ARENS REGULARITY AND DISCRETE GROUPS

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Let $G$ be a locally compact group. Let $A_p(G)$ be the Herz algebra of $G$ associated with $1 < p < \infty$. We show that if $A_p(G)$ is Arens regular, then $G$ is discrete. We also exhibit a number of sufficient conditions for such a group to be finite.

1. Introduction. Let $G$ be a locally compact group. For $1 < p < \infty$, let $A_p(G)$ denote the linear subspace of $C_0(G)$ consisting of all functions of the form $u(x) = \sum_{i=1}^{\infty} (f_i \ast \tilde{g}_i)^\vee$, where $f_i \in L_p(G)$, $g_i \in L_q(G)$, $\frac{1}{p} + \frac{1}{q} = 1$, $\sum_{i=1}^{\infty} \|f\|_p \|g\|_q < \infty$, $f^\vee(x) = f(x^{-1})$ and $\tilde{f}(x) = f(x^{-1})$. $A_p(G)$ is a commutative Banach algebra with respect to pointwise multiplication and the norm

$$\|u\|_{A_p(G)} = \inf \left\{ \sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q |u(x) = \sum_{i=1}^{\infty} (f_i \ast \tilde{g}_i)^\vee \right\}.$$  

When $p = 2$, $A_2(G)$ is the Fourier algebra of $G$ as introduced by Eymard in [7]. For general $p$, the algebras $A_p(G)$ were introduced and first studied by Herz [13].

In this paper we will study the structure of the second dual $A_p(G)^{**}$ as a Banach algebra with respect to the two Arens products. In particular, we will show that if $A_p(G)$ is Arens regular, then $G$ is discrete. When $p = 2$, we show that for a large class of groups, Arens regularity will imply finiteness.

2. Preliminaries. Let $G$ be a locally compact group with a fixed left Haar measure $\lambda$. For $1 \leq p \leq \infty$, let $L_p(G)$ be the usual Banach space of equivalence classes of $p$-integrable (or essentially bounded) functions on $G$. The algebras $A_p(G)$ for $1 < p < \infty$ will be as defined in §1. When $p = 2$ we will write $A(G)$ for $A_2(G)$.

For $1 < p < \infty$, let $PF_p(G)$ and $PM_p(G)$ denote the closure of $L_1(G)$, considered as an algebra of convolution operators on $L_p(G)$, with respect to the norm topology and the weak operator topology respectively in $\mathcal{B}(L_p(G))$, the bounded operators on $L_p(G)$. The space $PM_p(G)$ can be identified with the dual of $A_p(G)$ for each $1 < p < \infty$ [see 19, p. 94].
Let $B_p(G)$ denote the space of multipliers of $A_p(G)$. Then $B_p(G)$ with the norm

$$
\|u\|_{B_p(G)} = \sup\{\|uv\|_{A_p(G)} | v \in A_p(G), \|v\|_{A_p(G)} \leq 1\}
$$

is a commutative Banach algebra with respect to pointwise multiplication.

Let $A \subseteq G$ be closed. We will denote by $I_p(A)$ the closed ideal of $A_p(G)$ of the form \{\(u \in A_p(G) | u(x) = 0\) for every \(x \in A\)\}. Given an ideal $I \subseteq A_p(G)$, we denote by $Z(I)$ the set \{\(x \in G | u(x) = 0\) for every \(u \in I\)\}.

Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}^{**}$ can be be given two multiplications which extend the multiplication of $\mathcal{A}$ and for which $\mathcal{A}^{**}$ becomes a Banach algebra. These products were introduced by Arens in [1]. They are defined as follows:

