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**ARENS REGULARITY AND DISCRETE GROUPS**

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Let  $G$  be a locally compact group. Let  $A_p(G)$  be the Herz algebra of  $G$  associated with  $1 < p < \infty$ . We show that if  $A_p(G)$  is Arens regular, then  $G$  is discrete. We also exhibit a number of sufficient conditions for such a group to be finite.

**1. Introduction.** Let  $G$  be a locally compact group. For  $1 < p < \infty$ , let  $A_p(G)$  denote the linear subspace of  $C_0(G)$  consisting of all functions of the form  $u(x) = \sum_{i=1}^{\infty} (f_i * \tilde{g}_i)^\vee$ , where  $f_i \in L_p(G)$ ,  $g_i \in L_q(G)$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q < \infty$ ,  $f^\vee(x) = f(x^{-1})$  and  $\tilde{f}(x) = \overline{f(x^{-1})}$ .  $A_p(G)$  is a commutative Banach algebra with respect to pointwise multiplication and the norm

$$\|u\|_{A_p(G)} = \inf \left\{ \sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q \mid u(x) = \sum_{i=1}^{\infty} (f_i * \tilde{g}_i)^\vee \right\}.$$

When  $p = 2$ ,  $A_2(G)$  is the Fourier algebra of  $G$  as introduced by Eymard in [7]. For general  $p$ , the algebras  $A_p(G)$  were introduced and first studied by Herz [13].

In this paper we will study the structure of the second dual  $A_p(G)^{**}$  as a Banach algebra with respect to the two Arens products. In particular, we will show that if  $A_p(G)$  is Arens regular, then  $G$  is discrete. When  $p = 2$ , we show that for a large class of groups, Arens regularity will imply finiteness.

**2. Preliminaries.** Let  $G$  be a locally compact group with a fixed left Haar measure  $\lambda$ . For  $1 \leq p \leq \infty$ , let  $L_p(G)$  be the usual Banach space of equivalence classes of  $p$ -integrable (or essentially bounded) functions on  $G$ . The algebras  $A_p(G)$  for  $1 < p < \infty$  will be as defined in §1. When  $p = 2$  we will write  $A(G)$  for  $A_2(G)$ .

For  $1 < p < \infty$ , let  $PF_p(G)$  and  $PM_p(G)$  denote the closure of  $L_1(G)$ , considered as an algebra of convolution operators on  $L_p(G)$ , with respect to the norm topology and the weak operator topology respectively in  $\mathcal{B}(L_p(G))$ , the bounded operators on  $L_p(G)$ . The space  $PM_p(G)$  can be identified with the dual of  $A_p(G)$  for each  $1 < p < \infty$  [see 19, p. 94].

Let  $B_p(G)$  denote the space of multipliers of  $A_p(G)$ . Then  $B_p(G)$  with the norm

$$\|u\|_{B_p(G)} = \sup\{\|uv\|_{A_p(G)} \mid v \in A_p(G), \|v\|_{A_p(G)} \leq 1\}$$

is a commutative Banach algebra with respect to pointwise multiplication.

Let  $A \subseteq G$  be closed. We will denote by  $I_p(A)$  the closed ideal of  $A_p(G)$  of the form  $\{u \in A_p(G) \mid u(x) = 0 \text{ for every } x \in A\}$ . Given an ideal  $I \subseteq A_p(G)$ , we denote by  $Z(I)$  the set  $\{x \in G \mid u(x) = 0 \text{ for every } u \in I\}$ .

