EMBEDDING A 2-COMPLEX $K$ IN $\mathbb{R}^4$ WHEN $H^2(K)$ IS A CYCLIC GROUP

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We prove that every finite 2-dimensional cell complex with cyclic second cohomology embeds in $\mathbb{R}^4$ tamely.

1. Introduction. It has long been known that every compact PL (piecewise-linear) manifold embeds in euclidean space of double dimension. The analogous result, however, is not true for arbitrary simplicial complexes (see [2]). In [6] an obstruction to embedding $n$-complexes in $\mathbb{R}^{2n}$ was found. Since that obstruction is not homotopy invariant and is in general difficult to calculate, it is natural to ask if a certain class of $n$-complexes which can be easily described embeds in $\mathbb{R}^{2n}$. It has been known that every $n$-complex with cyclic $n$th cohomology embeds in $\mathbb{R}^{2n}$ if $n \neq 2$ (see [5]). If $n > 2$ one can use the techniques of [7] to prove it. The same techniques are much harder to apply when $n = 2$ and if they are successful they yield embeddings which are not smooth but only tame on each 2-cell (recall that an embedding $D^2 \to \mathbb{R}^4$ is tame if it can be extended to an embedding $D^2 \times D^2 \to \mathbb{R}^4$). At present the author does not even know whether every contractible 2-complex embeds in $\mathbb{R}^4$ piecewise smoothly.

In [4] it was shown that the case $n = 2$ really is different from other dimensions (§3). Here we establish a result analogous to other dimensions.

THEOREM. If $K$ is a finite 2-complex such that $H^2(K)$ is cyclic then $K$ can be embedded in $\mathbb{R}^4$.

Note. All homology and cohomology groups will be with integer coefficients; $\mathbb{Z}$ denotes the ring of integers.

The case $H^2(K) = 0$ was proved in [4]. The general case can be reduced to the case when $H^2(K)$ is infinite cyclic. This case is basically in two steps. First it is proved for the case when $H_2(K)$ is generated by an embedded orientable surface. For arbitrary $K$ with $H^2(K) = \mathbb{Z}$ the situation is reduced to the previous case by constructing a tower.
of maps and 2-complexes
\[ K_r \xrightarrow{p_r} K_{r-1} \xrightarrow{p_{r-1}} \cdots \xrightarrow{p_1} K_1 \xrightarrow{p_0} K_0 = K \]
such that \( K_{j-1} \) can be embedded in \( \mathbb{R}^4 \) if \( K_j \) can and such that \( K_r \) embeds in \( \mathbb{R}^4 \).

In what follows all embeddings of \( K \) in \( \mathbb{R}^4 \) will be smooth in the interior of each cell except for a finite number of points in the interiors of 2-cells where they will still be tame. Thus if we construct such an embedding of a subdivided \( K \) it will still be tame on the original \( K \). Therefore we can assume without loss of generality whenever it is convenient that \( K \) is either a simplicial complex or that all the attaching maps are homeomorphisms.

2. A special case. In what follows \( K \) will be a finite connected 2-complex.

**Lemma 1.** Suppose \( H^2(K) = \mathbb{Z} \) and suppose that \( H_2(K) \) is generated by an embedded orientable surface \( F \subset K \). Then \( K \) can be embedded in \( \mathbb{R}^4 \).

**Proof.** Let \( e_0 \) be a 2-cell of \( F \). Then the inclusion \((K - \text{int}(e_0), F - \text{int}(e_0)) \subset (K, F)\) gives rise to the following commutative diagram

\[
\begin{array}{ccc}
H^2(K, F) & \longrightarrow & H^2(K) \\
\downarrow & & \downarrow \\
H^2(K - \text{int}(e_0), F - \text{int}(e_0)) & \longrightarrow & H^2(K - \text{int}(e_0))
\end{array}
\]

in which both rows are exact. Since \( H^2(K) \rightarrow H^2(F) \) is an isomorphism the first homomorphism in the top row is trivial. The first vertical map is an isomorphism (by excision); therefore the first homomorphism in the bottom row is also trivial. This implies that \( H^2(K - \text{int}(e_0)) \) is 0.

By attaching 2-cells to \( K - \text{int}(e_0) \) we can obtain an acyclic 2-complex \( L \). Denote \( L \cup e_0 \) again by \( K \). Clearly if this \( K \) can be embedded in \( \mathbb{R}^4 \) so can the original 2-complex.

