ON LIPSCHITZ STABILITY FOR F.D.E

Yu Li Fu
ON LIPSCHITZ STABILITY FOR F. D. E.

YU-LI FU

Fozi M. Dannan and Saber Elaydi presented Lipschitz stability of O. D. E., and made a comparison between Lipschitz stability and Liapunov stability. In this paper, we will extend the concept of Lipschitz stability to the systems of functional differential equations (F.D.E.), and give some criteria via Liapunov's second method.

1. Definitions. We consider the system

\[(1.1) \quad \dot{x}(t) = f(t, x_t),\]

where \(x \in \mathbb{R}^n\), \(f: \mathbb{R} \times C([-r, 0], \mathbb{R}^n) \rightarrow \mathbb{R}^n\), \(f(t, 0) = 0\), \(f\) is continuous, \(x_t = x(t + \theta), \theta \in [-r, 0]\), \(r > 0\). The initial value condition associated with (1.1) is

\[(1.2) \quad x(\theta) = \phi(\theta), \quad \theta \in [-r, 0], \quad \phi(\theta) \in C([-r, 0], \mathbb{R}^n).\]

Set \(|\phi| = \sup_{\theta \in [-r, 0]} |\phi(\theta)|\), where \(|\cdot|\) is a norm in \(\mathbb{R}^n\). We always assume that the solution of (1.1) with (1.2) is existent and unique.

**Definition 1.** For the solution \(x(t)\) of (1.1) through \((t_0, \phi)\), (see [2, p. 38]), \((t_0, \phi) \in \mathbb{R}^+ \times C([-r, 0], \mathbb{R}^n), \mathbb{R}^+ \overset{\text{def}}{=} [0, +\infty)\), if there exists a constant \(\delta > 0\), which is independent of \(t_0\), and another constant \(M = M(\delta) > 0\), such that

\[(1.3) \quad |x_t| \leq M|\phi|, \quad \text{for } t \geq t_0, \text{ and } |\phi| < \delta,\]

then the zero solution of (1.1) is said to be Lipschitz uniformly stable. This is denoted by \((1.1) \in \text{Lip. U. S.}\).

**Definition 2.** If in Definition 1, \(\delta\) is allowed to be \(+\infty\), then the zero solution of (1.1) is said to be Lipschitz globally uniformly stable. This is denoted by \((1.1) \in \text{Lip. G. U. S.}\).

Obviously, if \(r = 0\), each definition above reduces to a definition for O.D.E.

If on \([t_0, T]\), where \(T\) is large enough, the solution of (1.1) through \((t_0, \phi)\) satisfies Definitions 1 and 2, it is said to be Lipschitz uniformly or globally uniformly stable on the large interval \([t_0, T]\).
2. Main results.

**Theorem 1.** For linear F.D.E.

\[
x'(t) = L(t, x_t),
\]

where \( L \) is a linear operator, the Lipschitz uniform stability of the zero solution is equivalent to Liapunov uniform stability of the zero solution.

**Proof.** If the zero solution (2.1) is Liapunov uniformly stable, from [2, p. 163], there exists a linear operator \( T(t, t_0) \), such that the solution of (2.1) through \((t_0, \phi)\) can be represented by

\[
x_t(t_0, \phi) = T(t, t_0)\phi,
\]

and there exists a constant \( M > 0 \), such that

\[
|T(t, t_0)| \leq M, \quad \text{for } t \geq t_0.
\]

This implies that the zero solution of (2.1) is Lipschitz uniformly stable.

On the other hand, it follows from Definition 1 that Lipschitz uniform stability implies Liapunov uniform stability.

The proof is complete.

**Theorem 2.** If there exists a continuous functional \( V(t, \psi) \geq 0 \), \((t, \psi) \in [t_0, +\infty) \times C([-r, 0], \mathbb{R}^n)\), for which:

(i) There exist nondecreasing continuous nonzero functions \( u, v, u(0) = 0, v(0) = 0 \), and \( v(s) \leq u(Ms) \) for all \( s > 0 \), where \( M \geq 1 \) is a constant, and

\[
u(|\psi|) \leq V(t, \psi) \leq v(|\psi|), \quad \text{for } \psi \in C([-r, 0], \mathbb{R}^n) \text{ and } t \geq t_0.
\]

(ii) For the solution \( x(t) \) of (1.1) through \((t_0, \phi)\), we suppose

\[
V'(t, x_t) \leq 0, \quad t \geq t_0,
\]

where

\[
V'(t, x_t) := \lim_{h \to 0^+} \sup_{t} \frac{1}{h}(V(t + h, x_{t+h}(t, \phi)) - V(t, \phi)).
\]

