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The finite continued fraction sequence of a reduced fraction a/b, with $0 \le a < b$, is the sequence $d = (d(1), d(2), \ldots, d(r))$ of positive integers such that $d(r) > 1$, and

$$a/b = 1/(d(1) + 1/(d(2) + \cdots + 1/d(r))).$$

In the standard terminology of continued fractions, this is written as $[0; d(1), d(2), \ldots, d(r)]$, which we abbreviate here to $[d(1), d(2), \ldots, d(r)]$. Thus $[1, 4, 2] = 1/(1 + 1/(4 + 1/2)) = 9/11$. The empty sequence corresponds to $0/1$. For any other fraction, there will be $r \ge 1$ *digits* (also known as partial quotients) $d(j)$ in this expansion $(1 \le j \le r)$. The largest of these we call $D(a/b)$ or $D(a, b)$. Thus $D(9/11) = D(9, 11) = 4$. The aim of this work is to elucidate the distribution of $D(a, b)$. Put informally, the main result is that $\text{Prob}[D(a, b) \le \alpha \log b] \approx \exp(-12/\alpha\pi^2)$. More precisely, it is shown that for all $\varepsilon > 0$, and uniformly in $\alpha > \varepsilon$ as $x \to \infty$,

$$\#\{(a, b) : 0 \le a < b \le x, \gcd(a, b) = 1, \text{ and } D(a, b) \le \alpha \log x\}$$
$$\approx (3/\pi^2)x^2 \exp(-12/\alpha\pi^2).$$

The question of how often there are exactly M digits exceeding $\alpha \log b$ in the continued fraction expansion of a reduced fraction a/b with $0 \le a < b \le x$ is also touched on. Evidence points to the estimate

$$(3/\pi^2)x^2(M!)^{-1}(12/\alpha\pi^2)^M \exp(-12/\alpha\pi^2)$$

for the number of such fractions.

Previous work in a similar vein includes a result of Galambos [**4, 5**] concerning the distribution of the continued fraction partial quotients (digits) of a randomly chosen *real* number in the interval $(0, 1)$. Corresponding to any irrational ξ in $(0, 1)$ there is a unique sequence $d_\xi = d = (d(1), d(2), \ldots)$ of positive integers such that

$$\xi = [d] = [d(1), d(2), \ldots] = 1/(d(1) + 1/(d(2) + \cdots)).$$

Galambos found that if X is a random variable uniformly distributed on $[0, 1]$ (in the statement of his result the random variable has the Gauss-Kuzmin distribution, but that was just a convenience), then

$$\lim_{r\to\infty} \text{Prob}\left(\max_{k\le r} d_X(k) < \alpha r\right) = e^{-1/\alpha \log 2}. \tag{1}$$

There is also a literature concerning the distribution of pairs (a, b) for which, in the finite continued fraction expansion $d = d_{a/b}$ of a/b, all of the $d(j)$ are bounded by some fixed N. It is known **[2, 3, 8]** that for each $N \geq 2$ there exists a real number $H(N)$, $0 < H(N) < 1$, such that the number of pairs (a, b) for which $b \leq x$ is on the order of $x^{2H(N)}$, uniformly in N as $x \to \infty$. For each fixed N, there is also **[9]** a constant $C(N) > 0$ such that this pair count is $\approx C(N)x^{2H(N)}$, but it is not known how fast the convergence to this asymptotic behavior is, or whether it is uniform in N. There is no evident reason to suspect that it would not be uniform, but in any event numerical evidence suggests that x must be fairly large before the asymptotic trend takes hold. Recently, the author also showed **[10]** that

$$\lim_{n\to\infty} N(1 - H(N)) = 6/\pi^2. \tag{2}$$

As usual, $\Phi(x)$ denotes $\sum_{n\leq x} \phi(n) = \#\{(a, b) : 0 \leq a < b \leq x$ and $\gcd(a, b) = 1\}$, so that

$$\Phi(x) \approx (3/\pi^2)x^2 \quad \text{as } x \to \infty. \tag{3}$$

Now let

$$\Phi(x, \alpha) := \#\{(a, b) : 0 \leq a < b \leq x,\ \gcd(a, b) = 1, \\ \text{and } D(a, b) \leq \alpha \log x\}.$$