Choose an embedding of \( F \cup K^{(1)} \) in \( \mathbb{R}^3 \times 0 \subset \mathbb{R}^4 \) which is smooth on \( F \) and on each edge of \( K \). Identify \( F \cup K^{(1)} \) with its image under this embedding. Then \( F \cup K^{(1)} \subset \mathbb{R}^3 \times 0 \). Let \( H \times 0 \) be a regular neighborhood of \( K^{(1)} \) in \( \mathbb{R}^3 \times 0 \). \( H \times 0 \) is a handlebody with spine \( K^{(1)} \). There is a natural projection \( p: \partial(H \times 0) \rightarrow K^{(1)} \) such that \( H \times 0 \) is the mapping cylinder of \( p \). Thus every point in \( H \times 0 \) can
be thought of as a class \([x, t]\) where \(x \in \partial H\), \(t \in I\) (\(= [0, 1])\), and \([x, 1] = \hat{p}(x)\). Let \(\hat{p} : H \rightarrow K^{(1)}\) be defined by \(\hat{p}([x, t]) = p(x)\).

Let \(U\) be a regular neighborhood of \(K^{(1)}\) in \(K\). \(\partial U\) is a union of circles \(C_0, \ldots, C_g\) where \(C_i\) corresponds to the 2-cell \(e_i\) of \(K\) and where \(g\) is the genus of \(H\) (because \(L\) is acyclic). Suppose \(\partial U \cap F = C_0 \cup \cdots \cup C_k\). Orient \(F\) and assume that \(C_0, \ldots, C_k\) have the induced orientation. Also choose orientations for the curves \(C_{k+1}, \ldots, C_g\). \(U\) and \(H\) can be chosen in such a way that \((H \times 0) \cap F = U \cap F\) and so that \(U \cap F = \hat{p}^{-1}(p(\partial U \cap F)) = \{[x, t] \in H \times 0; x \in \partial U \cap F, t \in I\}\). Embed \(C_{k+1} \cup \cdots \cup C_g\) smoothly in \(H \times 1\) in such a way that \(p|C_j : C_j \rightarrow K^{(1)}\) is the attaching map for \(e_j\). Let \(U_j = \{(x, t), (1-t) \in H \times [-1, 1]|x \in C_j, t \in I\}\). \(U_j\) is an embedding of the collar of \(e_j\) into \(H \times [0, 1]\). \((\bigcup_{j=k+1}^g U_j) \cup (F \cap H \times 0)\) is an embedding of \(U\) into \(H \times [-1, 1]\) which we can assume to be piecewise smooth.

Since \(L\) is acyclic, \(C_1, \ldots, C_g\) form a basis for \(H_i(\partial(H \times [-1, 1]))\). Let \(T\) be a maximal tree of \(K^{(1)}\) and let \(s_1, \ldots, s_g\) be the edges of \(K^{(1)} - T\). If \(m_i\) is the midpoint of \(s_i\) let

\[
S_i = (\hat{p}^{-1}(m_k) \times \{-1, 1\}) \cup p^{-1}(m_i) \times [-1, 1] \subset \partial(H \times [-1, 1]).
\]

Then \(S_i\) is an embedded 2-sphere. Choose an orientation for \(S_i\). For each \(i = 1, \ldots, g\) choose an oriented simple closed curve \(a_i\) in \(\partial(H \times [-1, 1])\) such that \(a_i \cdot S_j = \delta_{ij}\). Then \(\{a_1, \ldots, a_g\}\) is a basis for \(H_i(\partial(H \times [-1, 1]))\). Suppose \(C_i \sim \sum p_{ij} a_j, i = 1, \ldots, g\), in \(\partial(H \times [-1, 1])\) \((\sim\) stands for homologous). Then \(\text{det}(p_{ij}) = \pm 1\).

Let \(\Sigma_i\) be a union of suitably oriented disjoint copies of spheres \(S_1, \ldots, S_g\) representing the class \(\sum_{j=1}^g a_i j[S_j]\) in \(H_2(\partial(H \times [-1, 1]))\) where \((q_{ij}) = (p_{ji})^{-1}\). Then

\[
C_i \cdot \Sigma_j' = \sum_{k,l} p_{ik} q_{ji} a_{k} \cdot S_i = \sum_{k=1}^g p_{ik} q_{jk} = \delta_{ij}.
\]

