiv \pm 2\mu \pmod{7}$ (see [H1] or [N]),

(ii) $\lambda \equiv \pm \mu \pmod{13}$ (see [H1]),

(iii) $\mu \equiv 0 \pmod{4}$ and $\mu/\lambda \neq \pm 8$ (see [KL]).

In §5 below, we will prove:

**Theorem A.** $M(3\mu, \lambda)$ has a virtually $\mathbb{Z}$-representable fundamental group if $|\lambda| \notin \{\mu - 1, \mu + 1\}$.

In §6, we first give a simple proof of (iii) by explicitly constructing covers $N \rightarrow M(4\mu, \lambda)$, for which $\beta_1(N) \geq 1$. We show that $M(8, \pm 1)$ has a virtually $\mathbb{Z}$-representable fundamental group, the case not covered in [KL]. We then prove virtual $\mathbb{Z}$-representability for certain $M(2\mu, \lambda)$:

**Proposition C.** $M(2\mu, \lambda)$ has a virtually $\mathbb{Z}$-representable fundamental group if $\lambda \equiv \pm 7\mu \pmod{15}$.

In §7, we study singular boundary curve systems for $M$. In [H2], it is shown that $\{\alpha^3\}$, $\{\alpha\beta\}$ and $\{\alpha\beta^{-1}\}$ are singular boundary curve systems. We prove the following result:

**Theorem D.** $\{\alpha^2\beta\}$, $\{\alpha^2\beta^{-1}\}$, $\{\alpha^3\beta\}$ and $\{\alpha^3\beta^{-1}\}$ are singular boundary curve systems for $M$.

**Remark.** Our results, combined with (i)–(iii) above, show that approximately two-thirds of the surgeries on $K$ yield manifolds having virtually $\mathbb{Z}$-representable fundamental groups.
4. Construction of covers. For a given \((\mu, \lambda)\), we show that \(M(\mu, \lambda)\) has a virtually \(\mathbb{Z}\)-representable fundamental group by constructing a finite cover \(N \to M(\mu, \lambda)\) with \(\beta_1(N) \equiv \text{rank} \, H_1(N) \geq 1\). The cover \(N\) is obtained from a finite cover \(\tilde{M} \to M\) having the following two properties:

(i) The loop \(\alpha^\mu \beta^\lambda\) in \(\partial M\) lifts to loops in the components of \(\partial \tilde{M}\);

(ii) \(\beta_1(\tilde{M}) > \beta_0(\partial \tilde{M})\).

Property (i) guarantees that \(\tilde{M} \to M\) extends to an (unbranched) cover \(N \to M(\mu, \lambda)\) by Dehn filling on \(\tilde{M}\) and \(M\). Property (ii) guarantees that any manifold obtained by Dehn filling on \(\tilde{M}\) (hence \(N\)) has positive first Betti number.

Since \(M\) is a bundle over \(S^1\) with fiber \(T_0\) and characteristic homeomorphism \(g\), it follows that \(\tilde{M}\) is also a bundle over \(S^1\) with fiber \(F\) a cover of \(T_0\) and characteristic homeomorphism \(\tilde{g}\) a lifting of \(g^n\) for some integer \(n \geq 1\).

It is easy to show (see [H1]) that \(\tilde{M}\) satisfies property (ii) above if and only if \(\tilde{g}_*: H_1(F) \to H_1(F)\) fixes a non-boundary class in \(H_1(F)\).

We adopt the terminology of [H1] that \(\tilde{g}\) is homology reducible if it fixes such a non-boundary class in \(H_1(F)\).

Thus we will construct \(\tilde{M}\) by constructing a finite cover \(F \to T_0\) to which an appropriate power of \(g\) lifts to a homeomorphism \(\tilde{g}: F \to F\) that is homology reducible.

Since \(g = D_x \circ D_y\), it is difficult to tell, given a cover \(F \to T_0\), whether or not \(g^n\) lifts to a \(\tilde{g}\) that is homology reducible (in fact the matter of whether or not a given \(g^n\) even lifts is difficult to verify in practice). We will avoid these difficulties by using the fact that \(g^2\), \(g^3\) and \(g^4\) are isotopic to maps that are much easier to work with.

