Pacific Journal of Mathematics

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Vol. 151, No. 2 December 1991

PACIFIC JOURNAL OF MATHEMATICS
Vol. 151, No. 2, 1991

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Let G be a locally compact group. Let $A_p(G)$ be the Herz algebra of G associated with $1 < p < \infty$. We show that if $A_p(G)$ is Arens regular, then G is discrete. We also exhibit a number of sufficient conditions for such a group to be finite.

1. Introduction. Let G be a locally compact group. For $1 < p < \infty$, let $A_p(G)$ denote the linear subspace of $C_0(G)$ consisting of all functions of the form $u(x) = \sum_{i=1}^{\infty}(f_i * \tilde{g}_i)^{\vee}$, where $f_i \in L_p(G)$, $g_i \in L_q(G)$, $\frac{1}{p} + \frac{1}{q} = 1$, $\sum_{i=1}^{\infty} \|f\|_p \|g\|_q < \infty$, $f^{\vee}(x) = f(x^{-1})$ and $\tilde{f}(x) = \overline{f(x^{-1})}$. $A_p(G)$ is a commutative Banach algebra with respect to pointwise multiplication and the norm

$$\|u\|_{A_p(G)} = \inf\left\{\sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q \middle| u(x) = \sum_{i=1}^{\infty}(f_i * \tilde{g}_i)^{\vee}\right\}.$$

When $p = 2$, $A_2(G)$ is the Fourier algebra of G as introduced by Eymard in [7]. For general p, the algebras $A_p(G)$ were introduced and first studied by Herz [**13**].

In this paper we will study the structure of the second dual $A_p(G)^{**}$ as a Banach algebra with respect to the two Arens products. In particular, we will show that if $A_p(G)$ is Arens regular, then G is discrete. When $p = 2$, we show that for a large class of groups, Arens regularity will imply finiteness.

2. Preliminaries. Let G be a locally compact group with a fixed left Haar measure λ. For $1 \leq p \leq \infty$, let $L_p(G)$ be the usual Banach space of equivalence classes of p-integrable (or essentially bounded) functions on G. The algebras $A_p(G)$ for $1 < p < \infty$ will be as defined in §1. When $p = 2$ we will write $A(G)$ for $A_2(G)$.

For $1 < p < \infty$, let $PF_p(G)$ and $PM_p(G)$ denote the closure of $L_1(G)$, considered as an algebra of convolution operators on $L_p(G)$, with respect to the norm topology and the weak operator topology respectively in $\mathscr{B}(L_p(G))$, the bounded operators on $L_p(G)$. The space $PM_p(G)$ can be identified with the dual of $A_p(G)$ for each $1 < p < \infty$ [see **19**, p. 94].

Let $B_p(G)$ denote the space of multipliers of $A_p(G)$. Then $B_p(G)$ with the norm

$$\|u\|_{B_p(G)} = \sup\{\|uv\|_{A_p(G)} | v \in A_p(G),\ \|v\|_{A_p(G)} \leq 1\}$$

is a commutative Banach algebra with respect to pointwise multiplication.

Let $A \subseteq G$ be closed. We will denote by $I_p(A)$ the closed ideal of $A_p(G)$ of the form $\{u \in A_p(G) | u(x) = 0 \text{ for every } x \in A\}$. Given an ideal $I \subseteq A_p(G)$, we denote by $Z(I)$ the set $\{x \in G | u(x) = 0 \text{ for every } u \in I\}$.