The intersection number \(\Sigma' \cdot F\) is zero (it is the intersection of closed orientable surfaces in \(\mathbb{R}^4\)). Since \(\Sigma_i \cap F = \Sigma_i' \cap (C_0 \cup \cdots \cup C_k)\), the intersection number \(\Sigma_i' \cdot (C_0 \cup \cdots \cup C_k)\) in \(\partial(H \times [-1, 1])\) is also zero. Since \(\Sigma_i' \cdot (C_1 \cup \cdots \cup C_k) = 0\), for \(i > k\), it follows that \(\Sigma_i' \cdot C_0 = 0\), for \(i > k\). Therefore we can pipe together the intersections of \(\Sigma_i'\) with \(C_j, j = 0, \ldots, g\), along \(C_0 \cup \cdots \cup C_g\) to obtain for each \(i > k\) a surface \(\Sigma_i'' \subset \partial(H \times [-1, 1])\) such that \(\Sigma_i'' \cap F = \emptyset = \Sigma_i' \cap C_j, i \neq j\), and such that \(\Sigma_i'' \cap C_i\) is a point. Since all the \(\text{"pipes"}\) lie either in \(H \times 1\) or in a neighborhood of \(\partial H \times 0\) in \(\partial(H \times [-1, 1])\), one can
choose half of a symplectic basis for each \( H_1(Σ''_i), \ i > k \), represented by smooth simple closed curves in \( \partial H \times (0, 1] \cup H \times 1 \). Since \( M' = \mathbb{R}^3 \times [0, \infty) - \text{int}(H \times [-1, 1]) \) is simply connected, we can cap off these curves by regularly immersed discs in \( M' \). By performing surgeries along these discs change each \( Σ''_i, \ i > k \), into a singular 2-sphere \( Σ_i \). All the singularities lie in \( M' \). Furthermore, \( Σ_i \cap (U \cup F) = Σ_i \cap C_i \) is a point. Note also that \( Σ_i \cap Σ_j \cap \text{int}(H \times [-1, 1]) = \emptyset \), and that \( Σ_i \cdot Σ_j = 0 \), for \( i \neq j \), \( i, j > k \).

Cap off the curves \( C_{k+1}, \ldots, C_g \) by regularly immersed discs \( D'_{k+1}, \ldots, D'_g \), respectively, lying in \( \mathbb{R}^3 \times [1, \infty) \). This extends the embedding of \( F \cup U \) to a regular immersion of \( K \) into \( \mathbb{R}^4 \). Since \( D'_i \cdot Σ_j = δ_{ij} \) for all \( i, j > k \), we can use the spheres \( Σ_j \) to pipe off the intersections between the discs \( D'_{k+1}, \ldots, D'_g \), in order to get immersed discs \( D_{k+1}, \ldots, D_g \), respectively, such that \( D_i \cdot D_j = 0 \), for \( i \neq j \). Again \( Σ_i \cdot D_j = δ_{ij} \), for \( i, j > k \).

Let \( M \) be the union of \( M' \) and a regular neighborhood of \( Σ_{k+1} \cup \cdots \cup Σ_g \) which misses \( F \). Since \( Σ_j - M' \) is a union of embedded discs, for \( j = k + 1, \ldots, g \), \( M \) is simply connected. The discs \( D_{k+1}, \ldots, D_g \) and the classes \( x_i = [Σ_i] \in H_2(M), \ i > k \), satisfy the conditions of Theorem 3.1 of [3]. Applying Theorem 1.1 of [3] we get \( g - k \) tamely embedded discs \( B^2_{k+1}, \ldots, B^2_g \) in \( M \) such that \( B_j^2 \cap \partial M = C_j \). This, in turn, defines an embedding of \( K \) in \( \mathbb{R}^4 \).

3. The case \( H^2(K) = Z \). Let \( B \) be a ball of radius \( r \) and let \( F: B \times I \to B \) have the following properties: \( F_0 = \text{id}, \ F_t | \partial B = \text{id}, \) for \( t \in [0, 1] \), and \( F_t \) is a homeomorphism of \( B \) for \( t \in [0, 1] \). Then the homotopy \( H: B \times B^k \times I \to B \times B^k \) given by

\[
H((x, y), t) = (F(x, (1 - |y|)t), y)
\]

is the identity on \( \partial(B \times B^k) \). Furthermore, \( H_t \) is one-to-one on \( B \times B^k - B \times 0 \), for all \( t \in I \), and \( H_t | B \times 0 = F_t \times 0 \).

