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**THE REMAINDER TERMS ASPECT OF THE THEORY OF THE  
RIEMANN ZETA-FUNCTION**

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**Assuming the Riemann hypothesis for Riemann zeta-function  $\zeta(s)$ , let  $R(u)$  and  $S(t)$  denote the remainder terms for the prime number theorem (suitably normalized) and the zero counting formula for  $\zeta(s)$  respectively. We analyze the relation between  $R(u)$  and  $S(t)$ , which generalizes A. Guinand's work.**

**0. Introduction.** In this paper, the general form of transform  $T$  on  $L^2(0, \infty)$  defined by a kernel  $\varphi$  is given by

$$Tf(t) = \lim_{U \rightarrow \infty} \int_0^U f(u)\varphi(tu) du \quad \text{in } L^2(0, \infty)$$

for  $f \in L^2(0, \infty)$ . Throughout this paper, we assume the Riemann hypothesis for the Riemann zeta-function  $\zeta(s)$ . It is concerned with the remainder terms  $R(u)$  in the prime number theorem (suitably normalized) and  $S(t)$  in the zero counting formula for the Riemann zeta-function; see (2.1) and (2.4). Gallagher's version [2, Theorem 1] of the Guinand's summation formula, (0.0) below, gives a symmetric relation between  $R(u)$  and  $S(t)$ :

$$(0.0) \quad \int_0^\infty f(u) dR(u) = \int_0^\infty \hat{f}(t) dS(t)$$

where  $\hat{f}(t) = \int_0^\infty f(u) \cos tu du$ , for a suitable class of functions  $f(u)$  with "good" growth condition on  $f(u)$ . In particular, it is possible to take as  $f(\cdot)$  in (0.0) the characteristic function of the interval  $[0, u]$ , and on taking integration by parts, we get

$$(0.1) \quad \sqrt{\frac{2}{\pi}} \frac{R(u)}{u} = - \int_0^\infty \frac{S^*(t)}{t} k(tu) dt$$

where

$$S^*(t) = S(t) - S(0^+) \quad \text{and} \quad k(\theta) = \sqrt{\frac{2}{\pi}} \theta \frac{d}{d\theta} \frac{\sin \theta}{\theta}.$$

A. Guinand [3] observed that  $k(\theta)$  is the kernel of a Hankel transform of order  $3/2$ , which is an involution on  $L^2(0, \infty)$ . So we have

$$(0.2) \quad -\frac{S^*(t)}{t} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{R(u)}{u} k(tu) du,$$

since  $S(t) = O(\log t / \log \log t)$  due to Littlewood [6], and hence  $S^*(t)/t \in L^2(0, \infty)$ .

Equations (0.1) and (0.2) suggest a symmetric relation between  $R(u)$  and  $S(t)$ . For a function  $f(x)$ , the Riemann-Liouville fractional integral is defined by

$$(0.3) \quad I_\alpha f(x) = f_\alpha(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} f(y) dy & (\alpha > 0), \\ f(x) & (\alpha = 0); \end{cases}$$

and we have

$$I_{\alpha+\beta} f(x) = I_\alpha I_\beta f(x)$$

for  $\alpha \geq 0$ ,  $\beta \geq 0$ . For  $\alpha$  an integer  $f_\alpha(x)$  turns out to be an iterated integral. As a generalization of Guinand's work, we consider the Riemann-Liouville fractional integral on  $R(u)$  and  $S(t)$ , and construct the kernel

$$K(\alpha, m, \theta) = \sqrt{\frac{2}{\pi}} \theta^{\alpha+1} \left( \frac{d}{d\theta} \right)^{m+1} \frac{\sin_\alpha(\theta)}{\theta^{\alpha+1}} \quad (\theta > 0)$$

where  $m$  is an integer and  $\alpha$  is real, and the equation similar to (0.1)

$$(0.4) \quad \sqrt{\frac{2}{\pi}} \frac{R_\alpha(u)}{u^{m+1}} = (-1)^{m+1} \int_0^\infty \frac{S_m^*(t)}{t^{\alpha+1}} K(\alpha, m, tu) dt.$$

Note that  $S_1(t) = O(\log t / (\log \log t)^2)$  due to Littlewood [6], and hence

$$(0.5) \quad S_m^*(t) = S_m(t) - \frac{1}{m!} S(0^+) t^m = O(t^m) \quad (t \rightarrow +\infty)$$

for  $m \geq 1$ , and

$$(0.6) \quad S_m^*(t) = O(t^{m+1}) \quad (t \rightarrow 0^+)$$

for  $m \geq 0$ .

We give first in Theorem 1 the conditions on  $\alpha$ ,  $m$  for which equation (0.4) holds good, and then in Theorem 2 the conditions on  $\alpha$ ,  $m$  for which  $K(\alpha, m, \theta)$  defines a bounded and invertible transform on  $L^2(0, \infty)$ . In particular,  $K(m, m, \theta)$  for  $m \geq -1$  is an involution

on  $L^2(0, \infty)$ . Thus corresponding to (0.2) we will be able to show in Theorem 3

$$(0.7) \quad \frac{S_\alpha^*(t)}{t^{m+1}} = (-1)^{m+1} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{R_m(u)}{u^{\alpha+1}} K(\alpha, m, tu) du$$

for suitable  $\alpha$  and  $m$ , and furthermore

$$(0.8) \quad \int_0^\infty \left| \frac{S_m^*(t)}{t^{m+1}} \right|^2 dt = \frac{2}{\pi} \int_0^\infty \left| \frac{R_m(u)}{u^{m+1}} \right|^2 du$$

for  $m = 0, 1, 2, 3, \dots$ .