Let  $\mathcal{A}$  be a Banach algebra. Then  $\mathcal{A}^{**}$  can be given two multiplications which extend the multiplication of  $\mathcal{A}$  and for which  $\mathcal{A}^{**}$  becomes a Banach algebra. These products were introduced by Arens in [1]. They are defined as follows:

- (1a)  $\langle u \cdot T, v \rangle = \langle T, vu \rangle$  for every  $u, v \in \mathcal{A}, T \in \mathcal{A}^{**}$ ,
- (1b)  $\langle T \odot \Gamma, u \rangle = \langle \Gamma, u \cdot T \rangle$  for every  $u \in \mathcal{A}, T \in \mathcal{A}^*, \Gamma \in \mathcal{A}^{**}$ ,
- (1c)  $\langle \Gamma_1 \odot \Gamma_2, T \rangle = \langle \Gamma_2, T \odot \Gamma_1 \rangle$  for every  $T \in \mathcal{A}^*, \Gamma_1, \Gamma_2 \in \mathcal{A}^{**}$ ,
- (2a)  $\langle T \square u, v \rangle = \langle T, uv \rangle$  for every  $u, v \in \mathcal{A}, T \in \mathcal{A}^*$ ,
- (2b)  $\langle \Gamma \square T, u \rangle = \langle \Gamma, T \square u \rangle$  for every  $u \in \mathcal{A}, T \in \mathcal{A}^*, \Gamma \in \mathcal{A}^{**}$ ,
- (2c)  $\langle \Gamma_1 \square \Gamma_2, T \rangle = \langle \Gamma_1, \Gamma_2 \square T \rangle$  for every  $T \in \mathcal{A}^*, \Gamma_1, \Gamma_2 \in \mathcal{A}^{**}$ .

In general,  $\Gamma_1 \odot \Gamma_2 = \Gamma_2 \square \Gamma_1$  may fail for some  $\Gamma_1, \Gamma_2 \in \mathcal{A}^{**}$ . If  $\Gamma_1 \odot \Gamma_2 = \Gamma_1 \square \Gamma_2$  for every  $\Gamma_1, \Gamma_2 \in \mathcal{A}^{**}$ , then  $\mathcal{A}$  is said to be Arens regular.

Let  $\mathcal{A}$  be a commutative Banach algebra. Then  $u \cdot T = T \square u$ . Hence  $\mathcal{A}^*$  becomes a commutative Banach  $\mathcal{A}$ -bimodule. Moreover,  $\mathcal{A}$  is Arens regular if and only if  $\mathcal{A}^{**}$  is commutative with respect to either, and hence both, of the Arens products.

We call  $T \in \mathcal{A}^*$  weakly almost periodic if  $\mathcal{O}(T) = \{u \cdot T \mid \|u\|_{\mathcal{A}} \leq 1\}$  is relatively weakly compact.  $T$  is uniformly continuous if  $T$  is in the norm closure of  $\text{span}\{u \cdot T_1 \mid u \in \mathcal{A}, T_1 \in \mathcal{A}^*\}$ . When  $\mathcal{A} = A_p(G)$ , we denote the weakly almost periodic functionals by  $W_p(\widehat{G})$  and the uniformly continuous functionals by  $UCB_p(\widehat{G})$  (see [9]).

A locally compact group  $G$  is amenable if there exists  $m \in L_\infty(G)^*$  such that  $m(1) = \|m\| = 1$  and  $m(L_x f) = m(f)$  where  $L_x f(y) = f(x^{-1}y)$  for every  $x, y \in G$ . The functional  $m$  is called a left invariant mean on  $L_\infty(G)$ . All commutative locally compact groups and all compact groups are amenable.  $F_2$ , the free group on two generators, is not amenable.

A functional  $m \in PM_p(G)^*$  is called a topologically invariant mean on  $PM_p(G)$  if  $\|m\| = 1$  and  $m(uT) = u(e)m(T)$  for every  $u \in A_p(G)$ ,

$T \in PM_p(G)$ . For any locally compact group it is known that  $PM_p(G)$  has a T.I.M. [9, Proposition 2]. We can speak of a topologically invariant mean on any closed  $A_p(G)$ -submodule of  $PM_p(G)$  which contains the functional  $L_e$  ( $L_e(u) = u(e)$ ). It is known that  $W_p(\widehat{G})$  always has a unique invariant mean [9, Proposition 9].