Lemma 2. Let \( K \) be a finite 2-dimensional cell complex, such that all the 2-cells are attached via homeomorphisms. Let \( g \) be an embedding of \( K \) into \( \mathbb{R}^4 \). Then there exists a homotopy with compact support \( H: \mathbb{R}^4 \times I \to \mathbb{R}^4 \), such that \( H_0 = \text{id} \), and such that \( H_t \) is homeomorphism for \( t \in [0, 1] \), which does one of the following three types of deformations:

(i) for an edge \( s \) of \( K \), \( H_1 \) maps \( g(s) \) to a point and is 1-1 elsewhere;
(ii) for a 2-cell e with boundary a union of two edges s₁, s₂ having pairs of common endpoints, H is a deformation retraction of g(e) onto g(s₁), which is fixed on g(s₁).

(iii) for two 2-cells e₁, e₂ with e₁ ∩ e₂ being an arc A, H₁ maps g(e₁) homeomorphically onto g(e₂), and is 1-1 on g(K) − g(e₁ ∪ e₂). Furthermore, H is fixed on g(e₂).

If K₁ is the 2-complex obtained from K by the identifications defined by H₁ then H₁g : K → R⁴ factors through K₁. The factoring map K₁ → R⁴ is an embedding.

Proof. Define a homotopy F : 2B⁰ × I → 2B⁰ as follows:

For type (i) let k = 1, and let

\[ F(x, t) = \begin{cases} (1 - t)x & \text{for } |x| \leq 1, \\ (1 + t)x - 2tx/|x| & \text{for } 1 \leq |x| \leq 2. \end{cases} \]

F squeezes [−1, 1] to 0 and linearly stretches the rest of [−2, 2].

For type (ii) let k = 2, and let

\[ F((x, y), t) = \begin{cases} (x, y(1 - t)) & \text{for } |x| \leq 1, \ 0 \leq y \leq A(x), \\ (x, (1/(A(x) - B(x)))(A(x)(1 - t) - B(x))y + tA(x)B(x))) & \text{for } |x| \leq 1, \ A(x) \leq y \leq B(x), \\ (x, y) & \text{elsewhere}, \end{cases} \]

where \( A(x) = \sqrt{1 - x^2}, B(x) = \sqrt{4 - x^2}. \) F shrinks \( D^2 \cap R^2_+ \) to [−1, 1] × 0.

For type (iii) let k = 3 and define F as follows:

Let \( \delta : [0, 2\pi] \times I \to [0, 2\pi] \) be the homotopy

\[ \delta(\alpha, t) = \begin{cases} (1 - t)\alpha & \text{for } 0 \leq \alpha \leq \pi/2, \\ (1 + t/3)\alpha - 2\pi t/3 & \text{for } \alpha \geq \pi/2. \end{cases} \]

\( \delta \) shrinks \( [0, \pi/2] \) to 0 and stretches \([\pi/2, 2\pi]\) over \([0, 2\pi]\). A point in \( R^3 \) can be represented as a pair of a real and a complex number. Let

\[ F((x, r \exp(i\alpha)), t) = \begin{cases} (x, r \exp(i\delta(\alpha, t))) & \text{for } \rho \leq 1, \\ (x, r \exp(i[(2 - \rho)\delta(\alpha, t) + (\rho - 1)\alpha])) & \text{for } \rho \in [1, 2], \end{cases} \]

where \( \rho = \sqrt{x^2 + r^2}. \)

In each case \( F_\tau|\delta(2B^k) \) is identity for all \( t \in I. \)
For type (i) \( g(s) \) has a regular neighborhood \( N \) homeomorphic to \([-2, 2] \times B^3 \). Let \( \varphi : [-2, 2] \times B^3 \to N \) be a homeomorphism such that \( \varphi([-1, 1] \times 0) = g(s) \).

For type (ii) \( g(e) \) has a regular neighborhood \( N \) homeomorphic to \( 2D^2 \times B^2 \). Let \( \varphi : 2D^2 \times B^2 \to N \) be a homeomorphism such that \( \varphi((D^2 \cap \mathbb{R}^2 \times 0) = g(e) \), and such that \( \varphi([-1, 1] \times 0) = g(s_1) \).

For type (iii), since \( D = g(e_1 \cup e_2) \) is a tame disc such that its interior doesn’t intersect \( g(K) - D \), there exists a homeomorphism \( \varphi \) from \( 2B^3 \times [-1, 1] \) onto a regular neighborhood \( N \) of \( D \), satisfying the following two properties: \( \varphi(B^3 \times 0) \cap (g(K) - D) = \emptyset \), and \( \varphi \) maps \( \{(x, y, z, 0) \in B^3 \times 0 | y \geq 0, z \geq 0, yz = 0\} \) onto \( D \) so that \( g(A) = \varphi((x, 0, 0, 0) \in B^3 \times 0\} \).