5. In this section we prove the following:

**Theorem A.** \(M(3\mu, \lambda)\) has a virtually \(\mathbb{Z}\)-representable fundamental group if \(|\lambda| \notin \{\mu - 1, \mu + 1\}\).

We fix \(h = D_x^2 \circ D_y^{-4} \circ D_x \circ D_y^{-4} \circ D_x\). Recall that \(M_h = T_0 \times I/h\), \(\alpha_h = b \times I/h\), \(\beta = \partial T_0\) and \(M_h(\mu, \lambda)\) is the manifold obtained by Dehn filling on \(M_h\) with respect to the loop \(\alpha_h^\mu \beta^\lambda\) (see §2.2).

**Lemma 5.1.** \(M_h(\mu, \lambda) \to M(3\mu, \mu + \lambda) \cong M(3\mu, -\mu - \lambda)\) is a 3-fold cover.
Proof. Since $h$ and $g^3$ both have the same monodromy matrix \((\frac{13}{8} \frac{3}{5})\) $\in \text{SL}_2(\mathbb{Z})$ they are isotopic, and hence $M_h$ is bundle equivalent to the 3-fold cyclic cover, $M_{g^3}$, of $M$. Moreover the isotopy $H$ from $g^3$ to $h$ rotates $\partial T_0$ one turn counter-clockwise, since for any $z \in \pi_1(T_0, b)$,

$$g_\#^3(z) = (xyx^{-1}y^{-1})h_\#(z)(xyx^{-1}y^{-1})^{-1}.$$  

(It suffices to check this for $x, y \in \pi_1(T_0, b)$.) Thus the induced bundle isomorphism $H: M_h \to M_{g^3}$ sends the pair of loops $(\alpha_h, \beta)$ to $(\alpha^{g^3}_h \beta, \beta)$ which projects to $(\alpha^3 \beta, \beta)$ in $M$.  

Now Theorem 1 of [B] tells us that $M_h(\mu, \lambda)$ has a virtually $\mathbb{Z}$-representable fundamental group for $\mu \geq 1$, $|\lambda| \geq 2$ and, if $\lambda$ is odd, either $\lambda > 2$ or $-4\mu/3 < \lambda < -2$ or $\lambda < -4\mu$. Since $M_h(\mu, \lambda) \to M(3\mu, \mu + \lambda) \cong M(3\mu, -\mu - \lambda)$ is a cover, Theorem A above follows easily.

5.2. We illustrate Theorem A by constructing covers $N \to M(3\mu, \lambda), \beta_1(N) \geq 1$, for $\mu, \lambda$ odd. Consider the 16-fold cover $F \to T_0$ pictured in Figure 3. Let $F' \to T_0$ be the cover corresponding to the kernel of the map $\theta: \pi_1(T_0) \to \mathbb{Z}/4 \oplus \mathbb{Z}/4$ defined by $\theta([x]) = (1, 0)$ and $\theta([y]) = (0, 1)$. We obtain $F$ by making eight vertical cuts in $F'$ and identifying the left edge of each cut to the right edge of the cut 2 to the right (mod 4). $F$ is a surface of genus 5 with $\partial F$ consisting of eight circles, each projecting 2 to 1 onto $\beta$ in $T_0$. 

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**FIGURE 3**

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Both $D_x$ and $D_x^4$ lift to homeomorphisms of $F$. $D_x$ lifts to $\tilde{D}_x$ which can be viewed as $1/4$ "fractional" Dehn twists about the $\{x_i\}$. In particular $\tilde{D}_x$ fixes pointwise rows 1 and 3 while shifting rows 2 and 4 each three squares to the right (mod 4). $D_y^4$ lifts to $\tilde{D}_y$ which consists of performing simultaneous Dehn twists about the $\{y_i\}$.

Since both $D_x$ and $D_y^4$ lift to $F$, $h$ lifts to a homeomorphism $\tilde{h}: F \rightarrow F$. It is easy to see that $\tilde{h}$ fixes pointwise $\partial F$ and that $\tilde{h}$ is homology reducible since $\tilde{h}_*$ fixes the nonboundary class $[\gamma] + [\delta]$ in $H_1(F)$.