A Banach space  $X$  has the Radon-Nikodym Property of R.N.P. if every closed convex bounded subset  $C \subseteq X$  is dentable. That is, for every  $\varepsilon > 0$  there exists  $x \in C$  such that  $x \notin \overline{\text{co}}\{C \setminus B_\varepsilon(x)\}$  where  $B_\varepsilon(x) = \{y \in X \mid \|x - y\| < \varepsilon\}$ . See [23, §2] for further information on the R.N.P.

**3. Arens regularity.** We begin with the following useful lemma.

**LEMMA 3.1.** *Let  $G$  be a locally compact group for which  $A_p(G)$  is Arens regular. Then*

- (i) *If  $I$  is a closed ideal of  $A_p(G)$ , then  $I$  is Arens regular.*
- (ii) *If  $H$  is a closed subgroup of  $G$ , then  $A_p(H)$  is Arens regular.*
- (iii) *If  $K$  is a compact normal subgroup of  $G$ , then  $A_p(G/K)$  is Arens regular.*

*Proof.* (i) This follows from [4, p. 312, Corollary].

(ii) By appealing to [13] and by the following the arguments of [8, Lemma 3.8], we can show that  $A_p(H)$  is isometrically isomorphic to  $A_p(G)/I_p(H)$ . Hence by [4, p. 312, Corollary],  $A_p(G)$  is Arens regular.

(iii)  $A_p(G/K)$  is isometrically isomorphic to the closed subalgebra of  $A_p(G)$  consisting of functions which are constant on cosets of  $K$  [13, Proposition 6]. The Arens regularity of  $A_p(G/K)$  now follows immediately from [4, p. 312, Corollary]  $\square$

**THEOREM 3.2.** *Let  $G$  be a locally compact group for which  $A_p(G)$  is Arens regular. Then  $G$  is discrete.*

*Proof.* We first assume that  $G$  is separable. If  $A_p(G)$  is Arens regular, then  $W_p(\widehat{G}) = PM_p(G)$ . Hence  $UCB_p(\widehat{G}) \subseteq W_p(\widehat{G})$  and  $G$  is discrete by [9, Theorem 16].

Let  $G$  be an arbitrary locally compact group. Let  $U$  be an open neighborhood of  $\{e\}$  in  $G$  with compact closure. Then  $U$  generates an open  $\sigma$ -compact subgroup  $G_0$  of  $G$ . By Lemma (3.1)(ii),  $A_p(G_0)$  is also Arens regular. Since  $G_0$  is compactly generated either  $G_0$  is

discrete and we are done or there is a compact normal subgroup  $K$  in  $G_0$  such that  $\lambda(K) = 0$  and  $G_0/K$  is separable.

Assume the latter to be true. By Lemma 3.1(ii),  $A_p(G_0/K)$  is Arens regular. But  $G_0/K$  is separable and therefore must be discrete. Thus  $K$  is an open subgroup which contradicts the assumption that  $\lambda(K) = 0$ . Hence  $G_0$  must be discrete. Consequently, so must  $G$  be discrete.  $\square$

The next result generalizes [15, Theorem 3.7]. The proof is similar.

**LEMMA 3.3.** *Let  $G$  be a locally compact group. Then  $A_p(G)$  is an ideal in  $PM_p(G)^*$  if and only if  $G$  is discrete.*

*Proof.* Assume first that  $G$  is discrete. Then  $UCB_p(G) = PF_p(G)$  [9, Proposition 15]. Let  $u \in A_p(G)$  and  $m \in PM_p(G)^*$ . Let  $T \in PM_p(G)$ . Then

$$\begin{aligned} \langle m \odot u, T \rangle &= \langle m, uT \rangle \\ &= \langle v, uT \rangle \quad \text{for some } v \in W_p(G) = PF_p(G)^* \\ &= \langle vu, T \rangle. \end{aligned}$$

Therefore  $m \odot u = vu \in A_p(G)$ . Also,  $\langle u \odot m, T \rangle = \langle u, m \odot T \rangle = \langle m, uT \rangle = \langle m \odot u, T \rangle$ , so  $A_p(G)$  is in the center of  $PM_p(G)^*$ . Therefore  $A_p(G)$  is a closed two-sided ideal in  $PM_p(G)^*$ .