**1. Averaging operators.** To pursue the object of this paper, we have to introduce the averaging operator.

For  $f \in L^2(0, \infty)$ , we define the transforms  $A_\alpha$  and  $B_\beta$  by

$$A_\alpha f(x) = x^{-\alpha} \int_0^x y^{\alpha-1} f(y) dy \quad (\alpha > \frac{1}{2}),$$

$$B_\beta f(x) = x^{-\beta} \int_x^\infty y^{\beta-1} f(y) dy \quad (\beta < \frac{1}{2}).$$

LEMMA 1.1. *The  $A_\alpha$  and  $B_\beta$  are bounded operators on  $L^2(0, \infty)$  and*

$$\|A_\alpha\|_2 \leq (\alpha - \frac{1}{2})^{-1}, \quad \|B_\beta\|_2 \leq (\frac{1}{2} - \beta)^{-1}.$$

*Proof.* By (9.9.8) and (9.9.9) of [4], the results follow immediately.

LEMMA 1.2. *For all values of  $\alpha$  and  $\beta$  for which  $A_\alpha$  and  $B_\beta$  are defined, we have*

$$A_\alpha B_\beta = \frac{B_\beta + A_\alpha}{\alpha - \beta} = B_\beta A_\alpha,$$

$$A_\alpha A_\beta = \frac{A_\beta - A_\alpha}{\alpha - \beta} = A_\beta A_\alpha \quad (\alpha \neq \beta),$$

$$B_\alpha B_\beta = \frac{B_\beta - B_\alpha}{\beta - \alpha} = B_\beta B_\alpha \quad (\alpha \neq \beta).$$

*Proof.* Note first that the unitary involution  $J$ , defined by

$$Jf(x) = \frac{1}{x} f\left(\frac{1}{x}\right),$$

intertwines the operators  $A_\alpha$  and  $B_{1-\alpha}$ :  $JA_\alpha = B_{1-\alpha}J$ .

It suffices to prove the first two equations on the left. The first equation on the right then follows by conjugation by  $J$ , the second

equation on the right by symmetry, and the third pair of equations follows from the second pair by conjugation by  $J$ .

For continuous  $f(x)$  with compact support in  $(0, \infty)$ , we have

$$\begin{aligned} A_\alpha B_\beta f(x) &= x^{-\alpha} \int_0^x y^{\alpha-1} y^{-\beta} \int_y^\infty z^{\beta-1} f(z) dz dy \\ &= x^{-\alpha} \left[ \frac{y^{\alpha-\beta}}{\alpha-\beta} \int_y^\infty z^{\beta-1} f(z) dz \right]_{y=0}^{y=x} \\ &\quad + x^{-\alpha} \int_0^x \frac{y^{\alpha-\beta}}{\alpha-\beta} y^{\beta-1} f(y) dy, \\ A_\alpha A_\beta f(x) &= x^{-\alpha} \int_0^x y^{\alpha-1} y^{-\beta} \int_0^y z^{\beta-1} f(z) dz dy \\ &= x^{-\alpha} \left[ \frac{y^{\alpha-\beta}}{\alpha-\beta} \int_0^y z^{\beta-1} f(z) dz \right]_{y=0}^{y=x} \\ &\quad - x^{-\alpha} \int_0^x \frac{y^{\alpha-\beta}}{\alpha-\beta} y^{\beta-1} f(y) dy \end{aligned}$$

from which the first two equations on the left follows. The general cases then follow by continuity of bounded operators.

**LEMMA 1.3.** *The operator  $\text{Id} - (2\alpha - 1)A_\alpha$  is unitary on  $L^2(0, \infty)$  for each  $\alpha > \frac{1}{2}$ .*

*Proof.* We show first

$$(1.1) \quad A_\alpha^* = B_{1-\alpha} \quad \text{for } \alpha > 1/2.$$

It suffices by continuity to show that

$$(A_\alpha f, g) = (f, B_{1-\alpha} g)$$

for continuous functions  $f, g$  with compact support. The left is

$$\begin{aligned} \int_0^\infty x^{-\alpha} \int_0^x y^{\alpha-1} f(y) dy \bar{g}(x) dx \\ = - \int_0^\infty \int_0^x y^{\alpha-1} f(y) d \int_x^\infty z^{-\alpha} \bar{g}(z) dz. \end{aligned}$$

On integrating by parts, the integrated terms drop out, leaving

$$\begin{aligned} \int_0^\infty \int_x^\infty z^{-\alpha} \bar{g}(z) dz d \int_0^x y^{\alpha-1} f(y) dy \\ = \int_0^\infty f(x) x^{\alpha-1} \int_x^\infty z^{-\alpha} \bar{g}(z) dz dx, \end{aligned}$$

which is the right side.

Now for each real  $\alpha > 1/2$ , we have by (1.1)

$$(\text{Id} - (2\alpha - 1)A_\alpha)^* = \text{Id} - (2\alpha - 1)B_{1-\alpha}$$

and by Lemma 1.2

$$\begin{aligned} & (\text{Id} - (2\alpha - 1)B_{1-\alpha})(\text{Id} - (2\alpha - 1)A_\alpha) \\ &= \text{Id} - (2\alpha - 1)B_{1-\alpha} - (2\alpha - 1)A_\alpha + (2\alpha - 1)^2 \frac{A_\alpha + B_{1-\alpha}}{2\alpha - 1} \\ &= \text{Id}, \end{aligned}$$

which proves Lemma 1.3.

