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**EMBEDDING A 2-COMPLEX  $K$  IN  $\mathbb{R}^4$  WHEN  $H^2(K)$  IS A  
CYCLIC GROUP**

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## EMBEDDING A 2-COMPLEX $K$ IN $\mathbb{R}^4$ WHEN $H^2(K)$ IS A CYCLIC GROUP

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We prove that every finite 2-dimensional cell complex with cyclic second cohomology embeds in  $\mathbb{R}^4$  tamely.

**1. Introduction.** It has long been known that every compact PL (piecewise-linear) manifold embeds in euclidean space of double dimension. The analogous result, however, is not true for arbitrary simplicial complexes (see [2]). In [6] an obstruction to embedding  $n$ -complexes in  $\mathbb{R}^{2n}$  was found. Since that obstruction is not homotopy invariant and is in general difficult to calculate, it is natural to ask if a certain class of  $n$ -complexes which can be easily described embeds in  $\mathbb{R}^{2n}$ . It has been known that every  $n$ -complex with cyclic  $n$ th cohomology embeds in  $\mathbb{R}^{2n}$  if  $n \neq 2$  (see [5]). If  $n > 2$  one can use the techniques of [7] to prove it. The same techniques are much harder to apply when  $n = 2$  and if they are successful they yield embeddings which are not smooth but only tame on each 2-cell (recall that an embedding  $D^2 \rightarrow \mathbb{R}^4$  is tame if it can be extended to an embedding  $D^2 \times D^2 \rightarrow \mathbb{R}^4$ ). At present the author does not even know whether every contractible 2-complex embeds in  $\mathbb{R}^4$  piecewise smoothly.

In [4] it was shown that the case  $n = 2$  really is different from other dimensions (§3). Here we establish a result analogous to other dimensions.

**THEOREM.** *If  $K$  is a finite 2-complex such that  $H^2(K)$  is cyclic then  $K$  can be embedded in  $\mathbb{R}^4$ .*

*Note.* All homology and cohomology groups will be with integer coefficients;  $Z$  denotes the ring of integers.

The case  $H^2(K) = 0$  was proved in [4]. The general case can be reduced to the case when  $H^2(K)$  is infinite cyclic. This case is basically in two steps. First it is proved for the case when  $H_2(K)$  is generated by an embedded orientable surface. For arbitrary  $K$  with  $H^2(K) = Z$  the situation is reduced to the previous case by constructing a tower

of maps and 2-complexes

$$K_r \xrightarrow{p_r} K_{r-1} \xrightarrow{p_{r-1}} \cdots \xrightarrow{p_1} K_1 \xrightarrow{p_0} K_0 = K$$

such that  $K_{j-1}$  can be embedded in  $\mathbb{R}^4$  if  $K_j$  can and such that  $K_r$  embeds in  $\mathbb{R}^4$ .

In what follows all embeddings of  $K$  in  $\mathbb{R}^4$  will be smooth in the interior of each cell except for a finite number of points in the interiors of 2-cells where they will still be tame. Thus if we construct such an embedding of a subdivided  $K$  it will still be tame on the original  $K$ . Therefore we can assume without loss of generality whenever it is convenient that  $K$  is either a simplicial complex or that all the attaching maps are homeomorphisms.

**2. A special case.** In what follows  $K$  will be a finite connected 2-complex.

**LEMMA 1.** *Suppose  $H^2(K) = Z$  and suppose that  $H_2(K)$  is generated by an embedded orientable surface  $F \subset K$ . Then  $K$  can be embedded in  $\mathbb{R}^4$ .*

*Proof.* Let  $e_0$  be a 2-cell of  $F$ . Then the inclusion  $(K - \text{int}(e_0), F - \text{int}(e_0)) \subset (K, F)$  gives rise to the following commutative diagram

$$\begin{array}{ccccccc} H^2(K, F) & \longrightarrow & H^2(K) & \longrightarrow & H^2(F) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H^2(K - \text{int}(e_0), F - \text{int}(e_0)) & \longrightarrow & H^2(K - \text{int}(e_0)) & \longrightarrow & 0 & & \end{array}$$

in which both rows are exact. Since  $H^2(K) \rightarrow H^2(F)$  is an isomorphism the first homomorphism in the top row is trivial. The first vertical map is an isomorphism (by excision); therefore the first homomorphism in the bottom row is also trivial. This implies that  $H^2(K - \text{int}(e_0))$  is 0.