Let $\mathscr{A}$ be a Banach algebra. Then $\mathscr{A}^{**}$ can be be given two multiplications which extend the multiplication of $\mathscr{A}$ and for which $\mathscr{A}^{**}$ becomes a Banach algebra. These products were introduced by Arens in **[1]**. They are defined as follows:

(1a) $\langle u \cdot T, v\rangle = \langle T, vu\rangle$ for every $u, v \in \mathscr{A}$, $T \in \mathscr{A}^{**}$,
(1b) $\langle T \odot \Gamma, u\rangle = \langle \Gamma, u \cdot T\rangle$ for every $u \in \mathscr{A}$, $T \in \mathscr{A}^{*}$, $\Gamma \in \mathscr{A}^{**}$,
(1c) $\langle \Gamma_1 \odot \Gamma_2, T\rangle = \langle \Gamma_2, T \odot \Gamma_1\rangle$ for every $T \in \mathscr{A}^{*}$, $\Gamma_1, \Gamma_2 \in \mathscr{A}^{**}$,
(2a) $\langle T \Box u, v\rangle = \langle T, uv\rangle$ for every $u, v \in \mathscr{A}$, $T \in \mathscr{A}^{*}$,
(2b) $\langle \Gamma \Box T, u\rangle = \langle \Gamma, T \Box u\rangle$ for every $u \in \mathscr{A}$, $T \in \mathscr{A}^{*}$, $\Gamma \in \mathscr{A}^{**}$,
(2c) $\langle \Gamma_1 \Box \Gamma_2, T\rangle = \langle \Gamma_1, \Gamma_2 \Box T\rangle$ for every $T \in \mathscr{A}^{*}$, $\Gamma_1, \Gamma_2 \in \mathscr{A}^{**}$.

In general, $\Gamma_1 \odot \Gamma_2 = \Gamma_2 \Box \Gamma_2$ may fail for some $\Gamma_1, \Gamma_2 \in \mathscr{A}^{**}$. If $\Gamma_1 \odot \Gamma_2 = \Gamma_1 \Box \Gamma_2$ for every $\Gamma_1, \Gamma_2 \in \mathscr{A}$, then $\mathscr{A}$ is said to be Arens regular.

Let $\mathscr{A}$ be a commutative Banach algebra. Then $u \cdot T = T \Box u$. Hence $\mathscr{A}^{*}$ becomes a commutative Banach $\mathscr{A}$-bimodule. Moreover, $\mathscr{A}$ is Arens regular if and only if $\mathscr{A}^{**}$ is commutative with respect to either, and hence both, of the Arens products.

We call $T \in \mathscr{A}^{*}$ weakly almost periodic if $\mathscr{O}(T) = \{u \cdot T | \|u\|_{\mathscr{A}} \leq 1\}$ is relatively weakly compact. T is uniformly continuous if T is in the norm closure of $\operatorname{span}\{u \cdot T_1 | u \in \mathscr{A},\ T_1 \in \mathscr{A}^{*}\}$. When $\mathscr{A} = A_p(G)$, we denote the weakly almost periodic functionals by $W_p(\widehat{G})$ and the uniformly continuous functionals by $UCB_p(\widehat{G})$ (see **[9]**).

A locally compact group G is amenable if there exists $m \in L_\infty(G)^*$ such that $m(1) = \|m\| = 1$ and $m(L_x f) = m(f)$ where $L_x f(y) = f(x^{-1}y)$ for every $x, y \in G$. The functional m is called a left invariant mean on $L_\infty(G)$. All commutative locally compact groups and all compact groups are amenable. F_2, the free group on two generators, is not amenable.

A functional $m \in PM_p(G)^*$ is called a topologically invariant mean on $PM_p(G)$ if $\|m\| = 1$ and $m(uT) = u(e)m(T)$ for every $u \in A_p(G)$,

$T \in PM_p(G)$. For any locally compact group it is known that $PM_p(G)$ has a T.I.M. [**9**, Proposition 2]. We can speak of a topologically invariant mean on any closed $A_p(G)$-submodule of $PM_p(G)$ which contains the functional L_e $(L_e(u) = u(e))$. It is known that $W_p(\widehat{G})$ always has a unique invariant mean [**9**, Proposition 9].