Given \( \varphi \) and \( F \) for each type we define the desired homotopy \( H \) by

\[
H(x, t) = \begin{cases} 
  x & \text{for } x \in N, \\
  \varphi(F(u, (1 - |u|t), v) & \text{for } (u, v) \in 2B^k \times B^{4-k}, \\
  x = \varphi(u, v). & \end{cases}
\]

Suppose \( f : F \to K \) represents a generator of \( H_2(K) \). We can assume (by subdividing \( F \) and \( K \) appropriately) that \( f \) is simplicial and non-degenerate on each simplex (compare with [1], p. 11). We dealt with the case when \( f \) is an embedding in Lemma 1. Assume now that the singular set \( S \) of \( f \) (\( S \) is the closure of the set \{\( x \in F | f^{-1}(f(x)) \) contains more than one point\}) is non-empty. We will successively replace \( K \) by “nicer” complexes and finally reduce the problem of embeddability of \( K \) in \( \mathbb{R}^4 \) to the situation of Lemma 1.

**Case 1.** \( S \) is 0-dimensional.

If \( \Sigma = f(S) = \{v_1, \ldots, v_r\} \) then \( F_0 = f(F) \) is obtained from \( F \) by identifying the points of each set \( f^{-1}(y_j), j = 1, \ldots, t \). Suppose \( f^{-1}(v_1) = \{v_1, v_2, \ldots, v_l\} \). Construct \( F_1 \) from \( F \) by identifying the points of each set \( f^{-1}(v_1) - \{v_1\}, f^{-1}(v_2), \ldots, f^{-1}(v_r) \). Note that \( F_1 \) is not a surface. Clearly there exists a map \( f_1 \) making the following diagram commutative:

\[
\begin{array}{ccc}
F & \xrightarrow{f_1} & F_1 \\
\downarrow f & & \downarrow p_1 \\
F_0 & & 
\end{array}
\]
where \( p_1 : F_1 \to F_0 \) denotes the natural projection. The singular set \( S_1 \) of \( f_1 \) is equal to \( S - \{ v_1 \} \).

Attach the endpoints of an arc \( A \) to \( F_1 \) to \( w_1 \) and \( w_2 \), where \( w_i = f_1(v_i) \). The resulting space \( \hat{F}_1 \) is homotopy equivalent to \( F_0 \). For example, the map \( \hat{p}_1 : \hat{F}_1 \to F_0 \) defined to be \( p_1 \) on \( F_1 \) and sending \( A \) to \( y_1 \) is a homotopy equivalence. It is easy to find a homotopy inverse \( q : F_0 \to \hat{F}_1 \). Suppose \( \alpha : I \to A \) is a parametrization of \( A \) such that \( \alpha(0) = w_1 \). If \( \sigma \) is a simplex of dimension greater than zero in \( F_1 \), with vertex \( w_1 \), then \( \sigma \) is a cone over a simplex \( \tau \). Define

\[
q(x) = \begin{cases} 
  x & \text{for } x \notin p_1(\text{st}(w_1)), \\
  [u, 2t - 1] & \text{for } [u, t] \in \sigma = C(\tau), x = p_1([u, t]), \\
  t \in [1/2, 1], & \\
  \alpha(1 - 2t) & \text{for } t \in [0, 1/2].
\end{cases}
\]

Here \( \text{st}(w_1) \) denotes the star of \( w_1 \), and \( C(\tau) \) is the cone over \( \tau \) with the vertex \( w_1 \) corresponding to the value \( t = 0 \).

Clearly \( q \) is 1-1 on each 1-simplex of \( F_0 \). If \( L = K - F_0 \) then \( K \) is obtained from \( F_0 \) by attaching \( L \) along a graph \( G \) in \( F_0^{(1)} \). If \( \sigma \) is a cell attached to \( G \) via an attaching map \( \psi \), then attach \( \sigma \) to \( \hat{F}_1 \) via \( q\psi \). This gives us a new complex \( K_1 \) homotopy equivalent to \( K \) by an obvious extension \( q_1 : K_1 \to K \) of \( \hat{p}_1 \). By subdividing \( \text{st}(y_1) \) we can always assume that \( K_1 \) is again a simplicial complex with \( A \) one of its 1-simplices. \( H_2(K_1) \) is generated by the mapping \( f_1 : F \to K_1 \) which has one less point in its singular set than \( f \). Using Lemma 2 successively (one deformation of type (i) along \( A \) followed by a sequence of deformations of type (ii)) we see that if \( K_1 \) can be embedded in \( \mathbb{R}^4 \) then so can \( K \).