By attaching 2-cells to  $K - \text{int}(e_0)$  we can obtain an acyclic 2-complex  $L$ . Denote  $L \cup e_0$  again by  $K$ . Clearly if this  $K$  can be embedded in  $\mathbb{R}^4$  so can the original 2-complex.

Choose an embedding of  $F \cup K^{(1)}$  in  $\mathbb{R}^3 \times 0 \subset \mathbb{R}^4$  which is smooth on  $F$  and on each edge of  $K$ . Identify  $F \cup K^{(1)}$  with its image under this embedding. Then  $F \cup K^{(1)} \subset \mathbb{R}^3 \times 0$ . Let  $H \times 0$  be a regular neighborhood of  $K^{(1)}$  in  $\mathbb{R}^3 \times 0$ .  $H \times 0$  is a handlebody with spine  $K^{(1)}$ . There is a natural projection  $p: \partial(H \times 0) \rightarrow K^{(1)}$  such that  $H \times 0$  is the mapping cylinder of  $p$ . Thus every point in  $H \times 0$  can

be thought of as a class  $[x, t]$  where  $x \in \partial H$ ,  $t \in I$  ( $= [0, 1]$ ), and  $[x, 1] = p(x)$ . let  $\hat{p} : H \rightarrow K^{(1)}$  be defined by  $\hat{p}([x, t]) = p(x)$ .

Let  $U$  be a regular neighborhood of  $K^{(1)}$  in  $K$ .  $\partial U$  is a union of circles  $C_0, \dots, C_g$  where  $C_i$  corresponds to the 2-cell  $e_i$  of  $K$  and where  $g$  is the genus of  $H$  (because  $L$  is acyclic). Suppose  $\partial U \cap F = C_0 \cup \dots \cup C_k$ . Orient  $F$  and assume that  $C_0, \dots, C_k$  have the induced orientation. Also choose orientations for the curves  $C_{k+1}, \dots, C_g$ .  $U$  and  $H$  can be chosen in such a way that  $(H \times 0) \cap F = U \cap F$  and so that  $U \cap F = \hat{p}^{-1}(p(\partial U \cap F)) = \{[x, t] \in H \times 0; x \in \partial U \cap F, t \in I\}$ . Embed  $C_{k+1} \cup \dots \cup C_g$  smoothly in  $H \times 1$  in such a way that  $p|_{C_j} : C_j \rightarrow K^{(1)}$  is the attaching map for  $e_j$ . Let  $U_j = \{([x, t], 1 - t) \in H \times [-1, 1] | x \in C_j, t \in I\}$ .  $U_j$  is an embedding of the collar of  $e_j$  into  $H \times [0, 1]$ .  $(\bigcup_{j=k+1}^g U_j) \cup (F \cap H \times 0)$  is an embedding of  $U$  into  $H \times [-1, 1]$  which we can assume to be piecewise smooth.

Since  $L$  is acyclic,  $C_1, \dots, C_g$  form a basis for  $H_1(\partial(H \times [-1, 1]))$ . Let  $T$  be a maximal tree of  $K^{(1)}$  and let  $s_1, \dots, s_g$  be the edges of  $K^{(1)} - T$ . If  $m_i$  is the midpoint of  $s_i$  let

$$S_i = (\hat{p}^{-1}(m_k) \times \{-1, 1\}) \cup p^{-1}(m_i) \times [-1, 1] \subset \partial(H \times [-1, 1]).$$