A Banach space X has the Radon-Nikodym Property of R.N.P. if every closed convex bounded subset $C \subseteq X$ is dentable. That is, for every $\varepsilon > 0$ there exists $x \in C$ such that $x \notin \overline{\mathrm{co}}\{C \backslash B_\varepsilon(x)\}$ where $B_\varepsilon(x) = \{y \in X \mid \|x - y\| < \varepsilon\}$. See [**23**, §2] for further information on the R.N.P.

3. Arens regularity. We begin with the following useful lemma.

LEMMA 3.1. *Let G be a locally compact group for which $A_p(G)$ is Arens regular. Then*

(i) *If I is a closed ideal of $A_p(G)$, then I is Arens regular.*
(ii) *If H is a closed subgroup of G, then $A_p(H)$ is Arens regular.*
(iii) *If K is a compact normal subgroup of G, then $A_p(G/K)$ is Arens regular.*

Proof. (i) This follows from [**4**, p. 312, Corollary].

(ii) By appealing to [**13**] and by the following the arguments of [**8**, Lemma 3.8], we can show that $A_p(H)$ is isometrically isomorphic to $A_p(G)/I_p(H)$. Hence by [**4**, p. 312, Corollary], $A_p(G)$ is Arens regular.

(iii) $A_p(G/K)$ is isometrically isomorphic to the closed subalgebra of $A_p(G)$ consisting of functions which are constant on cosets of K [**13**, Proposition 6]. The Arens regularity of $A_p(G/K)$ now follows immediately from [**4**, p. 312, Corollary] □

THEOREM 3.2. *Let G be a locally compact group for which $A_p(G)$ is Arens regular. Then G is discrete.*

Proof. We first assume that G is separable. If $A_p(G)$ is Arens regular, then $W_p(\widehat{G}) = PM_p(G)$. Hence $UCB_p(\widehat{G}) \subseteq W_p(\widehat{G})$ and G is discrete by [**9**, Theorem 16].

Let G be an arbitrary locally compact group. Let U be an open neighborhood of $\{e\}$ in G with compact closure. Then U generates an open σ-compact subgroup G_0 of G. By Lemma (3.1)(ii), $A_p(G_0)$ is also Arens regular. Since G_0 is compactly generated either G_0 is discrete and we are done or there is a compact normal subgroup K in G_0 such that $\lambda(K) = 0$ and G_0/K is separable.

Assume the latter to be true. By Lemma 3.1(ii), $A_p(G_0/K)$ is Arens regular. But G_0/K is separable and therefore must be discrete. Thus K is an open subgroup which contradicts the assumption that $\lambda(K) = 0$. Hence G_0 must be discrete. Consequently, so must G be discrete. □

The next result generalizes [**15**, Theorem 3.7]. The proof is similar.

LEMMA 3.3. *Let G be a locally compact group. Then $A_p(G)$ is an ideal in $PM_p(G)^*$ if and only if G is discrete.*

Proof. Assume first that G is discrete. Then $UCB_p(G) = PF_p(G)$ [**9**, Proposition 15]. Let $u \in A_p(G)$ and $m \in PM_p(G)^*$. Let $T \in PM_p(G)$. Then

$$\begin{aligned}\langle m \odot u, T\rangle &= \langle m, uT\rangle \\ &= \langle v, uT\rangle \quad \text{for some } v \in W_p(G) = PF_p(G)^* \\ &= \langle vu, T\rangle .\end{aligned}$$

Therefore $m \odot u = vu \in A_p(G)$. Also, $\langle u \odot m, T\rangle = \langle u, m \odot T\rangle = \langle m, uT\rangle = \langle m\odot u, T\rangle$, so $A_p(G)$ is in the center of $PM_p(G)^*$. Therefore $A_p(G)$ is a closed two-sided ideal in $PM_p(G)^*$.