Conversely, assume that  $A_p(G)$  is an ideal in  $PM_p(G)^*$ . Let  $u_0 \in A_p(G)$  with  $u_0(e) = 1 = \|u_0\|_{A_p(G)}$ . Let

$$K = \{m \odot u_0 \mid m \in PM_p(G)^*, m(L_e) = 1 = \|m\|\}.$$

Since  $m \mapsto m \odot u_0$  is weak-\* to weak-\* continuous  $\{m \in PM_p(G)^* \mid m(L_e) = 1 = \|m\|\}$  is weak-\* compact, so is  $K$ . But  $K \subset A_p(G)$ , so  $K$  is weakly compact in  $A_p(G)$ . It is also clearly convex.

For each  $v \in A_p(G)$  with  $v(e) = \|v\|_{A_p(G)} = 1$ , define the operator  $\Gamma_v$  on  $K$  by  $\Gamma_v(w) = vw$ , for every  $w \in K$ . The operators  $\Gamma_v$  are pairwise commuting and  $(K, \text{weak})$  to  $(K, \text{weak})$  continuous. By the Kakutani-Markov fixed point theorem [8, p. 458], there exists some  $v_0 \in K$  such that  $\Gamma_v(v_0) = v_0$  for every  $v \in A_p(G)$  with  $\|v\|_{A_p(G)} = v(e) = 1$ . That is,  $vv_0 = v_0$ . Since  $v_0 \in K$ ,  $v_0(e) = 1$ . Let  $x \in G \sim \{e\}$ . Then there exists  $v_1 \in A_p(G)$  with  $v_1(e) = 1 = \|v_1\|_{A_p(G)}$  while  $v_1(x) = 0$ . Therefore  $v_0(x) = v_1(x)v_0(x) = 0$ . Hence  $v_0 = 1_{\{e\}}$  and  $G$  is discrete.  $\square$

**PROPOSITION 3.4.** *Let  $G$  be a locally compact group for which  $A_p(G)$  is Arens regular. If  $G$  is amenable, then  $PM_p(G)$  has the Radon-Nikodym Property.*

*Proof.* Since  $A_p(G)$  is Arens regular,  $A_p(G)$  is a two-sided ideal in its second dual. Since  $G$  is amenable,  $A_p(G)$  has a bounded approximate identity. It follows from [23, Corollary 3.2], that  $PM_p(G)$  has the Radon-Nikodym Property.  $\square$

**PROPOSITION 3.5.** *Let  $G$  be a locally compact group for which  $A_p(G)$  is Arens regular. Assume that  $G$  is amenable. Then the following are equivalent:*

- (i)  $A_p(G)$  is weakly sequentially complete,
- (ii)  $G$  is finite.

*Proof.* (i)  $\rightarrow$  (ii). If  $A_p(G)$  is weakly sequentially complete and  $G$  is amenable, then by [23, Corollary 3.9]  $A_p(G)$  is reflexive. It follows from [11, Theorem 4], that  $A_p(G)$  is finite dimensional and hence  $G$  is finite.

(ii)  $\rightarrow$  (i). If  $G$  is finite  $A_p(G)$  is finite dimensional.  $\square$

**COROLLARY 3.6.** *Let  $G$  be an amenable locally compact group. Then  $A(G)$  is Arens regular if and only if  $G$  is finite.*

*Proof.* Since  $A_2(G) = A(G)$  is the predual of a von Neumann algebra, it is weakly sequentially complete. If  $A(G)$  is Arens regular, then  $G$  is finite by Proposition 3.5.