1a) \((u \cdot T, v) = (T, vu)\) for every $u, v \in \mathcal{A}$, $T \in \mathcal{A}^{**}$,
1b) \((T \circ \Gamma, u) = (\Gamma, u \cdot T)\) for every $u \in \mathcal{A}$, $T \in \mathcal{A}^{*}$, $\Gamma \in \mathcal{A}^{**}$,
1c) \((\Gamma_1 \circ \Gamma_2, T) = (\Gamma_2, T \circ \Gamma_1)\) for every $T \in \mathcal{A}^*$, $\Gamma_1, \Gamma_2 \in \mathcal{A}^{**}$,
2a) \((T \square u, v) = (T, uv)\) for every $u, v \in \mathcal{A}$, $T \in \mathcal{A}^{*}$,
2b) \((\Gamma \square T, u) = (\Gamma, T \square u)\) for every $u \in \mathcal{A}$, $T \in \mathcal{A}^*$, $\Gamma \in \mathcal{A}^{**}$,
2c) \((\Gamma_1 \square \Gamma_2, T) = (\Gamma_1, \Gamma_2 \square T)\) for every $T \in \mathcal{A}^{*}$, $\Gamma_1, \Gamma_2 \in \mathcal{A}^{**}$.

In general, $\Gamma_1 \circ \Gamma_2 \neq \Gamma_2 \circ \Gamma_2$ may fail for some $\Gamma_1, \Gamma_2 \in \mathcal{A}^{**}$. If $\Gamma_1 \circ \Gamma_2 = \Gamma_1 \square \Gamma_2$ for every $\Gamma_1, \Gamma_2 \in \mathcal{A}$, then $\mathcal{A}$ is said to be Arens regular.

Let $\mathcal{A}$ be a commutative Banach algebra. Then $u \cdot T = T \square u$. Hence $\mathcal{A}^*$ becomes a commutative Banach $\mathcal{A}$-bimodule. Moreover, $\mathcal{A}$ is Arens regular if and only if $\mathcal{A}^{**}$ is commutative with respect to either, and hence both, of the Arens products.

We call $T \in \mathcal{A}^*$ weakly almost periodic if $\mathcal{E}(T) = \{u \cdot T | \|u\|_\mathcal{A} \leq 1\}$ is relatively weakly compact. $T$ is uniformly continuous if $T$ is in the norm closure of $\text{span}\{u \cdot T | u \in \mathcal{A}, T_1 \in \mathcal{A}^*\}$. When $\mathcal{A} = A_p(G)$, we denote the weakly almost periodic functionals by $W_p(\hat{G})$ and the uniformly continuous functionals by $UCB_p(\hat{G})$ (see [9]).

A locally compact group $G$ is amenable if there exists $m \in L_\infty(G)^*$ such that $m(1) = \|m\| = 1$ and $m(L_x f) = m(f)$ where $L_x f(y) = f(x^{-1}y)$ for every $x, y \in G$. The functional $m$ is called a left invariant mean on $L_\infty(G)$. All commutative locally compact groups and all compact groups are amenable. $F_2$, the free group on two generators, is not amenable.

A functional $m \in PM_p(G)^*$ is called a topologically invariant mean on $PM_p(G)$ if $\|m\| = 1$ and $m(uT) = m(e)m(T)$ for every $u \in A_p(G)$,
T ∈ PMₚ(G). For any locally compact group it is known that PMₚ(G) has a T.I.M. [9, Proposition 2]. We can speak of a topologically invariant mean on any closed Aₚ(G)-submodule of PMₚ(G) which contains the functional \( L_e \) \( (L_e(u) = u(e)) \). It is known that \( W_p(\hat{G}) \) always has a unique invariant mean [9, Proposition 9].

A Banach space \( X \) has the Radon-Nikodym Property of R.N.P. if every closed convex bounded subset \( C \subseteq X \) is dentable. That is, for every \( \varepsilon > 0 \) there exists \( x \in C \) such that \( x \not\in \overline{C \setminus B_\varepsilon(x)} \) where \( B_\varepsilon(x) = \{ y \in X | \|x - y\| < \varepsilon \} \). See [23, §2] for further information on the R.N.P.

3. Arens regularity. We begin with the following useful lemma.

**Lemma 3.1.** Let \( G \) be a locally compact group for which \( A_p(G) \) is Arens regular. Then

(i) If \( I \) is a closed ideal of \( A_p(G) \), then \( I \) is Arens regular.

(ii) If \( H \) is a closed subgroup of \( G \), then \( A_p(H) \) is Arens regular.