Conversely, assume that $A_p(G)$ is an ideal in $PM_p(G)^*$. Let $u_0 \in A_p(G)$ with $u(e) = 1 = \|u\|_{A_p(G)}$. Let

$$K = \{m \odot u_0 | m \in PM_p(G)^*, \ m(L_e) = 1 = \|m\|\}.$$

Since $m \mapsto m \odot u_0$ is weak-$*$ to weak-$*$ continuous $\{m \in PM_p(G)^* | m(L_e) = 1 = \|m\|\}$ is weak-$*$ compact, so is K. But $K \subset A_p(G)$, so K is weakly compact in $A_p(G)$. It is also clearly convex.

For each $v \in A_p(G)$ with $v(e) = \|u\|_{A_p(G)} = 1$, define the operator Γ_v on K by $\Gamma_v(w) = vw$, for every $w \in K$. The operators Γ_v are pairwise commuting and $(K$, weak) to $(K$, weak) continuous. By the Kakutani-Markov fixed point theorem [**8**, p. 458], there exists some $v_0 \in K$ such that $\Gamma_v(v_0) = v_0$ for every $v \in A_p(G)$ with $\|v\|_{A_p(G)} = v(e) = 1$. That is, $vv_0 = v_0$. Since $v_0 \in K$, $v_0(e) = 1$. Let $x \in G \sim \{e\}$. Then there exists $v_1 \in A_p(G)$ with $v_1(e) = 1 = \|v_1\|_{A_p(G)}$ while $v_1(x) = 0$. Therefore $v_0(x) = v_1(x)v_0(x) = 0$. Hence $v_0 = 1_{\{e\}}$ and G is discrete. □

PROPOSITION 3.4. *Let G be a locally compact group for which $A_p(G)$ is Arens regular. If G is amenable, then $PM_p(G)$ has the Radon-Nikodym Property.*

Proof. Since $A_p(G)$ is Arens regular, $A_p(G)$ is a two-sided ideal in its second dual. Since G is amenable, $A_p(G)$ has a bounded approximate identity. It follows from [**23**, Corollary 3.2], that $PM_p(G)$ has the Radon-Nikodym Property. □

PROPOSITION 3.5. *Let G be a locally compact group for which $A_p(G)$ is Arens regular. Assume that G is amenable. Then the following are equivalent*:

(i) *$A_p(G)$ is weakly sequentially complete,*
(ii) *G is finite.*

Proof. (i) $\to$ (ii). If $A_p(G)$ is weakly sequentially complete and G is amenable, then by [**23**, Corollary 3.9] $A_p(G)$ is reflexive. It follows from [**11**, Theorem 4], that $A_p(G)$ is finite dimensional and hence G is finite.

(ii) $\to$ (i). If G is finite $A_p(G)$ is finite dimensional. □

COROLLARY 3.6. *Let G be an amenable locally compact group. Then $A(G)$ is Arens regular if and only if G is finite.*

Proof. Since $A_2(G) = A(G)$ is the predual of a von Neumann algebra, it is weakly sequentially complete. If $A(G)$ is Arens regular, then G is finite by Proposition 3.5.

Conversely, if G is finite, then $A(G)$ is reflexive. Hence $A(G)^{**} = A(G)$ is commutative. Therefore $A(G)$ is Arens regular. □

Corollary 3.6 is due to Lau and Wong [**17**]. They consider only the case of amenable groups where it is known that $W_p(\widehat{G}) \subseteq UCB_p(\widehat{G})$ [**9**, Proposition 14]. For non-amenable groups, it is not known whether the above inclusion holds even for $p = 2$ and for G discrete. For $p = 2$, the following proposition sheds some light on the non-amenable case.

PROPOSITION 3.7. *Let G be a locally compact group for which $A(G)$ is Arens regular. Let H be an amenable subgroup of G. Then H is finite. In particular, G is periodic.*

Proof. By Lemma 3.1(ii), $A(H)$ is Arens regular. Hence by Corollary 3.6, H is finite.