Conversely, if  $G$  is finite, then  $A(G)$  is reflexive. Hence  $A(G)^{**} = A(G)$  is commutative. Therefore  $A(G)$  is Arens regular.  $\square$

Corollary 3.6 is due to Lau and Wong [17]. They consider only the case of amenable groups where it is known that  $W_p(\widehat{G}) \subseteq UCB_p(\widehat{G})$  [9, Proposition 14]. For non-amenable groups, it is not known whether the above inclusion holds even for  $p = 2$  and for  $G$  discrete. For  $p = 2$ , the following proposition sheds some light on the non-amenable case.

**PROPOSITION 3.7.** *Let  $G$  be a locally compact group for which  $A(G)$  is Arens regular. Let  $H$  be an amenable subgroup of  $G$ . Then  $H$  is finite. In particular,  $G$  is periodic.*

*Proof.* By Lemma 3.1(ii),  $A(H)$  is Arens regular. Hence by Corollary 3.6,  $H$  is finite.

Let  $x \in G$ . Then  $H = \langle x \rangle$ , the subgroup generated by  $x$  is commutative and hence amenable. Therefore  $H$  is finite and  $G$  is periodic.  $\square$

**COROLLARY 3.8.** *Let  $G$  be a discrete group which contains the free group on 2 generators. Then  $A(G)$  is not Arens regular.*

One of the most famous conjectures in the study of amenable groups was that a discrete group  $G$  would be amenable if and only if  $G$  did not contain a subgroup isomorphic to the free group on 2 generators. Ol'shanskii [18] has proved this conjecture to be false by constructing a non-amenable group  $G$  for which every non-trivial proper subgroup is infinite cyclic. It follows from Proposition 3.7 that  $A(G)$  is not Arens regular for this  $G$ . The natural question which arises is: Are there non-amenable periodic groups without infinite amenable subgroups?

Let  $\mathcal{X}$  be a class of groups such that if  $G \in \mathcal{X}$ , then any homomorphic image of  $G$  also belongs to  $\mathcal{X}$ . A group  $H$  is called a hyper- $\mathcal{X}$ -group if every homomorphic image  $H_1 \neq \{e\}$  of  $H$  has a normal  $\mathcal{X}$ -subgroup  $N \neq \{e\}$ .

**PROPOSITION 3.9.** *Let  $G$  be a discrete group which satisfies any of the following conditions:*

- (i)  $G$  is locally finite,
- (ii)  $G$  is isomorphic to a subgroup of  $GL(n, \mathbb{F})$  for some  $n$  and any field  $\mathbb{F}$ ,
- (iii)  $G$  is a 2-group,
- (iv)  $G$  is hyperfinite,
- (v)  $G$  has an involution  $x$  with  $|C_G(x)| < \infty$ ,
- (vi)  $G$  is hypercentral.

*Then  $A(G)$  is Arens regular if and only if  $G$  is finite.*

*Proof.* (i) If  $G$  is locally finite, then every finitely generated subgroup is finite and hence amenable. Therefore  $G$  is amenable [see 19, p. 121] and the result follows from Corollary 3.6.

(ii) If  $A(G)$  is Arens regular, then  $G$  is periodic. Hence  $G$  is locally finite [15, p. 60].

(iii) If  $G$  is an infinite 2-group, then  $G$  has an infinite abelian subgroup [15, p. 72]. Therefore  $A(G)$  is not Arens regular.

(iv) Assume that  $G$  is hyperfinite and that  $A(G)$  is Arens regular. Then by (iii) every elementary abelian 2-subgroup is finite. Hence  $G$  is finite [15, p. 6].

(v) If  $G$  is infinite, then [15, 2.1 Theorem] implies that  $G$  contains an infinite abelian subgroup which is impossible.

(vi) If  $G$  is hypercentral, then  $G$  is locally nilpotent [15, p. 10] and hence amenable.  $\square$

For  $p \neq 2$ , we are unable to show that  $UCB_p(\widehat{G}) = PM_p(G)$  implies that  $G$  is compact. Though we believe this to be true, this still remains the main stumbling block preventing the extension of Corollary 3.6 for  $p \neq 2$ .