From the results mentioned above, it follows that there exists $C > 0$ such that for all sufficiently large x,

$$(1/C)x^2 e^{-12/\alpha\pi^2} \leq \Phi(x, \alpha) \leq Cx^2 e^{-12/\alpha\pi^2} \tag{4}$$

whenever $\alpha \log x$ is an integer ≥ 2. In view of the results just mentioned, our main result below fits in nicely:

THEOREM 1. *Uniformly in $\alpha \geq 4/\log\log x$ as $x \to \infty$,*

$$\Phi(x, \alpha) = (3/\pi^2)x^2 e^{-12/\alpha\pi^2}(1 + O((\alpha^{-2} + 1)e^{24/\pi^2\alpha} \log\log x/\log x))$$

as $x \to \infty$.

The result can also be put in a form which refers to the Diophantine approximation properties of a/b rather than to its continued fraction expansion. Let

$$\delta(a, b) := \min_{1\leq k<b} \|ka/b\|,$$

where $\|u\|$ *denotes the distance from* u *to the nearest integer. Let*

(5) $$F(x, \alpha) := \#\{(a, b) : 0 \leq a < b \leq x,\ \gcd(a, b) = 1, \text{ and } \delta(a, b) > 1/\alpha \log x\}.$$

Then for fixed $\alpha > 0$, *as* $x \to \infty$,

(6) $$F(x, \alpha) \approx (3/\pi^2)x^2 e^{-12/\alpha\pi^2}.$$

The basic idea of the proof is to count $\Phi(x, \alpha)$ by inclusion and exclusion, throwing out all fractions with at least one digit too large—once for each such digit—then restoring those with at least two—once for each such pair of digits—and so on. Term by term, these counts are asymptotic to the corresponding term in the identity

(7) $$(3/\pi^2)x^2 e^{-12/\alpha\pi^2} = (3/\pi^2)x^2 \sum_{j=0}^{\infty} (-12/\pi^2\alpha)^j/(j!).$$

REMARK. A more sophisticated version of inclusion and exclusion yields an asymptotic estimate of the number of fractions with exactly M digits $\geq \alpha \log x$, and denominator $\leq x$. Let $\mu_M : \mathbb{N} \to \mathbb{Z}$ satisfy

$$\sum_{k=0}^{n} \binom{n}{k} \mu_M(k) = \{1 \text{ if } n \leq M, 0 \text{ if not}\}.$$

This defines μ_M recursively, and it is not hard to see that $\mu_M(0) = 1$,

$$\mu_M(j) = 0 \quad \text{if } 1 \leq j \leq M, \quad \text{and}$$
$$\mu_M(j) = (-1)^{(j-M)} \binom{j-1}{M} \quad \text{for } j > M.$$

Following the proof given here for the case $M = 0$, but with μ_M in place of $(-1)^j$, leads to a main term of

$$(3x^2/\pi^2)(1/M!)(12/\pi^2\alpha)^M e^{-(12/\pi^2\alpha)}.$$

2. Inclusion and exclusion. Let $V_r := \{v : \{1, 2, \ldots, r\} \to \mathbb{Z}^+\}$ be the set of all sequences of r positive integers, and let $V = \bigcup_{r=0}^{\infty} V_r$. For $v \in V_r$, let $\text{lex}(v) = r$, the lexicographic length of v.