Then \((1.1) \in \text{Lip.G.U.S.}\)

**Proof.** For the solution \( x(t) \) of (1.1) through \((t_0, \phi)\), from (ii) we have

\[
V(t, x_t) \leq V(t_0, \phi), \quad t \geq t_0.
\]
From (i) we obtain
\[ u(|x_t|) \leq V(t, x_t) \leq V(t_0, \phi) \leq v(|\phi|) \leq u(M|\phi|), \quad t \geq t_0. \]
Hence, \(|x_t| \leq M\|\phi\|\) holds for \(t \geq t_0\) and \(\|\phi\| < +\infty\). This completes the proof.

**Theorem 3.** Assume that

(i) There exists a functional \(V : [t_0, +\infty) \times C([-r, 0], \mathbb{R}^n) \rightarrow \mathbb{R}^+\), such that
\[ u(|\psi|) \leq V(t, \psi) \leq v(|\psi|), \quad \text{for } \psi \in C([-r, 0], \mathbb{R}^n), \ t \geq t_0, \]
and
\[ \limsup_{s \to 0^+} \frac{u^{-1}(v(s))}{s} \leq M, \]
where \(u, v\) is defined as in Theorem 2, respectively, and \(M \geq 1\) is a constant.

(ii) Condition (ii) of Theorem 2 is satisfied.

Then (1.1) \(\in\) Lip.U.S.

**Theorem 4.** Assume that there exists a continuous function \(g : [t_0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}^1\), and \(V : [t_0, +\infty) \times C([-r, 0], \mathbb{R}^n) \rightarrow \mathbb{R}^+\), for which

(i) \(V'_1(t, x_t) \leq g(t, V(t, x_t)), \ t \geq t_0, \) and \(h(|\psi|) \leq V(t, \psi) \leq v(|\psi|), \ \psi \in C([-r, 0], \mathbb{R}^n), \)
where \(x(t)\) is the solution of (1.1) through \((t_0, \phi)\), \(h(s), v(s)\) are continuous and nondecreasing nonzero functions for \(s \geq 0\), satisfying \(h(0) = 0, v(0) = 0, V(t, 0) = 0\).

(ii) The zero solution of comparison scalar O.D.E.
\[ (2.2) \quad u' = g(t, u), \quad (g(t, 0) = 0), \]
is Liapunov uniformly stable.

Then (1.1) \(\in\) Lip.U.S.

**Proof.** For any \(\varepsilon > 0\), there exists a \(\delta > 0\), such that
\[ |u(t)| \leq h(\varepsilon), \quad \text{for } |u_0| < \delta. \]

Taking \(u_0 = v(||\phi||)\), where \(\frac{\varepsilon}{N} \leq ||\phi|| \leq v^{-1}(\delta), \ \phi \in C([-r, 0], \mathbb{R}^n), \)
\(N = \text{const.} \geq 1\), we find that
\[ V(t, x_t) \leq u(t), \quad \text{for } t \geq t_0 \quad \text{and} \quad V(t_0, \phi) \leq v(||\phi||) = u_0, \]
by the theory of differential inequality.
Thus, we have
\[ h(|x_t|) \leq V(t, x_t) \leq u(t) \leq h(N\|\phi\|), \quad t \geq t_0, \quad \frac{\varepsilon}{N} \leq \|\phi\| \leq v^{-1}(\delta), \]
where \( x(t) \) is the solution of (1.1) through \((t_0, \phi)\).

The proof is complete.

3. Examples. (1) We consider a scalar model of infectious diseases
\[
(3.1) \quad x'(t) = f(t, x(t)) - f(t, x(t - r)), \quad x \geq 0, \quad t \in [t_0, T],
\]
where \( T \) is large enough, \( t_0 \geq 1, r > 0, f(t, x) \) is continuous and nonnegative, \( f(t, 0) = 0 \).