(iii) If \( K \) is a compact normal subgroup of \( G \), then \( A_p(G/K) \) is Arens regular.

**Proof.** (i) This follows from [4, p. 312, Corollary].

(ii) By appealing to [13] and by the following the arguments of [8, Lemma 3.8], we can show that \( A_p(H) \) is isometrically isomorphic to \( A_p(G)/I_p(H) \). Hence by [4, p. 312, Corollary], \( A_p(G) \) is Arens regular.

(iii) \( A_p(G/K) \) is isometrically isomorphic to the closed subalgebra of \( A_p(G) \) consisting of functions which are constant on cosets of \( K \) [13, Proposition 6]. The Arens regularity of \( A_p(G/K) \) now follows immediately from [4, p. 312, Corollary].

**Theorem 3.2.** Let \( G \) be a locally compact group for which \( A_p(G) \) is Arens regular. Then \( G \) is discrete.

**Proof.** We first assume that \( G \) is separable. If \( A_p(G) \) is Arens regular, then \( W_p(\hat{G}) = PM_p(G) \). Hence \( UCB_p(\hat{G}) \subseteq W_p(\hat{G}) \) and \( G \) is discrete by [9, Theorem 16].

Let \( G \) be an arbitrary locally compact group. Let \( U \) be an open neighborhood of \( \{ e \} \) in \( G \) with compact closure. Then \( U \) generates an open \( \sigma \)-compact subgroup \( G_0 \) of \( G \). By Lemma (3.1)(ii), \( A_p(G_0) \) is also Arens regular. Since \( G_0 \) is compactly generated either \( G_0 \) is
discrete and we are done or there is a compact normal subgroup \( K \) in \( G_0 \) such that \( \lambda(K) = 0 \) and \( G_0/K \) is separable.

Assume the latter to be true. By Lemma 3.1(ii), \( A_p(G_0/K) \) is Arens regular. But \( G_0/K \) is separable and therefore must be discrete. Thus \( K \) is an open subgroup which contradicts the assumption that \( \lambda(K) = 0 \). Hence \( G_0 \) must be discrete. Consequently, so must \( G \) be discrete. \( \square \)

The next result generalizes [15, Theorem 3.7]. The proof is similar.

**Lemma 3.3.** Let \( G \) be a locally compact group. Then \( A_p(G) \) is an ideal in \( PM_p(G)^* \) if and only if \( G \) is discrete.

**Proof.** Assume first that \( G \) is discrete. Then \( UCB_p(G) = PF_p(G) \) [9, Proposition 15]. Let \( u \in A_p(G) \) and \( m \in PM_p(G)^* \). Let \( T \in PM_p(G) \). Then

\[
\langle m \odot u, T \rangle = \langle m, uT \rangle = \langle v, uT \rangle \quad \text{for some} \quad v \in W_p(G) = PF_p(G)^* \\
= \langle vu, T \rangle.
\]

Therefore \( m \odot u = vu \in A_p(G) \). Also, \( \langle u \odot m, T \rangle = \langle u, m \odot T \rangle = \langle m, uT \rangle = \langle m \odot u, T \rangle \), so \( A_p(G) \) is in the center of \( PM_p(G)^* \). Therefore \( A_p(G) \) is a closed two-sided ideal in \( PM_p(G)^* \).

Conversely, assume that \( A_p(G) \) is an ideal in \( PM_p(G)^* \). Let \( u_0 \in A_p(G) \) with \( u(e) = 1 = \|u\|_{A_p(G)} \). Let

\[
K = \{m \odot u_0 \mid m \in PM_p(G)^*, \ m(L_e) = 1 = \|m\|\}.
\]

Since \( m \mapsto m \odot u_0 \) is weak-* to weak-* continuous \( \{m \in PM_p(G)^* \mid m(L_e) = 1 = \|m\|\} \) is weak-* compact, so is \( K \). But \( K \subset A_p(G) \), so \( K \) is weakly compact in \( A_p(G) \). It is also clearly convex.