Let  $h$  be a bounded measurable function on  $(0, \infty)$ . For each function  $f \in L^1(0, \infty)$ , we define

$$(Hf)(x) = \int_0^\infty f(y)h(xy) dy.$$

Provided in addition  $H$  is a bounded operator on  $L^1(0, \infty) \cap L^2(0, \infty)$ , i.e.

$$(1.2) \quad \|Hf\|_2 \ll \|f\|_2 \quad \text{for } f \in L^1(0, \infty) \cap L^2(0, \infty)$$

we may extend the operator  $H$  by continuity to a bounded operator on all of  $L^2(0, \infty)$ . In particular, in this case

$$(Hf)(x) = \lim_{Y \rightarrow \infty} \int_0^Y f(y)h(xy) dy \quad \text{in } L^2(0, \infty)$$

for each  $f \in L^2(0, \infty)$ .

**LEMMA 1.4.** *Let  $H$  be the bounded operator on  $L^2(0, \infty)$  defined as above by a bounded measurable kernel  $h$  satisfying (1.2). Then for each  $\alpha > 1/2$ , we have*

$$(1.3) \quad A_\alpha H = H B_{1-\alpha}.$$

Moreover, the operator (1.3) is defined as above by the kernel  $A_\alpha h$ .

*Proof.* We observe first that  $A_\alpha h$  is a bounded measurable function. Next, for  $f \in L^1 \cap L^2$ , we have

$$\begin{aligned} A_\alpha H f(x) &= x^{-\alpha} \int_0^x y^{\alpha-1} \int_0^\infty h(yz)f(z) dz dy \\ &= \int_0^\infty \left\{ x^{-\alpha} \int_0^x y^{\alpha-1} h(yz) dy \right\} f(z) dz \end{aligned}$$

and

$$\begin{aligned}
 HB_{1-\alpha}f(x) &= \int_0^\infty h(xy)y^{\alpha-1} \int_y^\infty z^{-\alpha} f(z) dz dy \\
 &= \int_0^\infty \left\{ z^{-\alpha} \int_0^z y^{\alpha-1} h(xy) dy \right\} f(z) dz,
 \end{aligned}$$

with both interchanges justified by Fubini's theorem. Each of the factors  $\{\dots\}$  reduces to  $A_\alpha h(xz) = A_\alpha g(x)$  where  $g(x) = h(xz)$ .

Denoting by  $H_\alpha$  the operator defined by  $A_\alpha h$  on  $L^1 \cap L^2$ , we thus have

$$(1.4) \quad H_\alpha f = A_\alpha Hf = HB_{1-\alpha}f$$

for  $f \in L^1 \cap L^2$ . For such  $f$ ,

$$\|H_\alpha f\|_2 = \|A_\alpha Hf\|_2 \leq \frac{1}{\alpha - 1/2} \|Hf\|_2 \ll \|f\|_2,$$

with an implicit constant depending on  $\alpha$ . It follows that  $H_\alpha$  is a bounded operator on  $L^1 \cap L^2$  and thus  $H_\alpha$  extends by continuity to a bounded operator on  $L^2$ . By continuity, the equation (1.4) now holds for  $f \in L^2$ , giving the assertions of Lemma 1.4.

**2. An explicit formula.** It is known that  $(s - 1)\zeta(s)$  is an entire function and

$$(2.0) \quad \frac{\zeta'}{\zeta}(s) = - \sum_{n=1}^\infty \frac{\Lambda(n)}{n^s} \quad (\text{Re } s > 1)$$

where  $\Lambda(n)$  is the von Mangoldt function.

Put

$$P(u) = -\frac{1}{2} \sum'_{n \leq e^u} \Lambda(n)n^{-1/2}, \quad Q(u) = -2 \sinh \frac{1}{2}u,$$

the prime on the summation here and in the following means if  $e^u$  is an integer, then the last term is weighted with  $1/2$ . A version of the prime number theorem states that  $\sum_{n \leq x} \Lambda(n) = x + O(x^{1/2} \log^2 x)$  see Davenport [1].

Define

$$(2.1) \quad R(u) = P(u) - Q(u).$$

We have that  $R(u) = O(u^2)$  which implies the prime number theorem with remainder term, and for  $\alpha \geq 0$

$$(2.2) \quad R_\alpha(u) = -\frac{1}{2} \frac{1}{\Gamma(\alpha+1)} \sum'_{n \leq e^u} \Lambda(n) n^{-1/2} (u - \log n)^\alpha \\ + \sinh_\alpha \left( \frac{1}{2} u \right) / \left( \frac{1}{2} \right)^{\alpha+1};$$

note that  $I_\alpha(f)|_{cy} = c^\alpha I_\alpha(g)|_y$  where  $g(y) = f(cy)$ .

Now define for  $t > 0$  not an ordinate of a zero of  $\zeta(s)$

$$N(t) = \# \left\{ \frac{1}{2} + iy \mid \zeta \left( \frac{1}{2} + iy \right) = 0, \quad 0 < y \leq t \right\},$$

and  $N(t) = \frac{1}{2}(N(t^+) + N(t^-))$  for  $t$  the imaginary part of a zero of  $\zeta(s)$ . By the argument principle, we see that for  $T > 0$  not an ordinate of a zero of  $\zeta(s)$

$$N(T) = \frac{1}{2\pi i} \int_{\Gamma_T} \frac{\zeta'}{\zeta}(s) ds$$

where  $\Gamma_T$  is the line running from  $\infty + iT$  to  $\frac{1}{2} + iT$  to  $\frac{1}{2} - iT$  to  $\infty - iT$ , and a Cauchy principal value is taken at each zero of  $\zeta(s)$  on  $\Gamma_T$ .