Repeating the same construction we get the following commutative diagram

\[
\begin{array}{ccccccccc}
F & \xrightarrow{f_j} & F_{j-1} & \xrightarrow{f_{j-1}} & \cdots & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 \\
\downarrow{p_j} & & \downarrow{p_{j-1}} & & \cdots & & \downarrow{p_2} & & \downarrow{p_1} \\
F_j & & F_{j-1} & & \cdots & & F_1 & & F_0 \\
\downarrow{\cap} & & \downarrow{\cap} & & \cdots & & \downarrow{\cap} & & \downarrow{\cap} \\
K_j & \xrightarrow{q_j} & K_{j-1} & \xrightarrow{q_{j-1}} & \cdots & \xrightarrow{q_2} & K_1 & \xrightarrow{q_1} & K_0 = K
\end{array}
\]

where the maps in the bottom row are homotopy equivalences, \( H_2(K_i) \) is generated by \( f_i : F \to F_i \subset K_i, \ i = 0, \ldots, j, \) and \( f_j \) is an
embedding. Furthermore, if \( K_i \) can be embedded in \( \mathbb{R}^4 \) so can \( K_{i-1} \), \( i = 1, \ldots, j \). Also \( K_j \) embeds in \( \mathbb{R}^4 \) by Lemma 1. This proves

**Proposition 1.** Suppose \( K \) is a finite simplicial complex. Suppose that \( H^2(K) = \mathbb{Z} \) and that \( H_2(K) \) is represented by a non-degenerate simplicial map \( f: F \to K \) of an orientable surface \( F \) into \( K \). If the singular set of \( f \) is 0-dimensional then \( K \) can be embedded in \( \mathbb{R}^4 \).

**Case 2.** \( S \) is 1-dimensional.

Then \( \Sigma = f(S) \) is also at most 1-dimensional. \( F_0 \) is obtained from \( F \) by identifying the points of each \( f^{-1}(y) \), \( y \in \Sigma^{(0)} \), and by identifying the components of each \( f^{-1}(\sigma) \) (by simplicial isomorphisms) where \( \sigma \) runs over the interiors of the edges of \( \Sigma \). Let \( f^{-1}(\sigma_0) \) be a union of open edges \( s_1, \ldots, s_r \), for some open edge \( \sigma_0 \in \Sigma \). Construct \( F_1 \) from \( F \) by identifying the points of each set \( f^{-1}(y) \), \( y \in \Sigma^{(0)} \), and by identifying the components of \( s_2 \cup \cdots \cup s_r \) and of the sets \( f^{-1}(\sigma) \) where \( \sigma \) runs over open 1-simplices of \( \Sigma - \sigma_0 \) (again via simplicial isomorphisms). As in Case 1 there exists a map \( f_1 \) making the diagram

\[
\begin{array}{ccc}
F & \xrightarrow{f_1} & F_1 \\
f \downarrow & & \downarrow p_1 \\
F_0 & & 
\end{array}
\]

commute where \( p_1: F_1 \to F_0 \) is the natural projection. The singular set \( S_1 \) of \( f_1 \) has one less edge than \( S: S_1 = S - s_1 \).

Attach a 2-cell \( D \) to \( z_1 \cup z_2 \subset F_1 \) via a homeomorphism where \( z_j = f_1(s_j) \). The resulting space \( \tilde{F}_1 \) is homotopy equivalent to \( F_0 \). The extension \( \tilde{p}_1: \tilde{F}_1 \to F_0 \) of \( p_1: F_1 \to F_0 \) which squeezes \( D \) to \( z_1 \) is a homotopy equivalence. Suppose, as before, that \( L = \overline{K - F_0} \) is attached to \( F_0 \) along a graph \( G \). Then \( \tilde{G} = p^{-1}(G) - z_1 \) is homeomorphic to \( G \) and \( L \) can be attached to \( \tilde{F}_1 \) along \( \tilde{G} \) in the obvious way to construct a 2-complex \( K_1 \) which is homotopy equivalent to \( K \). Let \( q_1: K_1 \to K \) be the obvious extension of \( \tilde{p}_1: \tilde{F}_1 \to F_0 \). \( H_2(K_1) \) is generated by \( f_1: F \to K_1 \) which has one less edge in its singular set than \( f \). Also, by using one deformation of type (ii) from Lemma 2 we see that if \( K_1 \) embeds in \( \mathbb{R}^4 \) then so does \( K \). As in Case 1 we
repeat the above procedure to get a commutative diagram