Then  $S_i$  is an embedded 2-sphere. Choose an orientation for  $S_i$ . For each  $i = 1, \dots, g$  choose an oriented simple closed curve  $a_i$  in  $\partial(H \times [-1, 1])$  such that  $a_i \cdot S_j = \delta_{ij}$ . Then  $\{a_1, \dots, a_g\}$  is a basis for  $H_1(\partial(H \times [-1, 1]))$ . Suppose  $C_i \sim \sum p_{ij} a_j$ ,  $i = 1, \dots, g$ , in  $\partial(H \times [-1, 1])$  ( $\sim$  stands for homologous). Then  $\det(p_{ij}) = \pm 1$ . Let  $\Sigma'_i$  be a union of suitably oriented disjoint copies of spheres  $S_1, \dots, S_g$  representing the class  $\sum_{j=1}^g q_{ij} [S_j]$  in  $H_2(\partial(H \times [-1, 1]))$  where  $(q_{ij}) = (p_{ji})^{-1}$ . Then

$$C_i \cdot \Sigma'_j = \sum_{k,l} p_{ik} q_{jl} a_k \cdot S_l = \sum_{k=1}^g p_{ik} q_{jk} = \delta_{ij}.$$

The intersection number  $\Sigma' \cdot F$  is zero (it is the intersection of closed orientable surfaces in  $\mathbb{R}^4$ ). Since  $\Sigma'_i \cap F = \Sigma'_i \cap (C_0 \cup \dots \cup C_k)$ , the intersection number  $\Sigma'_j \cdot (C_0 \cup \dots \cup C_k)$  in  $\partial(H \times [-1, 1])$  is also zero. Since  $\Sigma'_i \cdot (C_1 \cup \dots \cup C_k) = 0$ , for  $i > k$ , it follows that  $\Sigma'_i \cdot C_0 = 0$ , for  $i > k$ . Therefore we can pipe together the intersections of  $\Sigma'_i$  with  $C_j$ ,  $j = 0, \dots, g$ , along  $C_0 \cup \dots \cup C_g$  to obtain for each  $i > k$  a surface  $\Sigma''_i \subset \partial(H \times [-1, 1])$  such that  $\Sigma''_i \cap F = \emptyset = \Sigma''_i \cap C_j$ ,  $i \neq j$ , and such that  $\Sigma''_i \cap C_i$  is a point. Since all the "pipes" lie either in  $H \times 1$  or in a neighborhood of  $\partial H \times 0$  in  $\partial(H \times [-1, 1])$ , one can

choose half of a symplectic basis for each  $H_1(\Sigma''_i)$ ,  $i > k$ , represented by smooth simple closed curves in  $\partial H \times (0, 1] \cup H \times 1$ . Since  $M' = \mathbb{R}^3 \times [0, \infty) - \text{int}(H \times [-1, 1])$  is simply connected, we can cap off these curves by regularly immersed discs in  $M'$ . By performing surgeries along these discs change each  $\Sigma''_i$ ,  $i > k$ , into a singular 2-sphere  $\Sigma_i$ . All the singularities lie in  $M'$ . Furthermore,  $\Sigma_i \cap (U \cup F) = \Sigma'_i \cap C_i$  is a point. Note also that  $\Sigma_i \cap \Sigma_j \cap \text{int}(H \times [-1, 1]) = \emptyset$ , and that  $\Sigma_i \cdot \Sigma_j = 0$ , for  $i \neq j$ ,  $i, j > k$ .

Cap off the curves  $C_{k+1}, \dots, C_g$  by regularly immersed discs  $D'_{k+1}, \dots, D'_g$ , respectively, lying in  $\mathbb{R}^3 \times [1, \infty)$ . This extends the embedding of  $F \cup U$  to a regular immersion of  $K$  into  $\mathbb{R}^4$ . Since  $D'_i \cdot \Sigma_j = \delta_{ij}$  for all  $i, j > k$ , we can use the spheres  $\Sigma_j$  to pipe off the intersections between the discs  $D'_{k+1}, \dots, D'_g$ , in order to get immersed discs  $D_{k+1}, \dots, D_g$ , respectively, such that  $D_i \cdot D_j = 0$ , for  $i \neq j$ . Again  $\Sigma_i \cdot D_j = \delta_{ij}$ , for  $i, j > k$ .