For each \( v \in A_p(G) \) with \( v(e) = \|u\|_{A_p(G)} = 1 \), define the operator \( \Gamma_v \) on \( K \) by \( \Gamma_v(w) = vw \), for every \( w \in K \). The operators \( \Gamma_v \) are pairwise commuting and \( (K, \text{weak}) \) to \( (K, \text{weak}) \) continuous. By the Kakutani-Markov fixed point theorem [8, p. 458], there exists some \( v_0 \in K \) such that \( \Gamma_v(v_0) = v_0 \) for every \( v \in A_p(G) \) with \( \|v\|_{A_p(G)} = v(e) = 1 \). That is, \( vv_0 = v_0 \). Since \( v_0 \in K \), \( v_0(e) = 1 \). Let \( x \in G \sim \{e\} \). Then there exists \( v_1 \in A_p(G) \) with \( v_1(e) = 1 = \|v_1\|_{A_p(G)} \) while \( v_1(x) = 0 \). Therefore \( v_0(x) = v_1(x)v_0(x) = 0 \). Hence \( v_0 = 1_{\{e\}} \) and \( G \) is discrete. \( \square \)
PROPOSITION 3.4. Let $G$ be a locally compact group for which $A_p(G)$ is Arens regular. If $G$ is amenable, then $PM_p(G)$ has the Radon-Nikodym Property.

Proof. Since $A_p(G)$ is Arens regular, $A_p(G)$ is a two-sided ideal in its second dual. Since $G$ is amenable, $A_p(G)$ has a bounded approximate identity. It follows from [23, Corollary 3.2], that $PM_p(G)$ has the Radon-Nikodym Property. □

PROPOSITION 3.5. Let $G$ be a locally compact group for which $A_p(G)$ is Arens regular. Assume that $G$ is amenable. Then the following are equivalent:

(i) $A_p(G)$ is weakly sequentially complete,
(ii) $G$ is finite.

Proof. (i) $\implies$ (ii). If $A_p(G)$ is weakly sequentially complete and $G$ is amenable, then by [23, Corollary 3.9] $A_p(G)$ is reflexive. It follows from [11, Theorem 4], that $A_p(G)$ is finite dimensional and hence $G$ is finite.

(ii) $\implies$ (i). If $G$ is finite $A_p(G)$ is finite dimensional. □

COROLLARY 3.6. Let $G$ be an amenable locally compact group. Then $A(G)$ is Arens regular if and only if $G$ is finite.

Proof. Since $A_2(G) = A(G)$ is the predual of a von Neumann algebra, it is weakly sequentially complete. If $A(G)$ is Arens regular, then $G$ is finite by Proposition 3.5.

Conversely, if $G$ is finite, then $A(G)$ is reflexive. Hence $A(G)^{**} = A(G)$ is commutative. Therefore $A(G)$ is Arens regular. □

Corollary 3.6 is due to Lau and Wong [17]. They consider only the case of amenable groups where it is known that $W_p(\hat{G}) \subseteq UCB_p(\hat{G})$ [9, Proposition 14]. For non-amenable groups, it is not known whether the above inclusion holds even for $p = 2$ and for $G$ discrete. For $p = 2$, the following proposition sheds some light on the non-amenable case.

PROPOSITION 3.7. Let $G$ be a locally compact group for which $A(G)$ is Arens regular. Let $H$ be an amenable subgroup of $G$. Then $H$ is finite. In particular, $G$ is periodic.

Proof. By Lemma 3.1(ii), $A(H)$ is Arens regular. Hence by Corollary 3.6, $H$ is finite.
Let $x \in G$. Then $H = \langle x \rangle$, the subgroup generated by $x$ is commutative and hence amenable. Therefore $H$ is finite and $G$ is periodic.

**Corollary 3.8.** Let $G$ be a discrete group which contains the free group on 2 generators. Then $A(G)$ is not Arens regular.

One of the most famous conjectures in the study of amenable groups was that a discrete group $G$ would be amenable if and only if $G$ did not contain a subgroup isomorphic to the free group on 2 generators. Ol'shanskii [18] has proved this conjecture to be false by constructing a non-amenable group $G$ for which every non-trivial proper subgroup is infinite cyclic. It follows from Proposition 3.7 that $A(G)$ is not Arens regular for this $G$. The natural question which arises is: Are there non-amenable periodic groups without infinite amenable subgroups?