**PROPOSITION 3.10.** *Let  $G$  be an amenable locally compact group. Then  $G$  is discrete and  $A_p(G)$  is Arens regular if and only if  $PM_p(G)^* = B_p(G)$ .*

*Proof.* Assume that  $PM_p(G)^* = B_p(G)$ . Then since  $A_p(G)^{**}$  is commutative,  $A_p(G)$  is Arens regular. Hence  $G$  is discrete, by Theorem 3.2.

Conversely, if  $G$  is discrete, then  $UCB_p(\widehat{G}) = PF_p(G)$ . If  $G$  is Arens regular, then  $PF_p(G) = PM_p(G)$ . Hence  $PM_p(G)^* = B_p(G)$  as a Banach space. Let  $u, v \in B_p(G)$ . Let  $f \in l_1(G)$ . Then  $\langle u \odot v, f \rangle = \langle u, v f \rangle = \int uvf dx = \langle uv, f \rangle$ . Since  $PM_p(G) = PF_p(G)$ ,  $l_1(G)$  is norm dense in  $PM_p(G)$ . Therefore  $u \odot v = uv$  and the Arens multiplication agrees with the pointwise product on  $B_p(G)$ .  $\square$

**PROPOSITION 3.11.** *Let  $G$  be a countable amenable discrete group. If  $A_p(G)$  is Arens regular, then  $PM_p(G)$  is separable.*

*Proof.* By Proposition 3.4,  $PM_p(G)$  has the R.N.P. However, since  $G$  is countable,  $A_p(G)$  is separable. It follows that  $PM_p(G)$  is also separable [see 23, §2].  $\square$

When  $p = 2$ ,  $PM_p(G)$  is a von Neumann algebra. Since a separable von Neumann algebra is well known to be finite dimensional, we have another proof of Corollary 3.6. This follows since an infinite group must always have a countable infinite subgroup.

For amenable groups  $PM_p(G)$  can be identified with the multipliers of  $L_p(G)$ , that is, the algebra of all operators on  $L_p(G)$  which commute with convolution. The assumption of Arens regularity of  $A_p(G)$  implies that the closure of  $L_1(G)$  is the same with respect to both the norm topology and the weak operator topology on  $B(L_p(G))$ . This would seem to suggest that  $L_p(G)$  is finite dimensional and therefore



that  $G$  is finite. We are left to ponder the following two questions:

**PROBLEM 1.** If  $P_p(G)$  is separable for some  $1 < p < \infty$ , is  $G$  necessarily finite?

**PROBLEM 2.** If  $L_1(G)$  is norm dense in  $PM_p(G)$ , is  $G$  necessarily finite?

Let  $G$  be a discrete group. Let  $\{x_1, \dots, x_n\} = A$  be a finite subset of  $G$ . Then  $I_p(G \setminus A)$  is a closed finite dimensional ideal in  $A_p(G)$  and is therefore Arens regular. Moreover,  $I_p(G \setminus A)$  has an identity.

Conversely, if a non-zero closed ideal in  $A_2(G)$  is Arens regular, then this will be shown below to be sufficient to insure that  $G$  is discrete. If, in addition, we assume that  $I$  has a bounded approximate identity, then we will also show that  $I$  is reflexive and therefore infinite codimensional.

**THEOREM 3.12.** *Let  $G$  be a locally compact group. Let  $I$  be a closed non-zero ideal in  $A_p(G)$ . Assume that  $I$  is Arens regular. Then  $PM_p(G)$  has a unique topologically invariant mean.*

*Proof.* Let  $Z(I) = A \subset G$ . Since  $I$  is non-zero,  $A \neq G$ . Therefore  $G \setminus A$  is open. By translating if necessary, we can assume that  $G \setminus A$  is a neighborhood of  $e$ .