In view of  $\zeta(\bar{s}) = \overline{\zeta(s)}$ , we have

$$(2.3) \quad N(T) = M(T) + S(T)$$

where

$$(2.4) \quad M(T) = \frac{-1}{\pi} \int_0^T \operatorname{Re} \frac{\zeta'}{\zeta} \left( \frac{1}{2} + it \right) dt, \\ S(T) = \frac{1}{\pi} \int_\infty^{1/2} \operatorname{Im} \frac{\zeta'}{\zeta} (\sigma + iT) d\sigma = \frac{1}{\pi} \arg \zeta \left( \frac{1}{2} + iT \right).$$

The argument is defined by continuous horizontal movement from  $\infty + iT$  to  $\frac{1}{2} + iT$  starting with the value zero. Meanwhile, comparing (2.3) with the zero counting formula shown in Davenport [1], we see that

$$(2.5) \quad M(t) = \frac{1}{\pi} \left[ \arg \left( -\frac{1}{2} + it \right) + \arg(\pi^{-(1/2)(1/2+it)}) \right. \\ \left. + \arg \Gamma \left( \frac{5}{4} + i \frac{t}{2} \right) \right] \\ = \frac{t}{2\pi} \left( \log \frac{t}{2\pi} - 1 \right) + O(1) \quad \text{for large } t,$$



and

$$(2.6) \quad M'(t) \sim \frac{1}{2\pi} \log t.$$

Our first object is to construct the following “explicit formula.”

PROPOSITION 1. For  $\alpha \geq 0$ , we have

$$(\Delta) \quad R_\alpha(u) = \int_0^\infty \frac{\sin_\alpha(tu)}{t^{\alpha+1}} dS(t).$$

Before proceeding with the proof of Proposition 1, we need several lemmas.

LEMMA 2.1. If  $\alpha \geq 0$ ,  $u > 0$ , and  $f(v) = e^{sv}$ , then

$$(I_\alpha f)(u) = \frac{1}{\Gamma(\alpha)} \int_0^u (u-v)^{\alpha-1} e^{sv} dv = \frac{e^{su}}{s^\alpha} + L(\alpha, s, u)$$

for  $s$  on the slit plane cut along the line from the origin to  $-i\infty$ ; and  $s^\alpha$  is defined by analytic continuation starting with  $1^\alpha = 1$ . Moreover,  $(I_\alpha f)(u)$  is a holomorphic function on the entire plane of  $s$  with  $L(\alpha, s, u) \ll_\alpha u^{\alpha-1}/|s| + (|\sigma u|^{\alpha-1} + 1)/|s|^\alpha$ ,  $\sigma = \text{Re } s$ , and we define  $L(0, s, u) \equiv 0$ .

Proof. We start by considering, for  $\alpha > 0$ ,

$$\begin{aligned} e^{-su} \frac{1}{\Gamma(\alpha)} \int_0^u (u-v)^{\alpha-1} e^{sv} dv &= \frac{1}{\Gamma(\alpha)} \int_0^u (u-v)^{\alpha-1} e^{s(v-u)} dv \\ &= \frac{1}{s^\alpha} \frac{1}{\Gamma(\alpha)} \int_0^{su} v^{\alpha-1} e^{-v} dv \\ &= \frac{1}{s^\alpha} \frac{1}{\Gamma(\alpha)} \left\{ \Gamma(\alpha) - \int_{su}^{\sigma u} v^{\alpha-1} e^{-v} dv - \int_{\sigma u}^\infty v^{\alpha-1} e^{-v} dv \right\} \\ &= \frac{1}{s^\alpha} + O_\alpha \left( \frac{|su|^{\alpha-1}}{|s|^\alpha} e^{-\sigma u} + \frac{|\sigma u|^{\alpha-1} + 1}{|s|^\alpha} e^{-\sigma u} \right); \end{aligned}$$

the estimate of this remainder term will be given at the end of this section. This proves Lemma 2.1 by taking, for  $\alpha > 0$ ,

$$L(\alpha, s, u) = -\frac{1}{s^\alpha} \frac{e^{su}}{\Gamma(\alpha)} \left\{ \int_{su}^{\sigma u} v^{\alpha-1} e^{-v} dv + \int_{\sigma u}^\infty v^{\alpha-1} e^{-v} dv \right\}.$$

REMARK. We see by Lemma 2.1 that

$$(2.7) \quad \begin{aligned} \frac{1}{\Gamma(\alpha)} \int_0^u (u-v)^{\alpha-1} \sinh sv dv &= (\sinh_\alpha s)(u) \\ &= \frac{1}{2} \left[ \frac{e^{su}}{s^\alpha} + \frac{e^{-su}}{(-s)^\alpha} \right] + \tilde{L}(\alpha, s, u) \end{aligned}$$

where  $\tilde{L}(\alpha, s, u)$  is bounded by

$$O\left(\frac{u^{\alpha-1}}{|s|} + \frac{|\sigma u|^{\alpha-1} + 1}{|s|^\alpha}\right),$$

and  $\tilde{L}(0, s, u) \equiv 0$ .

LEMMA 2.2. Consider, for  $\alpha \geq 0$  and  $u > 0$ ,  $(\sinh_\alpha s)(u)$  as defined by (2.7) on the slit plane in Lemma 2.1. Then for  $c > 0$ ,  $\alpha \geq 0$ ,  $\eta \geq 0$

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{\pi i} \int_{c-iT}^{c+iT} \frac{(\sinh_\alpha s)(u)}{s} e^{-s\eta} ds \\ &= \begin{cases} \frac{1}{\Gamma(\alpha+1)}(u-\eta)^\alpha & (u > \eta), \\ 0 & (u < \eta) \text{ or } (u = \eta, \alpha > 0), \\ \frac{1}{2} & (u = \eta, \alpha = 0). \end{cases} \end{aligned}$$

*Proof.* The result follows by (1.5.3) of Titchmarsh [7] and (2.7).