Let $\mathcal{P}$ be a class of groups such that if $G \in \mathcal{P}$, then any homomorphic image of $G$ also belongs to $\mathcal{P}$. A group $H$ is called a hyper-$\mathcal{P}$-group if every homomorphic image $H_1 \neq \{e\}$ of $H$ has a normal $\mathcal{P}$-subgroup $N \neq \{e\}$.

**Proposition 3.9.** Let $G$ be a discrete group which satisfies any of the following conditions:

(i) $G$ is locally finite,

(ii) $G$ is isomorphic to a subgroup of $\text{GL}(n, \mathbb{F})$ for some $n$ and any field $\mathbb{F}$,

(iii) $G$ is a 2-group,

(iv) $G$ is hyperfinite,

(v) $G$ has an involution $x$ with $|C_G(x)| < \infty$,

(vi) $G$ is hypercentral.

Then $A(G)$ is Arens regular if and only if $G$ is finite.

**Proof.** (i) If $G$ is locally finite, then every finitely generated subgroup is finite and hence amenable. Therefore $G$ is amenable [see 19, p. 121] and the result follows from Corollary 3.6.

(ii) If $A(G)$ is Arens regular, then $G$ is periodic. Hence $G$ is locally finite [15, p. 60].

(iii) If $G$ is an infinite 2-group, then $G$ has an infinite abelian subgroup [15, p. 72]. Therefore $A(G)$ is not Arens regular.

(iv) Assume that $G$ is hyperfinite and that $A(G)$ is Arens regular. Then by (iii) every elementary abelian 2-subgroup is finite. Hence $G$ is finite [15, p. 6].
(v) If $G$ is infinite, then [15, 2.1 Theorem] implies that $G$ contains an infinite abelian subgroup which is impossible.

(vi) If $G$ is hypercentral, then $G$ is locally nilpotent [15, p. 10] and hence amenable.

For $p \neq 2$, we are unable to show that $UCB_p(\hat{G}) = PM_p(G)$ implies that $G$ is compact. Though we believe this to be true, this still remains the main stumbling block preventing the extension of Corollary 3.6 for $p \neq 2$.

**Proposition 3.10.** Let $G$ be an amenable locally compact group. Then $G$ is discrete and $A_p(G)$ is Arens regular if and only if $PM_p(G)^* = B_p(G)$.

**Proof.** Assume that $PM_p(G)^* = B_p(G)$. Then since $A_p(G)^{**}$ is commutative, $A_p(G)$ is Arens regular. Hence $G$ is discrete, by Theorem 3.2.

Conversely, if $G$ is discrete, then $UCB_p(\hat{G}) = PF_p(G)$. If $G$ is Arens regular, then $PF_p(G) = PM_p(G)$. Hence $PM_p(G)^* = B_p(G)$ as a Banach space. Let $u, v \in B_p(G)$. Let $f \in l_1(G)$. Then $\langle u \circ v, f \rangle = \int uvf \, dx = \langle uv, f \rangle$. Since $PM_p(G) = PF_p(G)$, $l_1(G)$ is norm dense in $PM_p(G)$. Therefore $u \circ v = uv$ and the Arens multiplication agrees with the pointwise product on $B_p(G)$.

**Proposition 3.11.** Let $G$ be a countable amenable discrete group. If $A_p(G)$ is Arens regular, then $PM_p(G)$ is separable.

**Proof.** By Proposition 3.4, $PM_p(G)$ has the R.N.P. However, since $G$ is countable, $A_p(G)$ is separable. It follows that $PM_p(G)$ is also separable [see 23, §2].

When $p = 2$, $PM_p(G)$ is a von Neumann algebra. Since a separable von Neumann algebra is well known to be finite dimensional, we have another proof of Corollary 3.6. This follows since an infinite group must always have a countable infinite subgroup.