Let $a_0(v) = 0$, $b_0(v) = 1$, $a_{-1}(v) = 1$, and $b_{-1}(v) = 0$. For $1 \leq i \leq r$ we define $a_i(v)$ and $b_i(v)$ recursively by the conditions

(8) $$a_i(v) = d_i a_{i-1}(v) + a_{i-2}(v),$$
$$b_i(v) = d_i b_{i-1}(v) + b_{i-2}(v),$$

where $d_i = v(i)$ is the ith entry in the sequence v. Let

$$\langle v\rangle := b_{\mathrm{lex}(v)}(v) = b_r(v),$$

say, let $[v] := a_r(v)/b_r(v)$, and $\{v\} := b_{r-1}(v)/b_r(v)$. By convention, if v is the empty sequence ($r = \mathrm{lex}(v) = 0$) then $\langle v\rangle = 1$ and $[v] = \{v\} = 0$.

There is a two-to-one correspondence T between V and $\{(a, b) : 0 \le a < b \text{ and } \gcd(a, b) = 1\}$. In one direction, we map $v \xrightarrow{T} (a_r, b_r)$ where $r = \mathrm{lex}(v)$. In the other direction, $\mathrm{cfx}(a, b)$ is defined as that one of the two v, mapped by T back to (a, b), for which the last entry $v(r)$ is greater than one. The other, call it $\tilde{v}$, is obtained by replacing $v(r)$ with $v(r) - 1$, and appending 1 as $\tilde{v}(r+1)$. Thus

$$\begin{aligned}(9)\quad &\#\{v \in V : \langle v\rangle \le x \text{ and } v(i) \le \alpha\log x - 1 \text{ for } 1 \le i \le \mathrm{lex}(v)\}\\ &\le 2\Phi(x, \alpha) \le \#\{v \in V : \langle v\rangle \le x \text{ and } v(i) \le \alpha\log x\\ &\qquad\qquad \text{for } 1 \le i \le \mathrm{lex}(v)\}.\end{aligned}$$

In (9) we don't get equality because there are some $v \in V$ such that $v(r) - 1 \le \alpha\log x < v(r)$. Given two sequences u, $w \in V$, we write uw for their concatenation. That is, uw denotes the sequence v such that $v(j) = u(j)$ for $j \le \mathrm{lex}(u)$, $\mathrm{lex}(v) = \mathrm{lex}(u) + \mathrm{lex}(w)$, and $v(j) = w(j - \mathrm{lex}(u))$ for $\mathrm{lex}(u) < j \le \mathrm{lex}(u) + \mathrm{lex}(w)$. With this notation, a well-known identity reads

$$\langle uw\rangle = \langle u\rangle\langle w\rangle(1 + \{u\}[w]). \tag{10}$$

Now if $d_j = u(j) > N$, (where in the subsequent application, $N = [\alpha\log x]$), then $\{u\} = b_{j-1}(u)/(d_j b_{j-1}(u) + b_{j-2}(u)) < 1/N$, so that for $u \in V_j$ with $u(j) > N$, and $w \in V$,

$$\langle u\rangle\langle w\rangle \le \langle uw\rangle \le (1 + 1/N)\langle u\rangle\langle w\rangle. \tag{11}$$

This gives us a way to estimate, for $1 \le l \le L$ say, the number of constructions of the form

$$v = u_1k_1u_2k_2\cdots u_lk_lu_{l+1}, \quad \text{with } u_1, u_2, \ldots, u_{l+1} \in V,$$

$k_1, k_2, \ldots, k_l \in V(1)$ or $\mathbb{Z}^+$ (which we equate by a sleight of notation), with all $k_i > N$, and with $\langle v\rangle \le x$. Note that since V includes the empty sequence, there need not be any genuine interposition between consecutive k_i. Note also that the same sequence, if more than l of the $v(i)$ are greater than N, can be expressed in the above form in more than one way.