Let $\tilde{M} = F \times I/\tilde{h}$. All Dehn fillings on $\tilde{M}$ have positive first Betti number. Moreover, since $\tilde{h}$ fixes pointwise $\partial F$, it follows that the loops $\alpha_h$, $\beta^2$ in $\partial M_h$ lift to loops $\tilde{\alpha}_i$, $\tilde{\beta}_i$ in the eight components of $\partial \tilde{M}$. Denote by $\tilde{M}(\mu, \lambda)$ the manifold obtained by Dehn filling on $\tilde{M}$ with respect to the curves $\tilde{\alpha}_i^\mu \tilde{\beta}_i^2$. Then the sequence of covers

$$\tilde{M} \left( \mu, \frac{\lambda - \mu}{2} \right) \rightarrow M_h(\mu, \lambda - \mu) \rightarrow M(3\mu, \lambda)$$

gives the desired cover of $M(3\mu, \lambda)$, $\mu, \lambda$ odd.

6. In this section we deal with the manifolds $M(2\mu, \lambda)$. Throughout §6, we fix $h = (R \circ D_y^{-3})$ where $R$ is the homeomorphism of $T_0$ induced by a $90^\circ$ counter-clockwise rotation of the square in Figure 2.

Let $M_h = T_0 \times I/h$. The loop $\alpha_h$ is represented in $T_0 \times I$ by the image of the curve $b \times I$ under a $90^\circ$ clockwise rotation of $\partial T_0 \times \{1\}$.

**Lemma 6.1.** $M_h$ is bundle equivalent to $M$, with the pair $(\alpha_h, \beta)$ mapping to $(\alpha, \beta)$.

**Proof.** Let $R'$ denote $R$ composed with a $90^\circ$ clockwise rotation of $\partial T_0$. Then $R'$ fixes $\partial T_0$ and induces on $\pi_1(T_0, b)$ the isomorphism $R'_#(x) = xyz^{-1}$, $R'#(y) = x^{-1}$. A calculation shows that, for any $z \in \pi_1(T_0, b)$,

$$g_#(z) = (D_x^{-1} \circ R' \circ D_y^{-3} \circ D_x)_#(z).$$

Thus the isotopy $H$ from $g$ to $D_x^{-1} \circ h \circ D_x$ rotates $\partial T_0$ only $90^\circ$ counter-clockwise and hence the bundle isomorphism $H \circ (D_x^{-1} \times \text{Id})$: $M_h \rightarrow M$ sends $(\alpha_h, \beta)$ to $(\alpha, \beta)$. $\square$

Now consider $M' = T_0 \times I/h^4$, the 4-fold cyclic cover of $M_h$. Note that $h^4$ fixes $\partial T_0$, so we define $(\alpha', \beta)$ for $M'$, where $\alpha' = b \times I/h^4$. 

Lemma 6.2. \( M' \to M \) is a 4-fold cover, sending the pair of loops \((\alpha', \beta) \) to the pair \((\alpha^4 \beta, \beta) \).

Proof. Note that the lift of \( \alpha^4_h \) to \( M' \) winds once around \( \partial T_0 \) in the clockwise direction and hence is represented by \( \alpha' \beta^{-1} \). Thus \( \alpha' \) projects to \( \alpha^4_h \beta \) in \( M_h \) which maps to \( \alpha^4 \beta \) in \( M \) by Lemma 6.1. \( \square \)

The following is an immediate consequence of Lemma 6.2 and will be used in §6.1:

Corollary 6.3. \( M'(\mu, \lambda) \to M(4\mu, \mu + \lambda) \approx M(4\mu, -\mu - \lambda) \) is a 4-fold cover.

6.1. Now we prove the following (see also [KL]):

Theorem B. \( M(4\mu, \lambda) \) has a virtually \( \mathbb{Z} \)-representable fundamental group.

We begin by considering the 9-fold cover \( S \to T_0 \) corresponding to the kernel of the map \( \theta : \pi_1(T_0) \to \mathbb{Z}/3 \oplus \mathbb{Z}/3 \) defined by \( \theta([x]) = (1, 0) \) and \( \theta([y]) = (0, 1) \). Note that both \( D_{y}^{-3} \) and \( R \) lift to \( S \).