Let  $M \in TIM_p(\widehat{G})$ . Let  $T \in I^\perp$ . We can find  $u \in A_p(G)$  such that  $u \in I$  and  $u(e) = 1$ . It follows that  $\langle uT, v \rangle = \langle T, uv \rangle = 0$  for every  $v \in A(G)$ . Hence  $uT = 0$ . But then  $0 = m(uT) = u(e)m(T) = m(T)$ . Therefore  $m \in I^{\perp\perp}$ . Since we can identify  $I^{\perp\perp}$  with  $I^{**}$ , we have  $TIM_p(\widehat{G}) \subseteq I^{**}$ .

Assume that  $m_1, m_2 \in TIM_p(\widehat{G})$ . It is easy to see that  $m_1 \odot m_2 = m_1$ . In fact, given any  $T \in PM_p(G)$  and any  $u \in A_p(G)$ , we have that  $\langle T \odot m_1, u \rangle = \langle m_1, uT \rangle = u(e)\langle m_1, T \rangle$ . Hence  $T \odot m_1 = \langle m_1, T \rangle L_e$ . Finally,  $\langle m_1 \odot m_2, T \rangle = \langle m_2, T \odot m_1 \rangle = \langle m_1, T \rangle \langle m_2, L_e \rangle = \langle m_1, T \rangle$ . However, since  $I$  is Arens regular,  $I^{**}$  is commutative. Therefore  $m_1 = m_1 \odot m_2 = m_2 \odot m_1 = m_2$ .  $\square$

**COROLLARY 3.13.** *Let  $G$  be a second countable locally compact group. Let  $I$  be a closed non-zero ideal in  $A(G)$ . If  $I$  is Arens regular, then  $G$  is discrete.*

*Proof.* By Theorem 3.12,  $PM_2(G)$  has a unique topologically invariant mean. Consequently,  $G$  is discrete [10, Theorem 1].  $\square$

**PROPOSITION 3.14.** *Let  $I$  be a proper closed ideal in  $A(G)$  with a bounded approximate identity. Then  $I$  is Arens regular if and only if  $I$  is reflexive.*

*Proof.* A reflexive ideal is clearly Arens regular.

Conversely, assume that  $I$  is Arens regular. Then  $G$  is a discrete group. As  $I$  has a bounded approximate identity, Cohen's Factorization Theorem [14, Corollary 32.26] implies that  $I = I^2 = \{uv \mid u, v \in I\}$ . Therefore  $I \circ I^{**} = (I \cdot I) \circ (I^{\perp\perp}) \subseteq I \circ (I \circ A(G)^{**}) \subseteq I \cdot A(G) \subseteq I$ . Hence  $I$  is an ideal in  $I^{**}$ . Also, since  $A(G)$  is weakly sequentially complete,  $I$  is weakly sequentially complete. It follows from [23, Corollary 3.7] and [23, Corollary 3.9] that  $I$  is reflexive.  $\square$

With Corollary 3.6 in mind, one might ask whether it is possible to have infinite dimensional ideals  $I$  which are Arens regular or reflexive. In [11, Theorem 5], Granirer shows that while in a non-discrete group  $A_2(G)$  has no non-zero reflexive ideals, (a fact that follows immediately from Corollary 3.13), every infinite discrete group is such that  $A_2(G)$  contains an ideal isomorphic to  $l_2$ .

We close this section with some results on the Arens regularity of some related Banach algebras.

**PROPOSITION 3.15.** *Let  $G$  be a locally compact group. Let  $\mathcal{A} = (B(G) \cap AP(G), \|\cdot\|_{B(G)})$ . Then  $\mathcal{A}$  is Arens regular if and only if  $AP(G)$  is finite dimensional.*

*Proof.*  $\mathcal{A}$  is isometrically isomorphic to  $A(G^{\text{ap}})$ , where  $G^{\text{ap}}$  denotes the almost periodic compactification of  $G$  [7, p. 203]. Since  $G^{\text{ap}}$  is a compact group, it is amenable. Therefore  $\mathcal{A}$  is Arens regular if and only if  $G^{\text{ap}}$  is finite. But  $G^{\text{ap}}$  is finite if and only if  $AP(G)$  is finite dimensional.