Our inclusion and exclusion argument is based on counting representations of v of the above form. For every integer $l \geq 0$, and every $v \in V$, let $\sigma(v, l, N)$ denote the number of ways in which v can be written as $u_1 k_1 \cdots u_l k_l u_{l+1}$, with all $k_i > N$. Then

$$\text{(12)} \qquad \sum_{l=0}^{\infty}(-1)^l \sigma(v, l, N) = \begin{cases} 1 & \text{if all } v(i) \leq N, \\ 0 & \text{if any } v(i) > N, \end{cases}$$

and $\sum_{l=0}^{K}(-1)^l \sigma(v, l, N)$ alternates about this, being $\geq \{1$ resp. $0\}$ for K even, and $\leq \{1$ resp $0\}$ for K odd. Now let

$$\Phi^-(x, \alpha) := \tfrac{1}{2}\#\{v \in V : \langle v \rangle \leq x \text{ and } v(i) \leq \alpha \log x - 1 \quad \text{for } 1 \leq i \leq \operatorname{lex}(v)\},$$

and

$$\Phi^+(x, \alpha) := \tfrac{1}{2}\#\{v \in V : \langle v \rangle \leq x \text{ and } v(i) \leq \alpha \log x \quad \text{for } 1 \leq i \leq \operatorname{lex}(v)\}.$$

Then with $N = [\alpha \log x]$ or $[\alpha \log x] - 1$ respectively,

$$\text{(13)} \qquad \begin{aligned} \Phi^{\pm}(x, \alpha) &= \frac{1}{2}\sum_{v \in V} \chi(\langle v \rangle \leq x) \sum_{l=0}^{\infty} (-1)^l \sigma(v, l, N) \\ &= \frac{1}{2}\sum_{l=0}^{\infty}(-1)^l \sum_{u_1 \in V} \sum_{k_1 > N} \sum_{u_2 \in V} \\ &\qquad \cdots \sum_{k_l > N} \sum_{u_{l+1} \in V} \chi(\langle u_1 k_1 \cdots u_l k_l u_{l+1} \rangle \leq x). \end{aligned}$$

Now let $W_l(x, N)$ denote the number of pairs $((u_1, u_2, \ldots, u_{l+1}), (k_1, k_2, \ldots, k_l))$ where the $u_i \in V$ and the $k_i > N$, and such that $\langle u_1 k_1 u_2 k_2 \cdots u_l k_l u_{l+1} \rangle \leq x$. Let $W_l'(x, N)$ denote the number of such pairs for which $\prod_{i=1}^{l} k_i \prod_{j=1}^{l+1} \langle u_j \rangle \leq x$. Then from (11), we see that

$$\text{(14)} \qquad W_l'((1 + 1/N)^{-2l} x, N) \leq W_l(x, N) \leq W_l'(x, n).$$

But

$$\text{(15)} \quad \begin{aligned} W_l'(x, N) = 2^{l+1} &\sum_{k_1 > N} \sum_{b_1 = 1}^{\infty} \sum_{k_2 > N} \sum_{b_2 = 1}^{\infty} \\ &\cdots \sum_{k_l > N} \sum_{b_{l+1} = 1}^{\infty} \prod_{i=1}^{l+1} \phi(b_i) \chi\left(\prod_{i=1}^{l+1} b_i \prod_{j=1}^{l} k_j \leq x\right). \end{aligned}$$

In view of this, it is natural to seek estimates for

$$A_m(y) := \sum_{b \in V_m} \prod_{i=1}^{m} \phi(b_i) \chi \left(\prod_{i=1}^{m} b_i \leq y \right),$$

and then to apply them with $y = (x/k_1 \cdots k_l)$ and $m = l+1$. Another way to write (15), using the definition above, is

$$(16) \qquad 2^m A_m(y) = \# \left\{ (u_1, u_2, \dots, u_m) : u_i \in V \right.$$
$$\left. \text{for } 1 \leq i \leq m \text{ and } \prod_{i=1}^{m} \langle u_i \rangle \leq y \right\}.$$

Once we have suitable estimates for $A_m(y)$, we will use (13), together with (17) below:

$$(17) \quad 2^{l+1} \sum_{k_1=N+1}^{\infty} \cdots \sum_{k_l=N+1}^{\infty} A_{l+1}((1+1/N)^{-2l} x/k_1 k_2 \cdots k_l)$$
$$\leq W_l(x, N) \leq 2^{l+1} \sum_{k_1=N+1}^{\infty} \cdots \sum_{k_l=N+1}^{\infty} A_{l+1}(x/k_1 k_2 \cdots k_l).$$