Assume that
\[
0 \leq \frac{\partial f}{\partial x} \leq \frac{1}{2N}, \quad \text{for a constant } N \geq T.
\]
Constructing
\[
V(t, \psi) = \psi^2(0) \frac{1}{t}, \quad u(s) = s^2 \frac{1}{N}, \quad v(s) = s^2,
\]
we have
\[
V_{(3.1)}'(t, x_t) = -\frac{x}{t} \left( \frac{x}{t} - 2 \frac{\partial f(t, \xi)}{\partial x} (x(t) - x(t - r)) \right)
= -\frac{x}{t} \left( \left( \frac{x}{t} - 2 \frac{\partial f(t, \xi)}{\partial x} x \right) + 2 \frac{\partial f(t, \xi)}{\partial x} x(t - r) \right)
\]
for \( t \in [t_0, T] \) and \( \xi \in [x(t), x(t - r)] \), and
\[
\frac{x}{t} - 2 \frac{\partial f(t, \xi)}{\partial x} x \geq \frac{x}{t} - 2 \frac{1}{2N} x \geq 0,
\]
for \( x \geq 0 \) and \( t_0 \leq t \leq T \), this implies that \( V_{(3.1)}'(t, x_t) \leq 0 \).

It follows from Theorem 2 that \((3.1) \in \text{Lip. G. U. S. on the large interval } [t_0, T]\).

(2) In (1.1), suppose that
\[
\psi^T(0)f(t, \psi) \leq K(t) \ln(\psi^T(0)\psi(0) + 1),
\]
for \( \psi \in \mathcal{C}([-r, 0], R^n), K(t) > 0, \int^{+\infty} K(t) \, dt < +\infty \).

Taking \( h(|\psi|) = v(|\psi|) = V(t, \psi) = \ln(\psi^T(0)\psi(0) + 1) \), we have
\[
V_{(1.1)}'(t, x_t) = \frac{2x^Tf(t, x_t)}{(x^Tx + 1)} \leq 2K(t)V(t, x_t),
\]
in view of that the zero solution of \( y' = 2K(t)y \) is Liapunov uniformly stable, it follows from Theorem 4 that \((1.1) \in \text{Lip. U. S.}\).
(3) For scalar D.D.E.

(3.2) \[ x'(t) = f(t, x(t-r)), \quad t \geq 0, \]

assume that

\[ |f(t, x(t-r))| \leq \frac{g(t)}{2} |x(t-r)|, \quad g(t) > 0, \quad \text{and} \]

\[ \int_0^{+\infty} g(s) \, ds < +\infty, \quad r > 0. \]

Constructing \( u(|\psi|) = \psi^2(0) \exp(- \int_0^{+\infty} g(s) \, ds) \), \( v(|\psi|) = \psi^2(0) \), and

\[
V(t, \psi) = \psi^2(0) \exp \left( - \int_0^t g(s) \, ds \right) \]
\[ + \frac{g(t)}{2} \left( \int_{-r}^0 \psi^2(\theta) \, d\theta \right) \exp \left( - \int_0^t g(s) \, ds \right), \]

we have

\[
V_{(3.2)}'(t, x_t)
\]
\[ = -\frac{g(t)}{2} (x^2(t) + x^2(t-r)) \exp \left( - \int_0^t g(s) \, ds \right) \]
\[ + 2x(t) f(t, x(t-r)) \exp \left( - \int_0^t g(s) \, ds \right) \]
\[ - \frac{g^2(t)}{2} \left( \int_{-r}^0 x^2(t+\theta) \, d\theta \right) \exp(- \int_0^t g(s) \, ds) \]
\[ \leq \frac{g(t)}{2} (-x^2(t) - x^2(t-r) + 2|x(t)||x(t-r)|) \]
\[ \times \exp \left( - \int_0^t g(s) \, ds \right) \leq 0, \]

\( t \geq t_0 \geq 0 \), and

\[ u(|x_t|) \leq V(t, x_t) \leq V(t_0, \phi) \leq v(|\phi|) \leq u(M|\phi|), \]

where \( M = \exp(\int_0^{+\infty} g(s) \, ds) > 1 \).

It follows from Theorem 2 that (3.2) \( \in \text{Lip. G. U. S.} \).

(4) For scalar D.D.E.

(3.3) \[ x'(t) = -4x^3(t) + 2x^3(t-r), \quad t \geq 0, \]

constructing

\[ V(\psi) = \frac{\psi^4(0)}{8} + \int_{-r}^0 \psi^6(\theta) \, d\theta, \quad u(s) = s^4, \quad v(s) = s^4 + s^6 r, \]
it is easy to conclude that
\[ V'(3,3)(x_t) \leq 0, \quad t \geq t_0 \geq 0, \]
and
\[ u(|x_t|) \leq V(x_t) \leq V(|\phi|) \leq ||\phi||^4 + ||\phi||^6r = v(||\phi||), \]
\[ ||\phi|| = \sup_{\theta \in [-r,0]} |\phi(\theta)|, \]
\[ \limsup_{s \to 0} \frac{u^{-1}(v(s))}{s} = \limsup_{s \to 0}(1 + s^2r)^{1/4} \leq M, \text{ where } M \geq 2. \]

It follows from Theorem 3 that (3.3) $\in$ Lip. U. S.