For amenable groups $PM_p(G)$ can be identified with the multipliers of $L_p(G)$, that is, the algebra of all operators on $L_p(G)$ which commute with convolution. The assumption of Arens regularity of $A_p(G)$ implies that the closure of $L_1(G)$ is the same with respect to both the norm topology and the weak operator topology on $B(L_p(G))$. This would seem to suggest that $L_p(G)$ is finite dimensional and therefore
that \( G \) is finite. We are left to ponder the following two questions:

**Problem 1.** If \( P_p(G) \) is separable for some \( 1 < p < \infty \), is \( G \) necessarily finite?

**Problem 2.** If \( L_1(G) \) is norm dense in \( PM_p(G) \), is \( G \) necessarily finite?

Let \( G \) be a discrete group. Let \( \{ x_1, \ldots, x_n \} = A \) be a finite subset of \( G \). Then \( I_p(G \setminus A) \) is a closed finite dimensional ideal in \( A_p(G) \) and is therefore Arens regular. Moreover, \( I_p(G \setminus A) \) has an identity.

Conversely, if a non-zero closed ideal in \( A_2(G) \) is Arens regular, then this will be shown below to be sufficient to insure that \( G \) is discrete. If, in addition, we assume that \( I \) has a bounded approximate identity, then we will also show that \( I \) is reflexive and therefore infinite codimensional.

**Theorem 3.12.** Let \( G \) be a locally compact group. Let \( I \) be a closed non-zero ideal in \( A_p(G) \). Assume that \( I \) is Arens regular. Then \( PM_p(G) \) has a unique topologically invariant mean.

**Proof.** Let \( Z(I) = A \subset G \). Since \( I \) is non-zero, \( A \neq G \). Therefore \( G \setminus A \) is open. By translating if necessary, we can assume that \( G \setminus A \) is a neighborhood of \( e \).

Let \( M \in TIM_p(\hat{G}) \). Let \( T \in I^{\perp} \). We can find \( u \in A_p(G) \) such that \( u \in I \) and \( u(e) = 1 \). It follows that \( \langle uT, v \rangle = \langle T, uv \rangle = 0 \) for every \( v \in A(G) \). Hence \( uT = 0 \). But then \( 0 = m(uT) = u(e)m(T) = m(T) \). Therefore \( m \in I^{\perp \perp} \). Since we can identify \( I^{\perp \perp} \) with \( I^{**} \), we have \( TIM_p(\hat{G}) \subseteq I^{**} \).

Assume that \( m_1, m_2 \in TIM_p(\hat{G}) \). It is easy to see that \( m_1 \circ m_2 = m_1 \). In fact, given any \( T \in PM_p(G) \) and any \( u \in A_p(G) \), we have that \( \langle T \circ m_1, u \rangle = \langle m_1, uT \rangle = u(e)\langle m_1, T \rangle \). Hence \( T \circ m_1 = \langle m_1, T \rangle L_e \).

Finally, \( \langle m_1 \circ m_2, T \rangle = \langle m_2, T \circ m_1 \rangle = \langle m_1, T \rangle \langle m_2, L_e \rangle = \langle m_1, T \rangle \). However, since \( I \) is Arens regular, \( I^{**} \) is commutative. Therefore \( m_1 = m_1 \circ m_2 = m_2 \circ m_1 = m_2 \).

**Corollary 3.13.** Let \( G \) be a second countable locally compact group. Let \( I \) be a closed non-zero ideal in \( A(G) \). If \( I \) is Arens regular, then \( G \) is discrete.

**Proof.** By Theorem 3.12, \( PM_2(G) \) has a unique topologically invariant mean. Consequently, \( G \) is discrete [10, Theorem 1].
**Proposition 3.14.** Let \( I \) be a proper closed ideal in \( A(G) \) with a bounded approximate identity. Then \( I \) is Arens regular if and only if \( I \) is reflexive.

*Proof.* A reflexive ideal is clearly Arens regular.