Let  $M$  be the union of  $M'$  and a regular neighborhood of  $\Sigma_{k+1} \cup \dots \cup \Sigma_g$  which misses  $F$ . Since  $\Sigma_j - M'$  is a union of embedded discs, for  $j = k + 1, \dots, g$ ,  $M$  is simply connected. The discs  $D_{k+1}, \dots, D_g$  and the classes  $x_i = [\Sigma_i] \in H_2(M)$ ,  $i > k$ , satisfy the conditions of Theorem 3.1 of [3]. Applying Theorem 1.1 of [3] we get  $g - k$  tamely embedded discs  $B^2_{k+1}, \dots, B^2_g$  in  $M$  such that  $B^2_j \cap \partial M = C_j$ . This, in turn, defines an embedding of  $K$  in  $\mathbb{R}^4$ .

**3. The case  $H^2(K) = Z$ .** Let  $B$  be a ball of radius  $r$  and let  $F: B \times I \rightarrow B$  have the following properties:  $F_0 = \text{id}$ ,  $F_t|_{\partial B} = \text{id}$ , for  $t \in [0, 1]$ , and  $F_t$  is a homeomorphism of  $B$  for  $t \in [0, 1)$ . Then the homotopy  $H: B \times B^k \times I \rightarrow B \times B^k$  given by

$$H((x, y), t) = (F(x, (1 - |y|)t), y)$$

is the identity on  $\partial(B \times B^k)$ . Furthermore,  $H_t$  is one-to-one on  $B \times B^k - B \times 0$ , for all  $t \in I$ , and  $H_t|_{B \times 0} = F_t \times 0$ .

**LEMMA 2.** *Let  $K$  be a finite 2-dimensional cell complex, such that all the 2-cells are attached via homeomorphisms. Let  $g$  be an embedding of  $K$  into  $\mathbb{R}^4$ . Then there exists a homotopy with compact support  $H: \mathbb{R}^4 \times I \rightarrow \mathbb{R}^4$ , such that  $H_0 = \text{id}$ , and such that  $H_t$  is homeomorphism for  $t \in [0, 1)$ , which does one of the following three types of deformations:*

(i) *for an edge  $s$  of  $K$ ,  $H_1$  maps  $g(s)$  to a point and is 1-1 elsewhere;*

(ii) for a 2-cell  $e$  with boundary a union of two edges  $s_1, s_2$  having pairs of common endpoints,  $H$  is a deformation retraction of  $g(e)$  onto  $g(s_1)$ , which is fixed on  $g(s_1)$ .

(iii) for two 2-cells  $e_1, e_2$  with  $e_1 \cap e_2$  being an arc  $A$ ,  $H_1$  maps  $g(e_1)$  homeomorphically onto  $g(e_2)$ , and is 1-1 on  $g(K) - g(e_1 \cup e_2)$ . Furthermore,  $H$  is fixed on  $g(e_2)$ .

If  $K_1$  is the 2-complex obtained from  $K$  by the identifications defined by  $H_1$  then  $H_1g: K \rightarrow \mathbb{R}^4$  factors through  $K_1$ . The factoring map  $K_1 \rightarrow \mathbb{R}^4$  is an embedding.

*Proof.* Define a homotopy  $F: 2B^k \times I \rightarrow 2B^k$  as follows:

For type (i) let  $k = 1$ , and let

$$F(x, t) = \begin{cases} (1-t)x & \text{for } |x| \leq 1, \\ (1+t)x - 2tx/|x| & \text{for } 1 \leq |x| \leq 2. \end{cases}$$

$F$  squeezes  $[-1, 1]$  to 0 and linearly stretches the rest of  $[-2, 2]$ .