(There are only finitely many nonzero terms in the sums of (17), as $A_m(y) = 0$ for $y < 1$.) But (13) can now be written as

$$(18) \qquad \Phi^{\pm}(x, \alpha) = \frac{1}{2} \sum_{l=0}^{\infty} (-1)^l W_l(x, N^{\pm}),$$

where $N = [\alpha \log x]$ or $[\alpha \log x] - 1$ for Φ^+ or Φ^- respectively.

We are now in a position to sketch out the proof of Theorem 1. First we obtain an estimate of the form (with $\lambda = 6/\pi^2$)

$$(19) \qquad A_m(y) \approx \tfrac{1}{2} y^2 \lambda^m (\log y)^{m-1}/(m-1)!,$$

by a study of the Dirichlet series

$$(20) \qquad \int_1^{\infty} t^{-s}\, dA_k(t) = \left(\sum_{n=1}^{\infty} n^{-s} \phi(n) \right)^k = (\zeta(s-1)/\zeta(s))^k.$$

Next, we estimate the sums of (17), which from (19) are given approximately by

$$(21) \quad 2^{l+1} \sum_{\substack{k_1=N+1 \\ k_1 k_2 \cdots k_l \leq x}}^{\infty} \cdots \sum_{k_l=N+1}^{\infty} \frac{1}{2} x^2 \lambda^{l+1} (l!)^{-1} \prod_{i=1}^{l} k_i^{-2} (\log(x/k_1 \cdots k_l))^l,$$

as

$$(22)\quad \frac{1}{2}x^2(2\lambda)^{l+1}(l!)^{-1} \times \iint\cdots\int_R \left(\log x - \sum_{i=1}^{l}\log t_i\right)^l t_1^{-2}\cdots t_l^{-2}\,dt_l\cdots dt_1,$$

where $R = \{(t_1, \ldots, t_l) : t_1 \geq N, \ldots, t_l \geq N \text{ and } t_1t_2\cdots t_l \leq x\}$. Calculus and simplifying estimates then reduce the integral expression above to about $\frac{1}{2}x^2(2\lambda)^{l+1}(l!)^{-1}\alpha^{-l}$. Finally, from (18) we expect to find that $2\Phi(x, \alpha)$ is given approximately by

$$(23)\quad \begin{aligned} x^2\lambda - \frac{1}{2}x^2\sum_{l=1}^{\infty}(-2\lambda)^{l+1}\alpha^{-l}/l! \\ = \lambda x^2\sum_{l=0}^{\infty}(-2\lambda/\alpha)^l/l! = \lambda x^2 e^{-2\lambda/\alpha}, \end{aligned}$$

which is roughly what is claimed in Theorem 1.

In §3 we give details for the estimation of $A_m(y)$. In §4 we give details of the resulting estimates of $W_l(x, N)$, and tie it all together.

3. Bounds for $A_m(y)$. Recall that $A_m(y) = \sum_{b_1b_2\cdots b_m\leq y}\prod_1^m \phi(b_i)$.

LEMMA 1. *There is a positive absolute constant* C *such that for* $1 \leq k \leq C\sqrt{\log y}$, $A_k(y) = (\frac{1}{2}\lambda^k \log^{k-1} y/(k-1)!)y^2(1 + O(k^2/\log y))$.

Proof. First we note that if Lemma 1 holds for integer $y \geq 3$, then it holds for real $y \geq 3$ as well. Also, the case $k = 1$ is the well-known result $\sum_{n\leq y}\phi(n) = \frac{1}{2}\lambda y^2(1 + O(1/\log y))$. Now let

$$(24)\quad \begin{aligned} f(s, k) = \left(\sum_{n=1}^{\infty} n^{-s}\phi(n)\right)^k \\ = (\zeta(s-1)/\zeta(s))^k = \sum_{n=1}^{\infty} a(k, n)n^{-s}, \text{ say.} \end{aligned}$$

The series representations of $f(s, k)$ are absolutely convergent, uniformly in $\mathrm{Re}(s) \geq c$ for each $c > 2$, and the zeta function representation provides the analytic continuation into the domain $\mathrm{Re}(s) \geq 4/3$, apart from a single pole of order k at $s = 2$.