Next we construct, for each \( d \geq 3 \), a cover \( F_d \to T_0 \) as follows: Let \( S_1, \ldots, S_d \) be copies of \( S \), each with eight cuts \( \{\tau_i\} \) as pictured in Figure 4. Glue the left edge of \( \tau_1 \) in \( S_i \) to the right edge of \( \tau_1 \) in \( S_{i+1} \) (mod \( d \)). Next glue the left edge of \( \tau_2 \) in \( S_i \) to the right edge of \( \tau_2 \) in \( S_{i-2} \) (mod \( d \)). Now glue the edges \( \tau_3, \ldots, \tau_8 \) so that the gluing is compatible with that of \( \tau_1, \tau_2 \) under a simultaneous counter-clockwise rotation by 90° of each \( S_i \). Note that the gluing of \( \tau_1 \) determines the pattern for \( \tau_3, \tau_5, \tau_7 \) while the gluing of \( \tau_2 \) determines that of the \( \tau_4, \tau_6, \tau_8 \). The surface \( F_3 \), with identifications for \( \tau_i \) numbered, is pictured in Figure 5. Some of the properties of the surface \( F_d \) are given in:

Lemma 6.4. The surface \( F_d \) is a 9\( d \)-fold cover of \( T_0 \). Each component \( \tilde{\beta}_i \) of \( \partial F_d \) projects \( r_i \) to 1 onto \( \beta = \partial T_0 \) for \( r_i \mid d \).

Now the loop \( x \) (resp. \( y \)) in \( T_0 \) is covered by \( 3d \) loops \( \tilde{x}_1, \ldots, \tilde{x}_{3d} \) (resp. \( 3d \) loops \( \tilde{y}_1, \ldots, \tilde{y}_{3d} \)) in \( F_d \) that project 3 to 1 onto \( x \) (resp. 3 to 1 onto \( y \)). Thus \( D_{y}^{-3} \) lifts to \( \hat{D}_{y}^{-1} \) consisting of simultaneous negative Dehn twists about the \( \{\tilde{y}_i\} \). It follows from the construction of \( F_d \) that \( R \) lifts to \( \hat{R} \), a simultaneous counter-clockwise rotation by 90° of each of the \( S_1, \ldots, S_d \) in \( F_d \). Thus \( h \approx R \circ D_{y}^{-3} \) and \( h^4 \) lift to \( \hat{h} \) and \( \hat{h}^4 \) on \( F_d \). Note that \( \hat{h}^4 \) fixes pointwise \( \partial F_d \).
LEMMA 6.5. $\tilde{h}^4: F_d \to F_d$ is homology reducible.

Proof. A portion of $F_d$ is pictured in Figure 6. The non-boundary class $[\gamma] + [\delta]$ in $H_1(F_d)$ corresponding to the loops $\gamma$, $\delta$ is fixed by $\tilde{h}^4_*$. Indeed, $\tilde{R}^4 = \text{Id}$ and $[\gamma] + [\delta]$ is fixed by $(\tilde{D}_y^{-1})_*$ since $\gamma$ and $\delta$ each intersect the same Dehn twist curves in $\{\tilde{y}_i\}$ with opposite orientations. 

Let $\tilde{M}_d = F_d \times I / \tilde{h}^4$. Now $\tilde{M}_d$ is, by construction, a $9d$-fold cover of $M'$, the 4-fold cyclic cover of $M_h$, hence $\tilde{M}_d \to M$ is a $36d$-fold covering space (see Lemma 6.2). Furthermore, Lemma 6.5 implies that any Dehn filling on $\tilde{M}_d$ yields a manifold with positive first Betti number.