For type (ii) let  $k = 2$ , and let

$$F((x, y), t) = \begin{cases} (x, y(1-t)) & \text{for } |x| \leq 1, 0 \leq y \leq A(x), \\ (x, (1/(A(x) - B(x)))(A(x)(1-t) - B(x))y + tA(x)B(x))) & \text{for } |x| \leq 1, A(x) \leq y \leq B(x), \\ (x, y) & \text{elsewhere,} \end{cases}$$

where  $A(x) = \sqrt{1-x^2}$ ,  $B(x) = \sqrt{4-x^2}$ .  $F$  shrinks  $D^2 \cap \mathbb{R}_+^2$  to  $[-1, 1] \times 0$ .

For type (iii) let  $k = 3$  and define  $F$  as follows:

Let  $\delta: [0, 2\pi] \times I \rightarrow [0, 2\pi]$  be the homotopy

$$\delta(\alpha, t) = \begin{cases} (1-t)\alpha & \text{for } 0 \leq \alpha \leq \pi/2, \\ (1+t/3)\alpha - 2\pi t/3 & \text{for } \alpha \geq \pi/2. \end{cases}$$

$\delta$  shrinks  $[0, \pi/2]$  to 0 and stretches  $[\pi/2, 2\pi]$  over  $[0, 2\pi]$ . A point in  $\mathbb{R}^3$  can be represented as a pair of a real and a complex number. Let

$$F((x, r \cdot \exp(i\alpha)), t) = \begin{cases} (x, r \cdot \exp(i\delta(\alpha, t))) & \text{for } \rho \leq 1, \\ (x, r \cdot \exp(i[(2-\rho)\delta(\alpha, t) + (\rho-1)\alpha])) & \text{for } \rho \in [1, 2], \end{cases}$$

where  $\rho = \sqrt{x^2 + r^2}$ .

In each case  $F_t|_{\partial(2B^k)}$  is identity for all  $t \in I$ .

For type (i)  $g(s)$  has a regular neighborhood  $N$  homeomorphic to  $[-2, 2] \times B^3$ . Let  $\varphi : [-2, 2] \times B^3 \rightarrow N$  be a homeomorphism such that  $\varphi([-1, 1] \times 0) = g(s)$ .

For type (ii)  $g(e)$  has a regular neighborhood  $N$  homeomorphic to  $2D^2 \times B^2$ . Let  $\varphi : 2D^2 \times B^2 \rightarrow N$  be a homeomorphism such that  $\varphi((D^2 \cap \mathbb{R}_+^2) \times 0) = g(e)$ , and such that  $\varphi([-1, 1] \times 0) = g(s_1)$ .

For type (iii), since  $D = g(e_1 \cup e_2)$  is a tame disc such that its interior doesn't intersect  $g(K) - D$ , there exists a homeomorphism  $\varphi$  from  $2B^3 \times [-1, 1]$  onto a regular neighborhood  $N$  of  $D$ , satisfying the following two properties:  $\varphi(B^3 \times 0) \cap (g(K) - D) = \emptyset$ , and  $\varphi$  maps  $\{(x, y, z, 0) \in B^3 \times 0 \mid y \geq 0, z \geq 0, yz = 0\}$  onto  $D$  so that  $g(A) = \varphi(\{(x, 0, 0, 0) \in B^3 \times 0\})$ .

Given  $\varphi$  and  $F$  for each type we define the desired homotopy  $H$  by

$$H(x, t) = \begin{cases} x & \text{for } x \in N, \\ \varphi(F(u, (1 - |v|t), v)) & \text{for } (u, v) \in 2B^k \times B^{4-k}, \\ x = \varphi(u, v). & \end{cases}$$

Suppose  $f: F \rightarrow K$  represents a generator of  $H_2(K)$ . We can assume (by subdividing  $F$  and  $K$  appropriately) that  $f$  is simplicial and non-degenerate on each simplex (compare with [1], p. 11). We dealt with the case when  $f$  is an embedding in Lemma 1. Assume now that the singular set  $S$  of  $f$  ( $S$  is the closure of the set  $\{x \in F \mid f^{-1}(f(x))\}$  contains more than one point}) is non-empty. We will successively replace  $K$  by "nicer" complexes and finally reduce the problem of embeddability of  $K$  in  $\mathbb{R}^4$  to the situation of Lemma 1.