*Proof of Proposition 1.* Consider, for  $c > 1$  and  $T > 0$  not an ordinate of a zero of  $\zeta(s)$ ,

$$J(T) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{(\sinh_\alpha(s - \frac{1}{2}))(u)}{s - \frac{1}{2}} \frac{\zeta'}{\zeta}(s) ds;$$

note that  $(\sinh_\alpha s)(u)/s$  is an even function of  $s$ .

Since  $\frac{\zeta'}{\zeta}(s) \ll (\log |t|)^2$  for  $s = \sigma + it (-\frac{1}{2} \leq \sigma)$  and a suitable sequence of  $t$  with infinity the limit of  $|t|$ , we have that by computation of residues and for large  $T$

$$\begin{aligned} J(T) &= -\frac{\sinh_\alpha\left(\frac{1}{2}u\right)}{\left(\frac{1}{2}\right)^{\alpha+1}} - \int_0^T \frac{(\sin_\alpha t)(u)}{t} dM(t) \\ &+ \sum_{\substack{0 < \gamma < T \\ \zeta(1/2+i\gamma)=0}} \frac{\sin_\alpha(\gamma u)}{\gamma^{\alpha+1}} + o(1) \\ &= -\frac{\sinh_\alpha\left(\frac{1}{2}u\right)}{\left(\frac{1}{2}\right)^{\alpha+1}} + \int_0^T \frac{\sin_\alpha(tu)}{t^{\alpha+1}} dS(t) + o(1), \end{aligned}$$

since  $(\sin_\alpha t)(u)/t = \sin_\alpha(tu)/t^{\alpha+1}$ . On the other hand, by using

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty} \Lambda(n) n^{-1/2} e^{-(s-1/2)\log n},$$

we have

$$J(T) = - \sum_{n=1}^{\infty} \Lambda(n) n^{-1/2} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\left(\sinh_{\alpha} \left(s - \frac{1}{2}\right)\right) (u)}{s - \frac{1}{2}} e^{-(s-1/2) \log n} ds.$$

By virtue of Lemma 2.2, (2.2), and making  $T \rightarrow \infty$ , we get

$$R_{\alpha}(u) = \int_0^{\infty} \frac{\sin_{\alpha}(tu)}{t^{\alpha+1}} dS(t).$$

This completes the proof of Proposition 1.

We now give the estimate of the remainder term in the proof of Lemma 2.1. It is based on the following

**PROPOSITION 2.** *If  $s = \sigma + it$  is on the slit plane cut along the line from the origin to  $-i\infty$ , then, for  $\alpha > 0$ ,*

$$\int_{\sigma+it}^{\infty+it} v^{\alpha-1} e^{-v} dv = O_{\alpha}(e^{-\sigma}(|s|^{\alpha-1} + 1))$$

with  $t \geq 0$ , or  $t \leq 0$  and  $\sigma > 0$ , and

$$\int_{\sigma+it}^{-\infty+it} v^{\alpha-1} e^v dv = O_{\alpha}(e^{-\sigma}(|s|^{\alpha-1} + 1))$$

with  $t \geq 0$ , or  $t \leq 0$  and  $\sigma < 0$ .

*Proof.* We only prove the first assertion, and the second assertion will follow by a similar argument.

If  $-1 \leq \sigma \leq 1$ , then, for  $|t| < 1$ ,

$$\int_{\sigma+it}^{\infty+it} v^{\alpha-1} e^{-v} dv = O(1) = O(e^{-\sigma}),$$

and for  $|t| > 1$ , by integrating by parts  $[\alpha]$  times,

$$\begin{aligned} \int_{\sigma+it}^{\infty+it} v^{\alpha-1} e^{-v} dv &= -e^{-v} v^{\alpha-1} \Big|_{v=\sigma+it}^{v=\infty+it} + (\alpha-1) \int_{\sigma+it}^{\infty+it} v^{\alpha-2} e^{-v} dv \\ &= O(e^{-\sigma} |s|^{\alpha-1}) + O\left(\int_{\sigma+it}^{\infty+it} |v|^{\alpha-[\alpha]-2} e^{-v} dv\right) \\ &= O(e^{-\sigma} |s|^{\alpha-1}) + O\left(e^{-\sigma} \int_{\sigma+it}^{\infty+it} |v|^{\alpha-[\alpha]-2} dv\right) \\ &= O(e^{-\sigma} |s|^{\alpha-1}). \end{aligned}$$

So, for  $-1 \leq \sigma \leq 1$ , we have

$$(2.8) \quad \int_{\sigma+it}^{\infty+it} v^{\alpha-1} e^{-v} dv = O(e^{-\sigma}(|s|^{\alpha-1} + 1)).$$

If  $1 < \sigma$ , then  $|s| > 1$  and

$$\begin{aligned} \int_{\sigma+it}^{\infty+it} v^{\alpha-1} e^{-v} dv &= -e^{-v} v^{\alpha-1} \Big|_{v=\sigma+it}^{v=\infty+it} + (\alpha-1) \int_{\sigma+it}^{\infty+it} v^{\alpha-2} e^{-v} dv \\ &= O(e^{-\sigma} |s|^{\alpha-1}) + O\left(\int_{\sigma+it}^{\infty+it} |v|^{\alpha-[\alpha]-2} e^{-v} dv\right) \\ &= O(e^{-\sigma} |s|^{\alpha-1}) + O\left(e^{-\sigma} \int_{\sigma+it}^{\infty+it} |v|^{\alpha-[\alpha]-2} dv\right) \\ &= O(e^{-\sigma} |s|^{\alpha-1}). \end{aligned}$$