Let $x \in G$. Then $H = \langle x \rangle$, the subgroup generated by x is commutative and hence amenable. Therefore H is finite and G is periodic. □

Corollary 3.8. *Let G be a discrete group which contains the free group on 2 generators. Then $A(G)$ is not Arens regular.*

One of the most famous conjectures in the study of amenable groups was that a discrete group G would be amenable if and only if G did not contain a subgroup isomorphic to the free group on 2 generators. Ol′shanskiĭ [**18**] has proved this conjecture to be false by constructing a non-amenable group G for which every non-trivial proper subgroup is infinite cyclic. It follows from Proposition 3.7 that $A(G)$ is not Arens regular for this G. The natural question which arises is: Are there non-amenable periodic groups without infinite amenable subgroups?

Let $\mathscr{X}$ be a class of groups such that if $G \in \mathscr{X}$, then any homomorphic image of G also belongs to $\mathscr{X}$. A group H is called a hyper-$\mathscr{X}$-group if every homomorphic image $H_1 \neq \{e\}$ of H has a normal $\mathscr{X}$-subgroup $N \neq \{e\}$.

Proposition 3.9. *Let G be a discrete group which satisfies any of the following conditions:*

(i) *G is locally finite,*
(ii) *G is isomorphic to a subgroup of* $\mathrm{GL}(n, \mathbb{F})$ *for some n and any field $\mathbb{F}$,*
(iii) *G is a 2-group,*
(iv) *G is hyperfinite,*
(v) *G has an involution x with $|C_G(x)| < \infty$,*
(vi) *G is hypercentral.*

Then $A(G)$ is Arens regular if and only if G is finite.

Proof. (i) If G is locally finite, then every finitely generated subgroup is finite and hence amenable. Therefore G is amenable [see **19**, p. 121] and the result follows from Corollary 3.6.

(ii) If $A(G)$ is Arens regular, then G is periodic. Hence G is locally finite [**15**, p. 60].

(iii) If G is an infinite 2-group, then G has an infinite abelian subgroup [**15**, p. 72]. Therefore $A(G)$ is not Arens regular.

(iv) Assume that G is hyperfinite and that $A(G)$ is Arens regular. Then by (iii) every elementary abelian 2-subgroup is finite. Hence G is finite [**15**, p. 6].

(v) If G is infinite, then [**15**, 2.1 Theorem] implies that G contains an infinite abelian subgroup which is impossible.

(vi) If G is hypercentral, then G is locally nilpotent [**15**, p. 10] and hence amenable. □

For $p \neq 2$, we are unable to show that $UCB_p(\widehat{G}) = PM_p(G)$ implies that G is compact. Though we believe this to be true, this still remains the main stumbling block preventing the extension of Corollary 3.6 for $p \neq 2$.

PROPOSITION 3.10. *Let G be an amenable locally compact group. Then G is discrete and $A_p(G)$ is Arens regular if and only if $PM_p(G)^* = B_p(G)$.*

Proof. Assume that $PM_p(G)^* = B_p(G)$. Then since $A_p(G)^{**}$ is commutative, $A_p(G)$ is Arens regular. Hence G is discrete, by Theorem 3.2.

Conversely, if G is discrete, then $UCB_p(\widehat{G}) = PF_p(G)$. If G is Arens regular, then $PF_p(G) = PM_p(G)$. Hence $PM_p(G)^* = B_p(G)$ as a Banach space. Let $u, v \in B_p(G)$. Let $f \in l_1(G)$. Then $\langle u \odot v, f\rangle = \langle u, vf\rangle = \int uvf\,dx = \langle uv, f\rangle$. Since $PM_p(G) = PF_p(G)$, $l_1(G)$ is norm dense in $PM_p(G)$. Therefore $u \odot v = uv$ and the Arens multiplication agrees with the pointwise product on $B_p(G)$. □

PROPOSITION 3.11. *Let G be a countable amenable discrete group. If $A_p(G)$ is Arens regular, then $PM_p(G)$ is separable.*

Proof. By Proposition 3.4, $PM_p(G)$ has the R.N.P. However, since G is countable, $A_p(G)$ is separable. It follows that $PM_p(G)$ is also separable [see **23**, §2]. □

When $p = 2$, $PM_p(G)$ is a von Neumann algebra. Since a separable von Neumann algebra is well known to be finite dimensional, we have another proof of Corollary 3.6. This follows since an infinite group must always have a countable infinite subgroup.