\[ F \]
\[ F_l \xrightarrow{p_1} F_{l-1} \xrightarrow{p_{l-1}} \cdots \xrightarrow{p_2} F_1 \xrightarrow{p_1} F_0 \]
\[ K_l \xrightarrow{q_i} K_{l-1} \xrightarrow{q_{l-1}} \cdots \xrightarrow{q_2} K_1 \xrightarrow{q_1} K_0 = K \]

where the bottom maps are homotopy equivalences, the singular set of \( f \) is 0-dimensional, and \( K_{i-1} \) embeds in \( \mathbb{R}^4 \) if \( K_i \) does, for \( i = 1, \ldots, l \). Combining this with Proposition 1 we get

**Proposition 2.** Suppose \( H^2(K) = \mathbb{Z} \), and suppose that a generator of \( H_2(K) \) is represented by a non-degenerate simplicial map \( f: F \to K \) where \( F \) is an orientable surface. If the singular set of \( f \) is 1-dimensional then \( K \) embeds in \( \mathbb{R}^4 \).

**Case 3.** \( S \) is 2-dimensional.

Choose a point \( b_\sigma \) in the interior of each 2-cell \( \sigma \) of \( F \). Let \( S_k \) be the collection of all open 2-cells \( \sigma \) such that \( f^{-1}(f(b_\sigma)) \) contains \( k \) points. Denote by \( Z_k \) the union of 2-cells \( \sigma \) such that \( \text{int}(\sigma) \in S_k \). Represent the homology class of \( f: F \to K \) by a linear combination \( \Sigma x_\sigma e \) where \( e \) runs over the 2-cells of \( K \). By choosing appropriate orientations for the 2-cells of \( f(F) \) we can assume that all the coefficients \( x_\sigma \) are non-negative. Furthermore, \( F \) can be chosen so that \( S_k = \{ f^{-1}(\text{int}(e)) | x_e = k \} \), for all \( k \) (see [2], p. 11). Let \( M = \max\{k | S_k \neq \emptyset \} \). Since \( S \) is 2-dimensional, \( M \) is greater than 1. \( S_M \) does not contain all the open 2-cells of \( F \) because the coefficients \( x_e \) have no common factor. Therefore there exists a 2-cell \( \sigma_1 \) such that \( \text{int}(\sigma_1) \in S_M \) and such that the intersection of \( \sigma_1 \) with \( F - Z_M \) contains an open edge \( s_1 \). Let \( \Sigma = f(S) \). Construct \( F_1 \) from \( F \)

1. by identifying the points of each \( f^{-1}(y) \), \( y \in \Sigma^{(0)} \),
2. by identifying the components of \( f^{-1}(\tau) \) where \( \tau \) runs over the open edges of \( \Sigma - f(s_1) \),
3. by identifying the components of \( f^{-1}(e) \) where \( e \) runs over all closed 2-cells of \( \Sigma - f(\sigma_1) \),
(4) by gluing together \( s_2, \ldots, s_m \) where \( s_1, \ldots, s_m \) are the components of \( f^{-1}(f(s_1)) \), and

(5) by gluing together \( \sigma_2, \ldots, \sigma_m \), where \( \sigma_1, \ldots, \sigma_m \) are closed 2-cells whose union is \( f^{-1}(f(\sigma_1)) \).

As before, let all the identifications be via simplicial isomorphisms. \( f \) can again be factored as \( p_1 f_1 \) where \( p_1 : F_1 \to F_0 \) is the natural projection. \( p_1 \) is a homotopy equivalence. If, as before, \( K \) is obtained from \( F_0 \) by attaching \( L = K - F_0 \) along a graph \( G \subset F_0 \), construct \( K_1 \) by attaching \( L \) to \( F_1 \) along \( p_1^{-1}(G) - f_1(s_1) \approx G \) in the obvious way. \( K_1 \) is homotopy equivalent to \( K \). Let \( q_1 : K_1 \to K \) be the natural extension of \( p_1 \). \( H_2(K) \) is generated by \( f_1 : F \to K_1 \). The singular set of \( f_1 \) has one less 2-simplex than \( S \). Also, by Lemma 2 (using type (iii) deformation) \( K \) embeds in \( \mathbb{R}^4 \) if \( K_1 \) does.