*Case 1.  $S$  is 0-dimensional.*

If  $\Sigma = f(S) = \{y_1, \dots, y_r\}$  then  $F_0 = f(F)$  is obtained from  $F$  by identifying the points of each set  $f^{-1}(y_j)$ ,  $j = 1, \dots, r$ . Suppose  $f^{-1}(y_1) = \{v_1, v_2, \dots, v_l\}$ . Construct  $F_1$  from  $F$  by identifying the points of each set  $f^{-1}(y_1) - \{v_1\}, f^{-1}(y_2), \dots, f^{-1}(y_r)$ . Note that  $F_1$  is not a surface. Clearly there exists a map  $f_1$  making the following diagram commutative:

$$\begin{array}{ccc} F & \xrightarrow{f_1} & F_1 \\ & \searrow f & \downarrow p_1 \\ & & F_0 \end{array}$$



embedding. Furthermore, if  $K_i$  can be embedded in  $\mathbb{R}^4$  so can  $K_{i-1}$ ,  $i = 1, \dots, j$ . Also  $K_j$  embeds in  $\mathbb{R}^4$  by Lemma 1. This proves

**PROPOSITION 1.** *Suppose  $K$  is a finite simplicial complex. Suppose that  $H^2(K) = Z$  and that  $H_2(K)$  is represented by a non-degenerate simplicial map  $f: F \rightarrow K$  of an orientable surface  $F$  into  $K$ . If the singular set of  $f$  is 0-dimensional then  $K$  can be embedded in  $\mathbb{R}^4$ .*

*Case 2.  $S$  is 1-dimensional.*

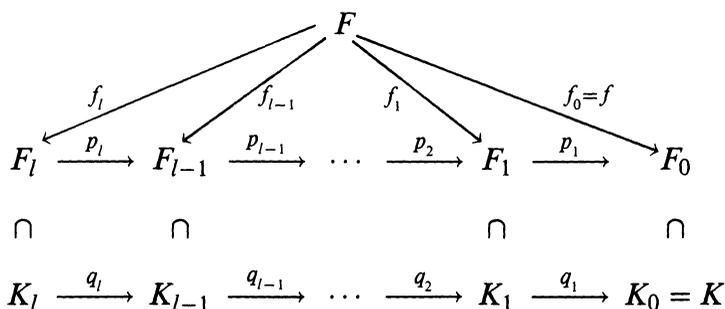
Then  $\Sigma = f(S)$  is also at most 1-dimensional.  $F_0$  is obtained from  $F$  by identifying the points of each  $f^{-1}(y)$ ,  $y \in \Sigma^{(0)}$ , and by identifying the components of each  $f^{-1}(\sigma)$  (by simplicial isomorphisms) where  $\sigma$  runs over the interiors of the edges of  $\Sigma$ . Let  $f^{-1}(\sigma_0)$  be a union of open edges  $s_1, \dots, s_r$ , for some open edge  $\sigma_0 \in \Sigma$ . Construct  $F_1$  from  $F$  by identifying the points of each set  $f^{-1}(y)$ ,  $y \in \Sigma^{(0)}$ , and by identifying the components of  $s_2 \cup \dots \cup s_r$  and of the sets  $f^{-1}(\sigma)$  where  $\sigma$  runs over open 1-simplices of  $\Sigma - \sigma_0$  (again via simplicial isomorphisms). As in Case 1 there exists a map  $f_1$  making the diagram

$$\begin{array}{ccc}
 F & \xrightarrow{f_1} & F_1 \\
 & \searrow f & \downarrow p_1 \\
 & & F_0
 \end{array}$$

commute where  $p_1: F_1 \rightarrow F_0$  is the natural projection. The singular set  $S_1$  of  $f_1$  has one less edge than  $S: S_1 = S - s_1$ .