For amenable groups $PM_p(G)$ can be identified with the multipliers of $L_p(G)$, that is, the algebra of all operators on $L_p(G)$ which commute with convolution. The assumption of Arens regularity of $A_p(G)$ implies that the closure of $L_1(G)$ is the same with respect to both the norm topology and the weak operator topology on $B(L_p(G))$. This would seem to suggest that $L_p(G)$ is finite dimensional and therefore that G is finite. We are left to ponder the following two questions:

PROBLEM 1. If $P_p(G)$ is separable for some $1 < p < \infty$, is G necessarily finite?

PROBLEM 2. If $L_1(G)$ is norm dense in $PM_p(G)$, is G necessarily finite?

Let G be a discrete group. Let $\{x_1, \dots, x_n\} = A$ be a finite subset of G. Then $I_p(G \backslash A)$ is a closed finite dimensional ideal in $A_p(G)$ and is therefore Arens regular. Moreover, $I_p(G \backslash A)$ has an identity.

Conversely, if a non-zero closed ideal in $A_2(G)$ is Arens regular, then this will be shown below to be sufficient to insure that G is discrete. If, in addition, we assume that I has a bounded approximate identity, then we will also show that I is reflexive and therefore infinite codimensional.

THEOREM 3.12. *Let G be a locally compact group. Let I be a closed non-zero ideal in $A_p(G)$. Assume that I is Arens regular. Then $PM_p(G)$ has a unique topologically invariant mean.*

Proof. Let $Z(I) = A \subset G$. Since I is non-zero, $A \neq G$. Therefore $G \backslash A$ is open. By translating if necessary, we can assume that $G \backslash A$ is a neighborhood of e.

Let $M \in TIM_p(\widehat{G})$. Let $T \in I^{\perp}$. We can find $u \in A_p(G)$ such that $u \in I$ and $u(e) = 1$. It follows that $\langle uT, v \rangle = \langle T, uv \rangle = 0$ for every $v \in A(G)$. Hence $uT = 0$. But then $0 = m(uT) = u(e)m(T) = m(T)$. Therefore $m \in I^{\perp\perp}$. Since we can identify $I^{\perp\perp}$ with I^{**}, we have $TIM_p(\widehat{G}) \subseteq I^{**}$.

Assume that $m_1, m_2 \in TIM_p(\widehat{G})$. It is easy to see that $m_1 \odot m_2 = m_1$. In fact, given any $T \in PM_p(G)$ and any $u \in A_p(G)$, we have that $\langle T \odot m_1, u \rangle = \langle m_1, uT \rangle = u(e)\langle m_1, T \rangle$. Hence $T \odot m_1 = \langle m_1, T \rangle L_e$. Finally, $\langle m_1 \odot m_2, T \rangle = \langle m_2, T \odot m_1 \rangle = \langle m_1, T \rangle \langle m_2, L_e \rangle = \langle m_1, T \rangle$. However, since I is Arens regular, I^{**} is commutative. Therefore $m_1 = m_1 \odot m_2 = m_2 \odot m_1 = m_2$. □

COROLLARY 3.13. *Let G be a second countable locally compact group. Let I be a closed non-zero ideal in $A(G)$. If I is Arens regular, then G is discrete.*

Proof. By Theorem 3.12, $PM_2(G)$ has a unique topologically invariant mean. Consequently, G is discrete [**10**, Theorem 1]. □

PROPOSITION 3.14. *Let I be a proper closed ideal in $A(G)$ with a bounded approximate identity. Then I is Arens regular if and only if I is reflexive.*

Proof. A reflexive ideal is clearly Arens regular.