Finally if  $\sigma < -1$ , then  $|s| > 1$  and

$$\begin{aligned} \int_{\sigma+it}^{\infty+it} v^{\alpha-1} e^{-v} dv &= \int_{\sigma+it}^{-1+it} v^{\alpha-1} e^{-v} dv + \int_{-1+it}^{\infty+it} v^{\alpha-1} e^{-v} dv \\ &= -e^{-v} v^{\alpha-1} \Big|_{v=\sigma+it}^{v=-1+it} + (\alpha-1) \int_{\sigma+it}^{-1+it} v^{\alpha-2} e^{-v} dv \\ &\quad + O(e^{-\sigma}(|1+it|^{\alpha-1} + 1)), \quad \text{by (2.8)} \\ &= O(e^{-\sigma}(|s|^{\alpha-1} + |1+it|^{\alpha-1})) \\ &\quad + O\left(e^{-\sigma} \int_{\sigma+it}^{-1+it} |v|^{\alpha-2} dv\right) \\ &\quad + O(e^{-\sigma}(|1+it|^{\alpha-1} + 1)) \\ &= O(e^{-\sigma}(|s|^{\alpha-1} + |1+it|^{\alpha-1} + 1)) \\ &= O(e^{-\sigma}(|s|^{\alpha-1} + 1)). \end{aligned}$$

This completes the proof of the first assertion of Proposition 2.

Now we obtain from Proposition 2

$$\int_{\sigma u}^{\infty} v^{\alpha-1} e^{-v} dv = O(e^{-\sigma u}(|\sigma u|^{\alpha-1} + 1))$$

needed in the proof of Lemma 2.1.

As for the following quantity

$$\int_{su}^{\sigma u} v^{\alpha-1} e^{-v} dv,$$

we apply the Cauchy integral theorem and get for  $\sigma > 0$ , or  $\sigma < 0$  and  $t > 0$ ,

$$\begin{aligned} \int_{su}^{\sigma u} v^{\alpha-1} e^{-v} dv &= \int_{su}^{\infty+itu} v^{\alpha-1} e^{-v} dv + \int_{\infty}^{\sigma u} v^{\alpha-1} e^{-v} dv \\ &= O(e^{-\sigma u}(|su|^{\alpha-1} + |\sigma u|^{\alpha-1} + 1)), \end{aligned}$$

by Proposition 2; and if  $t < 0$  and  $\sigma < 0$ , then by the Cauchy integral theorem

$$\begin{aligned} \int_{su}^{\sigma u} v^{\alpha-1} e^{-v} dv &= (-1)^{\alpha-1} \int_{-su}^{-\sigma u} v^{\alpha-1} e^v dv \\ &= (-1)^{\alpha-1} \left\{ \int_{-\infty}^{-\sigma u} v^{\alpha-1} e^v dv + \int_{-su}^{-\infty+itu} v^{\alpha-1} e^v dv \right\} \\ &= O(e^{-\sigma u}(|su|^{\alpha-1} + |\sigma u|^{\alpha-1} + 1)); \end{aligned}$$

by the second assertion of Proposition 2.

Hence we have given the estimates in the proof of Lemma 2.1.

**3. Theorems.** Consider equation  $(\Delta)$  in Proposition 1. On taking integration by parts  $m + 1$  times, we get formally

$$(3.1) \quad \sqrt{\frac{2}{\pi}} \frac{R_\alpha(u)}{u^{m+1}} = (-1)^{m+1} \int_0^\infty \frac{S_m^*(t)}{t^{\alpha+1}} K(\alpha, m, tu) dt$$

where

$$(3.2) \quad K(\alpha, m, \theta) = \sqrt{\frac{2}{\pi}} \theta^{\alpha+1} \left( \frac{d}{d\theta} \right)^{m+1} \frac{\sin_\alpha(\theta)}{\theta^{\alpha+1}} \quad (\theta > 0).$$

Since for  $\alpha \geq 0$ , by considering the Taylor series expansion of  $\sin \theta$ ,

$$(3.3) \quad \sin_\alpha \theta = \sum_{n=0}^\infty \frac{(-1)^n}{\Gamma(2n + 2 + \alpha)} \theta^{2n+1+\alpha} \quad (\theta > 0),$$

and the series on the right-hand side is defined for any real  $\alpha \in \mathbb{R}$ , we define (3.2) for any real  $\alpha \in \mathbb{R}$ .

In this section, we give suitable conditions on real  $\alpha$  and integer  $m$  for which  $K(\alpha, m, \theta)$  defines a bounded and invertible transform on  $L^2(0, \infty)$ . In view of  $S(t) = O(\log t / \log \log t)$ , (0.5), and (0.6); we see that

$$(3.4) \quad \frac{S_m^*(t)}{t^{\alpha+1}} \in L^2(0, \infty)$$

for  $\frac{1}{2} < \alpha - m + 1 < \frac{3}{2}$ ,  $m \geq 0$ ,  $\alpha \geq 0$ .

We show first

**THEOREM 1.** Equation (3.1) holds for  $\alpha > m - 1$ ,  $\alpha \geq 0$ ,  $m = 0, 1, 2, 3, \dots$

The following lemma is helpful in the proof of Theorem 1.

**LEMMA 3.1.** For a function  $f(x)$  on  $[0, \infty)$ , put

$$M(\alpha, x) = \sup_{0 \leq t \leq x} |f_\alpha(t)|.$$

Then the following estimate holds uniformly in  $\alpha$ :

$$|f_\alpha(t)| \ll M^{1-\alpha}(0, t) \cdot M^\alpha(1, t) \quad (0 \leq \alpha \leq 1).$$

*Proof.* Note first that if either  $g(x)$  or  $h(x)$  is monotonic on  $[a, b]$ , then

$$\int_a^b g(x) dh(x) \ll \sup_{a \leq x \leq b} |g(x)| \cdot \sup_{a \leq x \leq b} |h(x)|.$$

We may suppose  $f(x)$  is non-constant.