As in the previous two cases we can repeat the above procedure to get a commutative diagram

\[
\begin{array}{cccccc}
F_d & \xrightarrow{p_d} & F_{d-1} & \xrightarrow{p_{d-1}} & \cdots & \xrightarrow{p_2} & F_1 & \xrightarrow{p_1} & F_0 \\
\cap & & \cap & & \cap & & \cap & & \cap \\
K_d & \xrightarrow{q_d} & K_{d-1} & \xrightarrow{q_{d-1}} & \cdots & \xrightarrow{q_2} & K_1 & \xrightarrow{q_1} & K_0 = K
\end{array}
\]

where \( f_i : F \to K_i \) represents a generator of \( H_2(K_i), \ i = 0, \ldots, d \), where the singular set of \( f_d \) is 1-dimensional, and where \( K_{i-1} \) embeds in \( \mathbb{R}^4 \) if \( K_i \) does, for \( i = 1, \ldots, d \). Since, by Proposition 2, \( K_d \) embeds in \( \mathbb{R}^4 \) this proves the following result.

**Lemma 3.** If \( K \) is a finite 2-complex such that \( H^2(K) \) is infinite cyclic then \( K \) embeds in \( \mathbb{R}^4 \).

**4. Proof of the theorem.** Suppose \( H^2(K) = \mathbb{Z}/m\mathbb{Z} \). Then \( H_1(K) \) is isomorphic to the direct sum of \( \mathbb{Z}/m\mathbb{Z} \) and a free abelian group \( F \). Let \( x \in H_1(K) \) correspond to a generator of \( \mathbb{Z}/m\mathbb{Z} \). Since the second cohomology does not change if 1-cells are attached to \( K \), we can assume that \( K^{(1)} \) is connected. Therefore \( x \) can be represented by a closed curve \( C : S^1 \to K^{(1)} \). Denote by \( L \) the 2-complex obtained from \( K \) by attaching an additional 2-cell \( e \) using \( C \) as the attaching map. Let \( p \) be a point of \( \text{int}(e) \) and let \( y \) be a generator of \( H_1(\text{int}(e) - p) \). Since \( H_2(K) = 0 \) the Meyer-Vietoris sequence of
the pair \( \{L - p, \text{int}(e)\} \) gives rise to the following exact sequence:

\[
0 \to H_2(L) \to H_1(\text{int}(e) - p) \to H_1(K) \to H_1(L) \to 0.
\]

Because \( y \) gets mapped to \( x \), \( H_1(L) \) is free and \( H_2(L) \) is isomorphic to \( \mathbb{Z} \). Therefore \( H^2(L) = \mathbb{Z} \). By Lemma 3 \( L \) embeds in \( \mathbb{R}^4 \). Since \( K \subset L \) we also get an embedding of \( K \) into \( \mathbb{R}^4 \). This finishes the proof of the theorem.

**References**


Received January 26, 1990 and in revised form July 15, 1990.

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Selman Akbulut and Henry Churchill King, Rational structures on 3-manifolds .......................................................... 201
Mark Baker, On coverings of figure eight knot surgeries .................. 215
Christopher Michael Brislawn, Traceable integral kernels on countably generated measure spaces .................................. 229
William Chin, Crossed products and generalized inner actions of Hopf algebras .......................................................... 241
Tadeusz Figiel, William Buhmann Johnson and Gideon Schechtman, Factorizations of natural embeddings of $l^n_p$ into $L_r$. II .................. 261
David Howard Gluck, Character value estimates for groups of Lie type ...... 279
Charn-Huen Kan, Norming vectors of linear operators between $L_p$ spaces .............................................................. 309
Marko Kranjc, Embedding a 2-complex $K$ in $\mathbb{R}^4$ when $H^2(K)$ is a cyclic group .......................................................... 329
Ka-Lam Kueh, The remainder terms aspect of the theory of the Riemann zeta-function ....................................................... 341
J. A. Marti, Sur la rigidité comparée de fonctions, distributions, ou hyperfonctions analytiques par rapport à un groupe de variables ...... 359
Margherita Roggero and Paolo Valabrega, Chern classes and cohomology for rank 2 reflexive sheaves on $\mathbb{P}^3$ ................................................ 383