Conversely, assume that \( I \) is Arens regular. Then \( G \) is a discrete group. As \( I \) has a bounded approximate identity, Cohen's Factorization Theorem [14, Corollary 32.26] implies that \( I = I^2 = \{uv | u, v \in I\} \). Therefore \( I \circ I^{**} = (I \cdot I) \circ (I^{**}) \subseteq I \circ (I \circ \text{A}(G)^{**}) \subseteq I \cdot \text{A}(G) \subseteq I \). Hence \( I \) is an ideal in \( I^{**} \). Also, since \( \text{A}(G) \) is weakly sequentially complete, \( I \) is weakly sequentially complete. It follows from [23, Corollary 3.7] and [23, Corollary 3.9] that \( I \) is reflexive. \( \square \)

With Corollary 3.6 in mind, one might ask whether it is possible to have infinite dimensional ideals \( I \) which are Arens regular or reflexive. In [11, Theorem 5], Granirer shows that while in a non-discrete group \( A_2(G) \) has no non-zero reflexive ideals, (a fact that follows immediately from Corollary 3.13), every infinite discrete group is such that \( A_2(G) \) contains an ideal isomorphic to \( l_2 \).

We close this section with some results on the Arens regularity of some related Banach algebras.

**Proposition 3.15.** Let \( G \) be a locally compact group. Let \( \mathcal{A} = (B(G) \cap \text{AP}(G), \| \cdot \|_{B(G)}) \). Then \( \mathcal{A} \) is Arens regular if and only if \( \text{AP}(G) \) is finite dimensional.

*Proof.* \( \mathcal{A} \) is isometrically isomorphic to \( A(G^\text{ap}) \), where \( G^\text{ap} \) denotes the almost periodic compactification of \( G \) [7, p. 203]. Since \( G^\text{ap} \) is a compact group, it is amenable. Therefore \( \mathcal{A} \) is Arens regular if and only if \( G^\text{ap} \) is finite. But \( G^\text{ap} \) is finite if and only if \( \text{AP}(G) \) is finite dimensional.

The converse is obvious. \( \square \)

**Corollary 3.16.** Let \( G \) be a locally compact group. If \( B(G) \) is Arens regular, then \( \text{AP}(G) \) is finite dimensional.

Observe that \( \text{AP}(G) \cap B(G) \) is precisely the space of coefficient functions of the representation of \( G \) obtained by lifting the left regular representation of \( G^\text{ap} \) to \( G \). In this case, the representation is such that its coefficient functions form an algebra. For a general representation \( \pi \) of \( G \), this is so if and only if \( \pi \otimes \pi \) is quasi equivalent to a sub-representation of the representation \( \pi \) [2, Proposition 3.26].
Assume that \( G \) is a compact group. Let \( \pi \) be a continuous unitary representation of \( G \). Let \( \mathcal{A}_\pi \) denote the closed self-adjoint subalgebra of \( A(G) \) generated by the coefficients functions of \( \pi \). Then we have the following result:

**Proposition 3.17.** Let \( G \) be a compact group. Let \( \pi \) be a continuous unitary representation of \( G \). Then \( A_\pi \) is Arens regular if and only if \( \ker \pi \) is open.

*Proof.* \( \mathcal{A}_\pi \) is isometrically isomorphic with \( A(G/\ker \pi) \) [23]. Clearly \( G/\ker \pi \) is finite if and only if \( \ker \pi \) is open. The result follows immediately from Corollary 3.6. \( \square \)

**Corollary 3.18.** Let \( G \) be compact and connected. Then \( \mathcal{A}_\pi \) is Arens regular if and only if \( \pi \) is the trivial representation.

We wish to bring the reader's attention to two related results in the literature which unfortunately contain errors. The first result is the equivalence of the unique invariant mean on \( PM_2(G) \) with the discreteness of \( G \). The proof of this result is usually attributed to Renaud [22]. However the proof of [22, Proposition 8] contains a serious error which may well be impossible to repair. It would therefore appear that at present the equivalence of the discreteness of \( G \) with the existence of a unique invariant mean requires the assumption of second countability.

Secondly, in example 7.2 (b) of the deep paper [21], it is mistakenly stated that for every compact group \( A_p(G) \) is Arens regular.

**References**


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