Now for  $0 < \lambda < x$

$$\begin{aligned} f_\alpha(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} f(y) dy \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{x-\lambda} (x-y)^{\alpha-1} f(y) dy \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{x-\lambda}^x (x-y)^{\alpha-1} f(y) dy \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{x-\lambda} (x-y)^{\alpha-1} df_1(y) \\ &\quad - \frac{1}{\Gamma(\alpha+1)} \int_{x-\lambda}^x f(y) d(x-y)^\alpha \\ &\ll \alpha \lambda^{\alpha-1} M(1, x) + \lambda^\alpha M(0, x) \\ &\ll M^{1-\alpha}(0, x) M^\alpha(1, x), \end{aligned}$$

by taking  $\lambda = \alpha M(1, x)/M(0, x) < x$ .

This proves Lemma 3.1.

*Proof of Theorem 1.* It suffices to show that the integral constant

$$(3.5) \quad S_m^*(t) \left( \frac{d}{dt} \right)^m \frac{\sin_\alpha(tu)}{t^{\alpha+1}} \Big|_{t=0}^{t=\infty} = 0$$

for  $\alpha > m - 1$ ,  $\alpha \geq 0$ ,  $m \geq 0$ . Lemma 3.1 and the power series (3.3) give

$$(3.6) \quad \sin_\beta(\theta) \ll \begin{cases} \max(\theta^{\beta-1}, 1), & \theta \rightarrow +\infty, \\ \theta^{\beta+1}, & \theta \rightarrow 0^+ \end{cases}$$

for all  $\beta \in \mathbb{R}$ . Thus, the Leibniz formula yields

$$(3.7) \quad \left(\frac{d}{dt}\right)^m \frac{\sin_\alpha(tu)}{t^{\alpha+1}} \ll \sum_{j=0}^m \binom{m}{j} |\sin_{\alpha-j}(tu)| \cdot t^{-\alpha-1-m+j} \\ \ll \max\{t^{-m-2}, t^{-\alpha-1}\} \quad (t \rightarrow +\infty),$$

and the power series (3.3) gives

$$(3.8) \quad \left(\frac{d}{dt}\right)^m \frac{\sin_\alpha(tu)}{t^{\alpha+1}} \ll 1 \quad (t \rightarrow 0^+).$$

Now (3.5) follows immediately from (0.5), (0.6), and (3.7), (3.8). This proves Lemma 3.2 Theorem 1.

We next show the following.

**THEOREM 2.** *The kernel (3.2) defines a bounded and invertible transform on  $L^2(0, \infty)$  for  $m - \frac{1}{2} < \alpha < m + \frac{3}{2}$ ,  $m = -1, 0, 1, 2, \dots$ . In particular, for  $\alpha = m = -1, 0, 1, 2, \dots$ , it defines an involution on  $L^2(0, \infty)$ .*

The proof of Theorem 2 is based on the following lemma.

**LEMMA 3.2.** *We have*

$$K(\alpha + 1, m + 1, \theta) = (\text{Id} - (m + \alpha + 3)A_{m+2})K(\alpha, m, \theta)$$

for  $m + \alpha + 3 > 0$ ,  $-\infty < \alpha < +\infty$ ,  $m = -1, 0, 1, 2, 3, \dots$ .

*Proof.* Recall (3.2) and (3.3). We see that

$$K(\alpha, m, \theta) = \sqrt{\frac{2}{\pi}} \theta^{\alpha+1} \left(\frac{d}{d\theta}\right)^{m+1} \sum_{l=0}^{\infty} \frac{(-1)^l}{\Gamma(2l + \alpha + 2)} \theta^{2l}.$$

Thus

$$\begin{aligned}
 & (m + \alpha + 3)A_{m+2}K(\alpha, m, \theta) \\
 &= \sqrt{\frac{2}{\pi}}(m + \alpha + 3)\theta^{-m-2} \int_0^\theta \eta^{m+1} \eta^{\alpha+1} \left(\frac{d}{d\eta}\right)^{m+1} \\
 & \quad \cdot \sum_{l=0}^{\infty} \frac{(-1)^l}{\Gamma(2l + \alpha + 2)} \eta^{2l} d\eta \\
 &= \sqrt{\frac{2}{\pi}}\theta^{-m-2} \left\{ \eta^{m+\alpha+3} \left(\frac{d}{d\eta}\right)^{m+1} \sum_{l=0}^{\infty} \frac{(-1)^l}{\Gamma(2l + \alpha + 2)} \eta^{2l} \Big|_{\eta=0}^{\eta=\theta} \right. \\
 & \quad \left. - \int_0^\theta \eta^{m+\alpha+3} \left(\frac{d}{d\eta}\right)^{m+2} \sum_{l=0}^{\infty} \frac{(-1)^l}{\Gamma(2l + \alpha + 2)} \eta^{2l} d\eta \right\} \\
 &= K(\alpha, m, \theta) \\
 & \quad - \sqrt{\frac{2}{\pi}}\theta^{-m-2} \int_0^\theta \eta^{m+\alpha+3} \left(\frac{d}{d\eta}\right)^{m+2} \sum_{l=0}^{\infty} \frac{(-1)^l}{\Gamma(2l + \alpha + 2)} \eta^{2l} d\eta \\
 &= K(\alpha, m, \theta) - K(\alpha + 1, m + 1, \theta).
 \end{aligned}$$

This proves Lemma 3.2.