For the analysis ahead, it will be more convenient to first study $B_k(y) := \sum_{n=1}^{y} A_k(n)$, and to establish (for some fixed C, $0 < C < 1$), the following lemma.

LEMMA 2. *Uniformly in*

$$k \le C\sqrt{\log y}, \quad \text{as } y \to \infty,$$
$$B_k(y) = ((1/6)y^3\lambda^k(\log y)^{k-1}/(k-1)!)(1 + O(k^2/\log y)).$$

Before proving Lemma 2, we show how Lemma 1 follows from this secondary lemma.

Since $A_k(n)$ is increasing in n, for any integer m, $0 < m \le y$, we have

$$mA_k(y) \le B_k(y+m) - B_k(y). \tag{25}$$

Now from Lemma 2,

$$\begin{aligned}(26)\quad & B_k(y+m) - B_k(y)\\ &= ((1/6)\lambda^k/(k-1)!)((y+m)^3\log^k(y+m) - y^3\log^k y)\\ &\quad + O(k^2y^3\lambda^k\log^{k-2}y/(k-1)!).\end{aligned}$$

Taking $m = [ky/\log y]$, and bearing in mind that $k \le C\sqrt{\log y}$, this gives

$$\begin{aligned}(27)\quad & B_k(y+m) - B_k(y)\\ &= \left(\frac{1}{2}m\lambda^k y^2\log^{k-1}y/(k-1)!\right)(1 + O(k^2/\log y)).\end{aligned}$$

Thus

$$A_k(y) \le \left(\frac{1}{2}\lambda^k y^2\log^{k-1}y/(k-1)!\right)(1 + O(k^2/\log y)). \tag{28}$$

A similar calculation, starting from $A_k(y) \ge B_k(y) - B_k(y-m)$, gives a reversed version of (28). Taken together, these constitute the conclusion of Lemma 1.

We now turn to the proof of Lemma 2. By Perron's formula, for $c > 2$ we have

$$B_k(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(y^{s+1}/s(s+1))f(s,k)\,ds. \tag{29}$$

It is well known that $\zeta(s) = O(\theta^{-1}|s|^\theta)$, uniformly in $0 < \theta \le 1/2$ and $\mathrm{Re}(s) = 1-\theta$. With $\theta = 1/2k$, it follows that for some fixed $C_1 > 1$, and uniformly in $k \ge 2$, $\mathrm{Re}(s) = 2 - 1/2k$,

$$f(s,k) = O(C_1^k k^k |s|^{1/2}). \tag{30}$$

Although it is not essential to the proof, it will be convenient to have $C_1 = 4$. A little detail work, starting with the formula [11]

$$\zeta(s) = s/(s-1) + s\sum_{n=1}^{\infty}\int_0^1 u(n+u)^{-s-1}\,du$$

valid for $\mathrm{Re}(s) > 0$, is now in order. For $n \le |s|$ in the sum, one uses integration by parts, and with the obvious bounds for the other terms, this gives, for $s = 1-\theta$,

$$|\zeta(s)| \le \theta^{-1} + \theta^{-1}(1+|s|^{\theta}) + (1/2)|s|^{\theta}/(1-\theta).$$

For $s > 1$, $|\zeta(s)| \le \zeta(\sigma)$, so with $\theta = 1/2k$, the claim that C_1 can be 4 holds provided

$$4k + 2k|s|^{\theta} + \left(\frac{1}{2-2\theta}\right)|s|^{\theta} \le 4k\zeta(2-1/2k)|s|^{\theta}.$$

But $4k\zeta(2-1/2k) > 4k\zeta(2)+1$ since $\zeta'(\sigma) < -1/4$ for $1 < \sigma \le 2$, so we just need

$$4 \le (4\zeta(2)-2)|s|^{\theta}.$$

The worst case is $k = 2$, $s = 3/4$, and even then $4 \le 4.2619\ldots$.