We complete the proof of Theorem B by constructing, for each $(4\mu, \lambda)$ coprime, a cover $N \to M(4\mu, \lambda)$, $\beta_1(N) \geq 1$, gotten by Dehn filling on an appropriate $\tilde{M}_d$. Since $M(0, \pm 1)$ itself has positive first Betti number, we exclude this case.
Recall that $M'(\mu, \lambda - \mu) \to M(4\mu, \lambda) \cong M(4\mu, -\lambda)$ is a 4-fold cover by Corollary 6.3. Since $(4\mu, \lambda) \neq (0, \pm 1)$, by changing the sign of $\lambda$ if necessary, we can assume that either $\lambda = \mu = \pm 1$ or $|\lambda - \mu| \geq 3$. In the first case the loop $\alpha'$ in $\partial M'$ lifts to loops $\{c_i\}$ in $\partial M_d$ for any $d$. In the second case the loop $(\alpha')^\mu \beta^{\lambda - \mu}$ in $\partial M$ lifts to loops $\{c_i\}$ in $\partial \tilde{M}_d$ for $d = |\lambda - \mu|$. In both cases we obtain $N \to M(4\mu, \lambda)$ by Dehn filling on $\tilde{M}_d$ with respect to the loops $\{c_i\}$ in $\partial \tilde{M}_d$. This completes the proof of Theorem B.

As an example, consider the case $M(8, -1) \cong M(8, 1)$. Then $M'(2, -3) \to M(8, 1)$ and the $(2, -3)$ loop in $M'$ lifts to loops $\{c_i\}$ in the boundary components of $\tilde{M}_3$. $N$ is gotten by Dehn filling on $\tilde{M}_3$ with respect to the loops $\{c_i\}$ (see Figure 5).

6.2. **Proposition C.** $M(2\mu, \lambda)$ has a virtually $\mathbb{Z}$-representable fundamental group if $\lambda \equiv \pm 7\mu \pmod{15}$.

Consider the 9-fold cover $S \to T_0$ described in §6.1, and construct a new cover $F \to T_0$ by making eight cuts in $S$ and identifying the edges as shown in Figure 7. The surface $F$ has genus 4 and $\partial F$ consists of 3 circles: $\beta_1$ that projects 5-1 onto $\beta$, $\beta_2$ projecting 3-1 onto $\beta$, and $\beta_3$ projecting 1-1 onto $\beta$. The loop $x$ (resp. $y$) in $T_0$ is covered by the three loops $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$ (resp. $\tilde{y}_1$, $\tilde{y}_2$, $\tilde{y}_3$) which project 3-1 onto $x$ (resp. onto $y$).

It follows from the construction of $F$ that $R$ lifts to $\tilde{R}$ the homeomorphism induced by a $90^\circ$ counter-clockwise rotation, and that $D_{y}^{-3}$ lifts to $\tilde{D}_{y}^{-1}$ given by simultaneous negative Dehn twists about the $\{\tilde{y}_i\}$. Hence $h (= R \circ D_{y}^{-3})$ lifts to $\tilde{h}$.
**Lemma 6.6.** \( \tilde{h}^2 \) is homology reducible.

**Proof.** \( \tilde{h}^2 \) fixes the non-boundary class \( [\tilde{x}_2] - [\tilde{x}_3] \) in \( H_1(F) \) (see Figure 7).

Let \( \tilde{M} = F \times I/\tilde{h}^2 \). Then \( \tilde{M} \) is an 18-fold cover of \( M_h \). Since \( \tilde{h}^2 \) rotates each component \( \tilde{\beta}_i \) of \( \partial F \) one half turn counter-clockwise, we can choose on each component \( T_i \subset \partial \tilde{M} \) loops \( (\tilde{\alpha}_i, \tilde{\beta}_i) \) where \( (\alpha_1, \beta_1) \) projects to \( (\alpha_h^2\beta^{-2}, \beta^5) \), \( (\alpha_2, \beta_2) \) projects to \( (\alpha_h^2\beta^{-1}, \beta^3) \) and \( (\alpha_3, \beta_3) \) projects to \( (\alpha_h^3, \beta) \) in \( M_h \).

Now our proposition follows, since by the above paragraph any loop in \( \partial M_h \) of the form \( \alpha_h^{2\mu}\beta^\lambda, \lambda \equiv -7\mu \) (mod 15), lifts to loops \( \{c_i\} \) in each component \( T_i \) of \( \partial \tilde{M} \). Dehn filling on \( \tilde{M} \) with respect to the loops \( \{c_i\} \) provides a cover \( N \rightarrow M_h(2\mu, \lambda) \cong M(2\mu, \lambda) \), the last isomorphism by Lemma 6.1.