Attach a 2-cell  $D$  to  $z_1 \cup z_2 \subset F_1$  via a homeomorphism where  $z_j = f_1(s_j)$ . The resulting space  $\widehat{F}_1$  is homotopy equivalent to  $F_0$ . The extension  $\widehat{p}_1: \widehat{F}_1 \rightarrow F_0$  of  $p_1: F_1 \rightarrow F_0$  which squeezes  $D$  to  $z_1$  is a homotopy equivalence. Suppose, as before, that  $L = \overline{K} - \overline{F_0}$  is attached to  $F_0$  along a graph  $G$ . Then  $\widehat{G} = p^{-1}(G) - z_1$  is homeomorphic to  $G$  and  $L$  can be attached to  $\widehat{F}_1$  along  $\widehat{G}$  in the obvious way to construct a 2-complex  $K_1$  which is homotopy equivalent to  $K$ . Let  $q_1: K_1 \rightarrow K$  be the obvious extension of  $\widehat{p}_1: \widehat{F}_1 \rightarrow F_0$ .  $H_2(K_1)$  is generated by  $f_1: F \rightarrow K_1$  which has one less edge in its singular set than  $f$ . Also, by using one deformation of type (ii) from Lemma 2 we see that if  $K_1$  embeds in  $\mathbb{R}^4$  then so does  $K$ . As in Case 1 we

repeat the above procedure to get a commutative diagram



where the bottom maps are homotopy equivalences, the singular set of  $f_l$  is 0-dimensional, and  $K_{i-1}$  embeds in  $\mathbb{R}^4$  if  $K_i$  does, for  $i = 1, \dots, l$ . Combining this with Proposition 1 we get

**PROPOSITION 2.** *Suppose  $H^2(K) = Z$ , and suppose that a generator of  $H_2(K)$  is represented by a non-degenerate simplicial map  $f: F \rightarrow K$  where  $F$  is an orientable surface. If the singular set of  $f$  is 1-dimensional then  $K$  embeds in  $\mathbb{R}^4$ .*

*Case 3.  $S$  is 2-dimensional.*

Choose a point  $b_\sigma$  in the interior of each 2-cell  $\sigma$  of  $F$ . Let  $S_k$  be the collection of all open 2-cells  $\sigma$  such that  $f^{-1}(f(b_\sigma))$  contains  $k$  points. Denote by  $Z_k$  the union of 2-cells  $\sigma$  such that  $\text{int}(\sigma) \in S_k$ . Represent the homology class of  $f: F \rightarrow K$  by a linear combination  $\sum x_e e$  where  $e$  runs over the 2-cells of  $K$ . By choosing appropriate orientations for the 2-cells of  $f(F)$  we can assume that all the coefficients  $x_e$  are non-negative. Furthermore,  $F$  can be chosen so that  $S_k = \{f^{-1}(\text{int}(e)) | x_e = k\}$ , for all  $k$  (see [2], p. 11). Let  $M = \max\{k | S_k \neq \emptyset\}$ . Since  $S$  is 2-dimensional,  $M$  is greater than 1.  $S_M$  does not contain all the open 2-cells of  $F$  because the coefficients  $x_e$  have no common factor. Therefore there exists a 2-cell  $\sigma_1$  such that  $\text{int}(\sigma_1) \in S_M$  and such that the intersection of  $\sigma_1$  with  $\overline{F - Z_M}$  contains an open edge  $s_1$ . Let  $\Sigma = f(S)$ . Construct  $F_1$  from  $F$

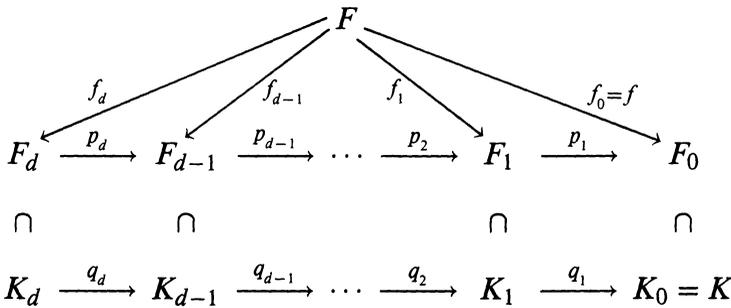
- (1) by identifying the points of each  $f^{-1}(y)$ ,  $y \in \Sigma^{(0)}$ ,
- (2) by identifying the components of  $f^{-1}(\tau)$  where  $\tau$  runs over the open edges of  $\Sigma - f(s_1)$ ,
- (3) by identifying the components of  $f^{-1}(e)$  where  $e$  runs over all closed 2-cells of  $\Sigma - f(\sigma_1)$ ,