*Proof of Theorem 2.* Note that  $K(\alpha, -1, \theta) = \sqrt{\frac{2}{\pi}} \sin_\alpha(\theta)$  and  $K(-1, -1, \theta) = \sqrt{\frac{2}{\pi}} \cos \theta$ . Lemma 3 and Theorem 1 of Kueh [5] show that  $K(\alpha, -1, \theta)$  ( $-\frac{3}{2} < \alpha < \frac{1}{2}$ ) defines a bounded and invertible transform on  $L^2(0, \infty)$ . Thus Theorem 2 follows immediately from Lemma 3.2 and Lemmas 1.1, 1.2, 1.3, 1.4. We see also that the kernel  $K^{-1}(\alpha, m, \theta)$  of the inverse of the transform defined by  $K(\alpha, m, \theta)$  satisfies, for the same condition as in Lemma 3.2,

$$(3.9) \quad K^{-1}(\alpha + 1, m + 1, \theta) = (\text{Id} - (m + \alpha + 3)A_{\alpha+2})K^{-1}(\alpha, m, \theta)$$

and

$$K^{-1}(\alpha, -1, \theta) = \sqrt{\frac{2}{\pi}} \sin\left(\theta - \frac{\pi}{2}\right) \quad \left(-\frac{3}{2} < \alpha < \frac{1}{2}\right).$$

By Theorems 1, 2, and (3.4), equation (0.8) holds.

Finally, we prove the following

**THEOREM 3.** Equation (0.7) holds for  $\alpha > m - 1$ ,  $\alpha \geq 0$ ,  $m = 0, 1, 2, \dots$



We need the estimate

$$(3.10) \quad R_1(u) = -\frac{\pi}{2}M'(0)u + O(1)$$

in the proof of Theorem 3. Taking  $\alpha = 1$  in Proposition 1, we obtain

$$R_1(u) = \int_0^\infty \frac{1 - \cos tu}{t^2} dS(t).$$

Note that  $S(t) = N(t) - M(t)$ . So

$$\begin{aligned} R_1(u) &= \int_0^\infty \frac{1 - \cos tu}{t^2} dN(t) - \int_0^\infty \frac{1 - \cos tu}{t^2} M'(t) dt \\ &= -M'(0) \int_0^\infty \frac{1 - \cos tu}{t^2} dt \\ &\quad - \int_0^\infty \frac{1 - \cos tu}{t^2} (M'(t) - M'(0)) dt + O(1) \\ &= -\frac{\pi}{2}M'(0)u + O(1), \end{aligned}$$

since

$$\int_0^\infty \frac{1 - \cos tu}{t^2} dt = \frac{\pi}{2}u,$$

and  $M'(t)$  is an even function making  $M'(t) - M'(0) = O(t^2)$  as  $t \rightarrow 0^+$ , and  $M'(t) \sim \frac{1}{2\pi} \log t$  for large  $t$ , by (2.6).

*Proof of Theorem 3.* By Theorem 2,

$$\frac{S_1^*(t)}{t^2} = \sqrt{\frac{2}{\pi}} \lim_{U \rightarrow \infty} \int_0^U \frac{R_1(u)}{u^2} K(1, 1, tu) du.$$

Now estimate (3.10) makes the above integral converge in the ordinary sense. Thus

$$(3.11) \quad S_1^*(t) = \frac{2}{\pi} \int_0^\infty R_1(u) \left(\frac{d}{du}\right)^2 \frac{\sin_1(tu)}{u^2} du.$$

In addition,

$$\int_0^U R_1(u) \left(\frac{d}{du}\right)^2 \frac{\sin_1(tu)}{u^2} du$$

is bounded uniformly with respect to  $t$  in any compact set as  $U \rightarrow \infty$ . So, after applying fractional integral operator on both sides of (3.11), we get for  $\alpha \geq 0$

$$(3.12) \quad S_{\alpha+1}^*(t) = \frac{2}{\pi} \int_0^\infty R_1(u) \left(\frac{d}{du}\right)^2 \frac{\sin_{\alpha+1}(tu)}{u^{\alpha+2}} du.$$

Now set

$$(3.13) \quad f(t) = \frac{2}{\pi} \int_0^\infty R_1(u) \left( \frac{d}{du} \right)^2 \frac{\sin_\alpha(tu)}{u^{\alpha+1}} du \quad (\alpha > 0).$$

By using Lemma 2 of Kueh [5] and (3.6), we see that similarly for  $\alpha > 0$

$$\int_0^U R_1(u) \left( \frac{d}{du} \right)^2 \frac{\sin_\alpha(tu)}{u^{\alpha+1}} du$$

is bounded uniformly with respect to  $t$  in any compact set as  $U \rightarrow \infty$ . Hence, on integrating both sides of (3.13), we get

$$f_1(t) = \frac{2}{\pi} \int_0^\infty R_1(u) \left( \frac{d}{du} \right)^2 \frac{\sin_{\alpha+1}(tu)}{u^{\alpha+2}} du \quad (\alpha > 0)$$

and, in view of (3.12),

$$(3.14) \quad S_{\alpha+1}^*(t) = f_1(t) \quad (\alpha > 0).$$

Thus, on differentiating both sides of (3.14), we get

$$\begin{aligned} S_\alpha^*(t) &= f(t) = \frac{2}{\pi} \int_0^\infty R_1(u) \left( \frac{d}{du} \right)^2 \frac{\sin_\alpha(tu)}{u^{\alpha+1}} du \\ &= -\frac{2}{\pi} \int_0^\infty R(u) \frac{d}{du} \frac{\sin_\alpha(tu)}{u^{\alpha+1}} du \quad (\alpha > 0). \end{aligned}$$

We now repeat the same argument as in Theorem 1 and get

$$\frac{S_\alpha^*(t)}{t^{m+1}} = (-1)^{m+1} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{R_m(u)}{u^{\alpha+1}} K(\alpha, m, tu) du$$

with  $\alpha > m - 1$ ,  $\alpha > 0$ ,  $m = 0, 1, 2, \dots$ . Also by (0.2), the above equation holds for  $\alpha = m = 0$ .

This completes the proof of Theorem 3.

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