**6.3. Remark.** By similar arguments, we can show that \( M(2\mu, \lambda) \) has a virtually \( \mathbb{Z} \)-representable fundamental group if \( \lambda \equiv \pm 3\mu \) (mod 7). These cases have been done in [H1] and [N] by different methods (see §3). Consider the cover \( F \rightarrow T_0 \) in Figure 8, obtained from 3 copies of \( S \) by removing the interiors of the four shaded regions in each copy of \( S \) and identifying the edges as numbered. The reader should check the following: \( \partial F \) consists of 3 circles, each projecting 7 to 1 onto \( \beta = \partial T_0 \); \( \tilde{h}^2 \) lifts to a homology reducible map \( \tilde{h}^2: F \rightarrow F \); and the loop \( \alpha_h^{2\mu}\beta^\lambda, \lambda \equiv \pm 3\mu \) (mod 7), in \( M \) lifts to loops in \( \tilde{M} = F \times I/\tilde{h}^2 \).

**7. Singular boundary curve systems for \( M \).** In this section we study singular incompressible surfaces in \( M \). Given a cover \( N \rightarrow M(\mu, \lambda) \)
obtained by Dehn filling on $\widetilde{M} \rightarrow M$, then $\beta_1(N) \geq \beta_1(\widetilde{M}) - \beta_0(\partial \widetilde{M})$. Hempel shows ([H2]) that this inequality is strict if and only if there is an incompressible, boundary incompressible surface $F$ in $\widetilde{M}$ such that $\partial F$ consists of a non-empty collection of Dehn filling curves. This surface $F$ projects to a singular surface in $M$ whose boundary curves are $\alpha^\mu \beta^\lambda$, and we say that $\{\alpha^\mu \beta^\lambda\}$ is a singular boundary curve system for $M$.

In [H2] the curves $\{\alpha^3\}$, $\{\alpha, \beta\}$, and $\{\alpha \beta^{-1}\}$ are shown to be singular boundary curve systems. We show:

**Theorem D.** The curves $\{\alpha^2 \beta\}$, $\{\alpha^2 \beta^{-1}\}$, $\{\alpha^3 \beta\}$ and $\{\alpha^3 \beta^{-1}\}$ are singular boundary curve systems for $M$.

(a) The curves $\alpha^3 \beta^{\pm 1}$: We use the 3-fold cover $M_h \rightarrow M$ for $h = D_x^2 \circ D_y^{-4} \circ D_x \circ D_y^{-4} \circ D_x$ described in §5.

By Lemma 5.1, $M_h(1, 0) \rightarrow M(3, 1)$ is a 3-fold covering. Now consider the 8-fold cover $F \rightarrow T_0$, pictured in Figure 9, to which $h$ lifts (see §5). Denote this lift by $\tilde{h}$. Note that $\tilde{h}$ fixes pointwise the eight components of $\partial F$ and that $\tilde{h}$ is not homology reducible.

Let $\widetilde{M} = F \times I/\tilde{h}$. By construction, the loop $\alpha_h$ in $M_h$ lifts to eight loops $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_8$ in $\partial \widetilde{M}$—indexed so that the loops $(\tilde{\alpha}_i, \tilde{\beta}_i)$ lie in the $i$th boundary torus of $\widetilde{M}$. Thus the loops $\tilde{\alpha}_i$ project to $\alpha^3 \beta$ in $\partial M$ and Dehn filling on $\widetilde{M}$ with respect to the $\{\tilde{\alpha}_i\}$ gives a cover $N \rightarrow M(3, 1) \cong M(3, -1)$.

**Lemma 7.1.** There exist relations among $\{[\tilde{\alpha}_i]\}$ in $H_1(\widetilde{M})$; hence $\beta_1(N) > \beta_1(\widetilde{M}) - \beta_0(\widetilde{M})$.