(4) by gluing together  $s_2, \dots, s_m$  where  $s_1, \dots, s_m$  are the components of  $f^{-1}(f(s_1))$ , and

(5) by gluing together  $\sigma_2, \dots, \sigma_m$ , where  $\sigma_1, \dots, \sigma_m$  are closed 2-cells whose union is  $f^{-1}(f(\sigma_1))$ .

As before, let all the identifications be via simplicial isomorphisms.  $f$  can again be factored as  $p_1 f_1$  where  $p_1: F_1 \rightarrow F_0$  is the natural projection.  $p_1$  is a homotopy equivalence. If, as before,  $K$  is obtained from  $F_0$  by attaching  $L (= \overline{K - F_0})$  along a graph  $G \subset F_0$ , construct  $K_1$  by attaching  $L$  to  $F_1$  along  $p_1^{-1}(G) - f_1(s_1) \approx G$  in the obvious way.  $K_1$  is homotopy equivalent to  $K$ . Let  $q_1: K_1 \rightarrow K$  be the natural extension of  $p_1$ .  $H_2(K)$  is generated by  $f_1: F \rightarrow K_1$ . The singular set of  $f_1$  has one less 2-simplex than  $S$ . Also, by Lemma 2 (using type (iii) deformation)  $K$  embeds in  $\mathbb{R}^4$  if  $K_1$  does.

As in the previous two cases we can repeat the above procedure to get a commutative diagram



where  $f_i: F \rightarrow K_i$  represents a generator of  $H_2(K_i)$ ,  $i = 0, \dots, d$ , where the singular set of  $f_d$  is 1-dimensional, and where  $K_{i-1}$  embeds in  $\mathbb{R}^4$  if  $K_i$  does, for  $i = 1, \dots, d$ . Since, by Proposition 2,  $K_d$  embeds in  $\mathbb{R}^4$  this proves the following result.

**LEMMA 3.** *If  $K$  is a finite 2-complex such that  $H^2(K)$  is infinite cyclic then  $K$  embeds in  $\mathbb{R}^4$ .*

**4. Proof of the theorem.** Suppose  $H^2(K) = Z/mZ$ . Then  $H_1(K)$  is isomorphic to the direct sum of  $Z/mZ$  and a free abelian group  $F$ . Let  $x \in H_1(K)$  correspond to a generator of  $Z/mZ$ . Since the second cohomology does not change if 1-cells are attached to  $K$ , we can assume that  $K^{(1)}$  is connected. Therefore  $x$  can be represented by a closed curve  $C: S^1 \rightarrow K^{(1)}$ . Denote by  $L$  the 2-complex obtained from  $K$  by attaching an additional 2-cell  $e$  using  $C$  as the attaching map. Let  $p$  be a point of  $\text{int}(e)$  and let  $y$  be a generator of  $H_1(\text{int}(e) - p)$ . Since  $H_2(K) = 0$  the Meyer-Vietoris sequence of

the pair  $\{L - p, \text{int}(e)\}$  gives rise to the following exact sequence:

$$0 \rightarrow H_2(L) \rightarrow H_1(\text{int}(e) - p) \rightarrow H_1(K) \rightarrow H_1(L) \rightarrow 0.$$

Because  $y$  gets mapped to  $x$ ,  $H_1(L)$  is free and  $H_2(L)$  is isomorphic to  $Z$ . Therefore  $H^2(L) = Z$ . By Lemma 3  $L$  embeds in  $\mathbb{R}^4$ . Since  $K \subset L$  we also get an embedding of  $K$  into  $\mathbb{R}^4$ . This finishes the proof of the theorem.

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