Now let Γ be the linear path from $3-i\infty$ to $3+i\infty$, and let $\Gamma_{N,k}$ be the counterclockwise circuit of the rectangle with corners $3-iN$, $3+iN$, $(2-1/2k)+iN$, and $(2-1/2k)-iN$. Then

$$(31)\quad \lim_{N\to\infty}\left(\int_{\Gamma} - \int_{\Gamma_{N,k}}\right)(y^{s+1}f(s,k)/s(s+1))\,ds$$
$$= \int_{2-1/2k-i\infty}^{2-1/2k+i\infty}(y^{s+1}f(s,k)/s(s+1))\,ds = E_1(y,k), \text{ say.}$$

In view of (30), $E_1(y,k) = O(C_1^k k^k y^{3-1/2k})$. For $k^2 \le C_1^{-2}\log y$, a simple calculation now shows that

$$E_1(y,k) = O(y^3\lambda^k(\log y)^{k-2}/k^2(k-1)!),$$

which is the error allowed for in Lemma 2.

REMARK. The argument fails here without some hypothesis on C_1.

This brings us to the kernel of the matter: we must evaluate the integral over $\Gamma_{N,k}$ to within $O(y^3\lambda^k(\log y)^{k-2}/k^2(k-1)!)$.

Let $\beta(s,k) = (s-2)^k f(s,k)/s(s+1)$. Then

$$(32)\qquad \frac{1}{2\pi i}\int_{\Gamma_{N,k}}(y^{s+1}f(s,k)/s(s+1))\,ds$$
$$= y^3\int_{\Gamma_{N,k}}((s-2)^{-k}\beta(s,k)y^{s-2}/2\pi i)\,ds,$$

and the latter integral is, by the residue theroem, equal to the $(s-2)^{k-1}$ coefficient, say $T_{k-1}(y)$, in the Taylor series expansion of $\beta(s, k)y^{s-2}$ about 2. To estimate this, we first note that for a complex analytic function ξ on a disk of radius r, if $|\xi| \leq K$ on the disk, then by the Plancherel formula, $|(d^j/ds^j)\xi(s)| \leq K j! r^{-j}$ at the center.

Now $(s-2)\zeta(s-2)/\zeta(s-1) = \lambda(1+a(s-2)+O(s-2)^2)$, uniformly in $|s-2| \leq 1/2$, say. Thus for arbitrary j, $1 \leq j \leq k$, on the disk $|s-2| \leq j/2k$, we have

$$\beta(s, k) = O(\lambda^k \exp(O(j))), \tag{33}$$

so that from the observation above, if $D_j(k) = (d^j/ds^j)\beta(s, k)$ evaluated at $s = 2$, then for $j \leq k$,

$$D_j(k) = O((2k/j)^j j! \lambda^k \exp(O(j))). \tag{34}$$

(For $j = 0$, we have $D_j(k) = \lambda^k/6$, of course.) Now

$$T_{k-1}(y) = ((k-1)!)^{-1}(d^{k-1}/ds^{k-1})(y^{s-2}\beta(s, k)),$$

evaluated at $s = 2$. Expanding the iterated derivative of a product as in the binomial theorem, we get

$$(k-1)! T_{k-1}(y) = \sum_{j=0}^{k-1} \binom{k-1}{j} (\log y)^{k-1-j} D_j(k). \tag{35}$$