The converse is obvious.  $\square$

**COROLLARY 3.16.** *Let  $G$  be a locally compact group. If  $B(G)$  is Arens regular, then  $AP(G)$  is finite dimensional.*

Observe that  $AP(G) \cap B(G)$  is precisely the space of coefficient functions of the representation of  $G$  obtained by lifting the left regular representation of  $G^{\text{ap}}$  to  $G$ . In this case, the representation is such that its coefficient functions form an algebra. For a general representation  $\pi$  of  $G$ , this is so if and only if  $\pi \otimes \pi$  is quasi equivalent to a sub-representation of the representation  $\pi$  [2, Proposition 3.26].

Assume that  $G$  is a compact group. Let  $\pi$  be a continuous unitary representation of  $G$ . Let  $\mathcal{A}_\pi$  denote the closed self-adjoint subalgebra of  $A(G)$  generated by the coefficients functions of  $\pi$ . Then we have the following result:

**PROPOSITION 3.17.** *Let  $G$  be a compact group. Let  $\pi$  be a continuous unitary representation of  $G$ . Then  $A_\pi$  is Arens regular if and only if  $\ker \pi$  is open.*

*Proof.*  $\mathcal{A}_\pi$  is isometrically isomorphic with  $A(G/\ker \pi)$  [23]. Clearly  $G/\ker \pi$  is finite if and only if  $\ker \pi$  is open. The result follows immediately from Corollary 3.6.  $\square$

**COROLLARY 3.18.** *Let  $G$  be compact and connected. Then  $\mathcal{A}_\pi$  is Arens regular if and only if  $\pi$  is the trivial representation.*

We wish to bring the reader's attention to two related results in the literature which unfortunately contain errors. The first result is the equivalence of the unique invariant mean on  $PM_2(G)$  with the discreteness of  $G$ . The proof of this result is usually attributed to Renaud [22]. However the proof of [22, Proposition 8] contains a serious error which may well be impossible to repair. It would therefore appear that at present the equivalence of the discreteness of  $G$  with the existence of a unique invariant mean requires the assumption of second countability.

Secondly, in example 7.2 (b) of the deep paper [21], it is mistakenly stated that for every compact group  $A_p(G)$  is Arens regular.

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# Pacific Journal of Mathematics

Vol. 151, No. 2      December, 1991

<b>Michael G. Eastwood and A. M. Pilato</b> , On the density of twistor elementary states .....	201
<b>Brian E. Forrest</b> , Arens regularity and discrete groups .....	217
<b>Yu Li Fu</b> , On Lipschitz stability for F.D.E .....	229
<b>Douglas Austin Hensley</b> , The largest digit in the continued fraction expansion of a rational number .....	237
<b>Uwe Kaiser</b> , Link homotopy in $\mathbb{R}^3$ and $S^3$ .....	257
<b>Ronald Leslie Lipsman</b> , The Penney-Fujiwara Plancherel formula for abelian symmetric spaces and completely solvable homogeneous spaces .....	265
<b>Florin G. Radulescu</b> , Singularity of the radial subalgebra of $\mathcal{L}(F_N)$ and the Pukánszky invariant .....	297
<b>Albert Jeu-Liang Sheu</b> , The structure of twisted $SU(3)$ groups .....	307
<b>Morwen Thistlethwaite</b> , On the algebraic part of an alternating link .....	317
<b>Thomas (Toma) V. Tonev</b> , Multi-tuple hulls .....	335
<b>Arno van den Essen</b> , A note on Meisters and Olech's proof of the global asymptotic stability Jacobian conjecture .....	351
<b>Hendrik J. van Maldeghem</b> , A characterization of the finite Moufang hexagons by generalized homologies .....	357
<b>Bun Wong</b> , A note on homotopy complex surfaces with negative tangent bundles .....	369
<b>Chung-Tao Yang</b> , Any Blaschke manifold of the homotopy type of $CP^n$ has the right volume .....	379