(2) **Theorem 5.** If the system (1.1) $\in$ Lip. U. S., then the zero solution of the system
\[ y' = D\phi f(t, x_t(t, 0, f))y_t, \]
is Lipschitz uniformly stable.

In fact, by the results of [2, page 46] and the definition of the derivative operator $D\phi$, we obtain that the solution of (4.1) through $(t_0, \psi)$ is in the form of
\[ y = D\phi x(t_0, 0, f)\psi = x(t_0, \psi, f) - x(t_0, 0, f) - w(0, \psi), \]
where
\[ \psi \in C([-r, 0], R^n), \quad t_0 \geq 0, \quad x(t_0, 0, f) = 0, \]
\[ \lim_{||\psi|| \to 0} \frac{||w(0, \psi)||}{||\psi||} = 0. \]

Hence, there exists a constant $\eta > 0$, such that $D\phi x(t_0, 0, f)$ is uniformly bounded, whenever $||\psi|| < \eta$. This implies that (4.1) $\in$ Lip. U. S.

Theorem 5 means that (4.1) $\in$ Lip. U. S. is a necessary condition for (1.1) $\in$ Lip. U. S. We can conclude that Lipschitz uniform stability is not equivalent to Liapunov uniform stability for nonlinear systems [1].
REFERENCES


Received May 31, 1990.

Fu ZHOU NORMAL COLLEGE
Fu ZHOU
Fu JIAN 350005, P. R. CHINA
Editors

V. S. Varadarajan
(Managing Editor)
University of California
Los Angeles, CA 90024-1555-05

Herbert Clemens
University of Utah
Salt Lake City, UT 84112

Thomas Enright
University of California, San Diego
La Jolla, CA 92093

Associate Editors

R. Arens
E. F. Beckenbach
(1906–1982)
B. H. Neumann
F. Wolf
K. Yoshida
(1904–1989)

Supporting Institutions

University of Arizona
University of Oregon
University of British Columbia
University of Southern California
California Institute of Technology
Stanford University
University of California
University of Hawaii
Montana State University
University of Tokyo
University of Nevada, Reno
University of Utah
New Mexico State University
Washington State University
Oregon State University
University of Washington

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the 1991 Mathematics Subject Classification scheme which can be found in the December index volumes of Mathematical Reviews. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024-1555-05.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics (ISSN 0030-8730) is published monthly except for July and August. Regular subscription rate: $190.00 a year (10 issues). Special rate: $95.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) is published monthly except for July and August. Second-class postage paid at Carmel Valley, California 93924, and additional mailing offices. Postmaster: send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

Published by Pacific Journal of Mathematics, a non-profit corporation
Copyright © 1991 by Pacific Journal of Mathematics
Michael G. Eastwood and A. M. Pilato, On the density of twistor elementary states .................................................. 201
Brian E. Forrest, Arens regularity and discrete groups ......................... 217
Yu Li Fu, On Lipschitz stability for F.D.E ........................................ 229
Douglas Austin Hensley, The largest digit in the continued fraction expansion of a rational number ........................................ 237
Uwe Kaiser, Link homotopy in \( \mathbb{R}^3 \) and \( S^3 \) .............................. 257
Ronald Leslie Lipsman, The Penney-Fujiwara Plancherel formula for abelian symmetric spaces and completely solvable homogeneous spaces ........................................................................ 265
Florin G. Radulescu, Singularity of the radial subalgebra of \( \mathcal{L}(F_N) \) and the Pukánszky invariant ................................................................. 297
Albert Jeu-Liang Sheu, The structure of twisted \( SU(3) \) groups ............... 307
Morwen Thistlethwaite, On the algebraic part of an alternating link ........... 317
Thomas (Toma) V. Tonev, Multi-tuple hulls ........................................... 335
Arno van den Essen, A note on Meisters and Olech’s proof of the global asymptotic stability Jacobian conjecture .................................................. 351
Hendrik J. van Maldeghem, A characterization of the finite Moufang hexagons by generalized homologies ................................................. 357
Bun Wong, A note on homotopy complex surfaces with negative tangent bundles ............................................................................. 369
Chung-Tao Yang, Any Blaschke manifold of the homotopy type of \( CP^n \) has the right volume ........................................................................ 379