Conversely, assume that I is Arens regular. Then G is a discrete group. As I has a bounded approximate identity, Cohen's Factorization Theorem [**14**, Corollary 32.26] implies that $I = I^2 = \{uv | u, v \in I\}$. Therefore $I \circ I^{**} = (I \cdot I) \circ (I^{\perp\perp}) \subseteq I \circ (I \circ A(G)^{**}) \subseteq I \cdot A(G) \subseteq I$. Hence I is an ideal in I^{**}. Also, since $A(G)$ is weakly sequentially complete, I is weakly sequentially complete. It follows from [**23**, Corollary 3.7] and [**23**, Corollary 3.9] that I is reflexive. □

With Corollary 3.6 in mind, one might ask whether it is possible to have infinite dimensional ideals I which are Arens regular or reflexive. In [**11**, Theorem 5], Granirer shows that while in a non-discrete group $A_2(G)$ has no non-zero reflexive ideals, (a fact that follows immediately from Corollary 3.13), every infinite discrete group is such that $A_2(G)$ contains an ideal isomorphic to l_2.

We close this section with some results on the Arens regularity of some related Banach algebras.

PROPOSITION 3.15. *Let G be a locally compact group. Let $\mathscr{A} = (B(G) \cap AP(G), \|\cdot\|_{B(G)})$. Then $\mathscr{A}$ is Arens regular if and only if $AP(G)$ is finite dimensional.*

Proof. $\mathscr{A}$ is isometrically isomorphic to $A(G^{\mathrm{ap}})$, where G^{ap} denotes the almost periodic compactification of G [7, p. 203]. Since G^{ap} is a compact group, it is amenable. Therefore $\mathscr{A}$ is Arens regular if and only if G^{ap} is finite. But G^{ap} is finite if and only if $AP(G)$ is finite dimensional.

The converse is obvious. □

COROLLARY 3.16. *Let G be a locally compact group. If $B(G)$ is Arens regular, then $AP(G)$ is finite dimensional.*

Observe that $AP(G) \cap B(G)$ is precisely the space of coefficient functions of the representation of G obtained by lifting the left regular representation of G^{ap} to G. In this case, the representation is such that its coefficient functions form an algebra. For a general representation π of G, this is so if and only if $\pi \otimes \pi$ is quasi equivalent to a sub-representation of the representation π [**2**, Proposition 3.26].

Assume that G is a compact group. Let π be a continuous unitary representation of G. Let $\mathscr{A}_\pi$ denote the closed self-adjoint subalgebra of $A(G)$ generated by the coefficients functions of π. Then we have the following result:

PROPOSITION 3.17. *Let G be a compact group. Let π be a continuous unitary representation of G. Then A_π is Arens regular if and only if* $\ker\pi$ *is open.*

Proof. $\mathscr{A}_\pi$ is isometrically isomorphic with $A(G/\ker\pi)$ [**23**]. Clearly $G/\ker\pi$ is finite if and only if $\ker\pi$ is open. The result follows immediately from Corollary 3.6. □

COROLLARY 3.18. *Let G be compact and connected. Then $\mathscr{A}_\pi$ is Arens regular if and only if π is the trivial representation.*

We wish to bring the reader's attention to two related results in the literature which unfortunately contain errors. The first result is the equivalence of the unique invariant mean on $PM_2(G)$ with the discreteness of G. The proof of this result is usually attributed to Renaud [**22**]. However the proof of [**22**, Proposition 8] contains a serious error which may well be impossible to repair. It would therefore appear that at present the equivalence of the discreteness of G with the existence of a unique invariant mean requires the assumption of second countability.

Secondly, in example 7.2 (b) of the deep paper [**21**], it is mistakenly stated that for every compact group $A_p(G)$ is Arens regular.

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Received July 12, 1990 and in revised form November 2, 1990.

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Pacific Journal of Mathematics
Vol. 151, No. 2 December, 1991