**Proof.** We have $[\tilde{\alpha}_2] - [\tilde{\alpha}_1] = [\tilde{\alpha}_6] - [\tilde{\alpha}_5]$ in $H_1(\widetilde{M})$. One computes $[\tilde{\alpha}_j] - [\tilde{\alpha}_i]$ as follows. Let $\sigma_{ij}$ be a simple path in $F \times \{0\}$ from [Figure 8]
\( \tilde{\alpha}_i \cap F \) to \( \tilde{\alpha}_j \cap F \). Then the disk \( \sigma_{ij} \times I \subset F \times I \) provides the relation 
\[
[\tilde{\alpha}_j] - [\tilde{\alpha}_i] = [\tilde{h}(\sigma_{ij}) * \sigma_{ij}^{-1}]
\]
where * denotes path composition.

Now \( \sigma_{12} \) and \( \sigma_{56} \) can be chosen as in Figure 9, and 
\[
[\tilde{h}(\sigma_{12}) * \sigma_{12}^{-1}] = [\tilde{h}(\sigma_{56}) * \sigma_{56}^{-1}]
\]
in \( H_1(\tilde{M}) \) since \( \tilde{D}_x \) fixes \( \sigma_{12} \) and \( \sigma_{56} \) pointwise and they both intersect the Dehn twist curve \( \tilde{y}_2 \).

(b) The curves \( \alpha^2 \beta^\pm 1 \): Consider the bundle \( M_f = T_0 \times I / f \) for \( f = (D_x^{-1} \circ D_y^5)^2 \).

**Lemma 7.2.** \( M_f \) is a 2-fold cover of \( M \). The pair \( (\alpha_f, \beta) \) maps to the pair \( (\alpha^2 \beta^{-1}, \beta) \).

**Proof.** Let \( g' = (D_x^{-1} \circ D_y^2 \circ D_x^{-1}) \circ g^2 \circ (D_x^{-1} \circ D_y^2 \circ D_x^{-1})^{-1} \). We have, for any \( z \in \pi_1(T_0, b) \),
\[
g^\#_f(z) = (x y x^{-1} y^{-1})^{-1} f^\#_f(z)(x y x^{-1} y^{-1}).
\]
Thus the isotopy \( H \) between \( g' \) and \( f \) rotates \( \partial T \) one full turn clockwise, so that the bundle isomorphism \( \{ (D_x^{-1} \circ D_y^2 \circ D_x^{-1})^{-1} \times \text{Id} \} \circ H: M_f \to M_g \) sends the pair \( (\alpha_f, \beta) \) to \( (\alpha_g \beta^{-1}, \beta) \) which projects to \( (\alpha^2 \beta^{-1}, \beta) \) in \( M \). \( \square \)

By Lemma 7.2 \( M_f(1, 0) \to M(2, -1) \) is a 2-fold cover. Now consider the 10-fold cover \( F \to T_0 \), pictured in Figure 10 to which \( f \)
lifts. Denote the lift of $f$ by $\hat{f}$ and let $M = F \times I / \hat{f}$. Now $\hat{f}$ fixes pointwise the eight boundary circles of $F$. Denote by $\tilde{\alpha}_i$ a lift of $\alpha_i$ to $\partial \hat{M}$, indexed so that the loops $(\tilde{\alpha}_i, \tilde{\beta}_i)$ lie on the $i$th boundary torus of $\hat{M}$. Thus the loops $\tilde{\alpha}_i$ project to $\alpha^2 \beta^{-1}$ in $M$ and Dehn filling on $\hat{M}$ with respect to $\{\tilde{\alpha}_i\}$ gives a cover $N \to M(2, -1) \cong M(2, 1)$.

**Lemma 7.3.** There exist relations among $\{[\tilde{\alpha}_i]\}$ in $h_1(\hat{M})$; hence $\beta_1(N) > \beta(\hat{M}) - \beta_0(\hat{M})$.

**Proof.** We have $[\tilde{\alpha}_2] - [\tilde{\alpha}_1] = [\tilde{\alpha}_6] - [\tilde{\alpha}_5]$ in $H_1(\hat{M})$ by an argument identical to that in Lemma 7.1. \hfill $\Box$

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