The main term here, corresponding to $j = 0$, is $(1/6)\lambda^k \log^{k-1} y$. For $j \geq 1$, we have, in the sum above,

$$\begin{aligned}\binom{k-1}{j} &(\log y)^{k-1-j} D_j(k) \\ &= O((k^j/j!)(\log y)^{k-1-j}(2k/j)^j j! \lambda^k \exp(O(j))) \\ &= (\log y)^{k-1} \cdot O_\varepsilon(k^{2j}(\log y)^{-j}(j^{-(1-\varepsilon)j})).\end{aligned} \tag{36}$$

Thus, for $k^2 \leq \log y$,

$$T_{k-1}(y) = (\lambda^k \log^{k-1} y/(k-1)!)(1 + O(k^2/\log y)), \tag{37}$$

which completes the proof of Lemma 2. With $C_1 = 4$, C in Lemma 1 becomes $1/4$. We need another estimate for the case of large k.

LEMMA 3. *For $k \geq 1$ and $y \geq 1$,*

$$A_k(y) \leq 4^{k+1} y^{2+3/2\pi^2}.$$

Proof. First note that this is trivial from the definition if $1 \le y < 4$, or if $k = 1$. Now in (29), take $c = 2 + 3/2\pi^2$. From this, it follows that with $s = (2 + 3/2\pi^2) + i\tau$,

$$B_k(y) \le (y^{c+1}/2\pi) \int_{-\infty}^{\infty} |f(s, k)|/|s(s+1)|\, d\tau. \tag{38}$$

For $\sigma = \mathrm{Re}(s) > 2$, by the product repesentation of the zeta function, and elementary properties of the linear fractional $(1 + zp^{-\sigma})/(1 + zp^{1-\sigma})$ on the circle $|z| = 1$, we have

$$|\zeta(s-1)/\zeta(s)| \le |\zeta(\sigma - 1)/\zeta(\sigma)|.$$

Thus from (38), $B_k(y) \le (\frac{1}{2\pi})y^{c+1}(\zeta(c-1)/\zeta(c))^k \cdot \pi/c$. Now taking $m = [y/4]$ in (26) gives

$$A_k(y) \le (3/5)(7/2)^k m^{-1}((y+m)^{c+1} - y^{c+1}). \tag{39}$$

Since this expression is increasing in m/y,

$$A_k(y) \le (3/5)(7/2)^k 4((5/4)^{c+1} - 1)y^c,$$

which for $k \ge 2$ is $\le 3 \cdot 4^k y^c < 4^{k+1} y^c$. This proves Lemma 3.

4. Estimation of $W_l(x, N)$. From (14), (15) and Lemma 1, we have

$$\begin{aligned} W_l(x, N) \le \frac{x^2 \lambda^{l+1} 2^{l+1}}{2(l!)} & \sum_{\substack{k_1 k_2 \cdots k_l \le x \exp(-16(l+1)^2) \\ k_i > N \text{ for } 1 \le i \le l}} k_1^{-2} k_2^{-2} \cdots k_l^{-2} \left(\log x - \sum_1^l \log k_i\right)^l \\ & \cdot \left(1 + O\left(l^2 / \left(\log x - \sum_1^l \log k_i\right)\right)\right) \\ & + 2^{l+1} \sum_{\substack{x \exp(-16(l+1)^2) < k_1 k_2 \cdots k_l \le x \\ k_i > N \text{ for } 1 \le i \le l}} A_{l+1}(x/k_1 k_2 \cdots k_l). \end{aligned} \tag{40}$$

In the second term here, $u = \log x - \sum_1^l \log k_i < 16(l+1)^2$, so that Lemma 1 is not applicable. Happily, for this term there is no need of sharp estimates. We get a crude, but adequate, bound from

Lemma 4. *For* $l \geq 1$, $N \geq 8$ *and* $x \geq (N+1)^l e^{16l^2}$,

$$2^{l+1} \sum_{\substack{x\exp(-16(l+1)^2)<k_1k_2\cdots k_l\leq x \\ k_i>N \text{ for } 1\leq i\leq l}} A_{l+1}(x/k_1k_2\cdots k_l)$$
$$\ll ((16)^l x \exp(16l^2(1+3/(2\pi^2)))(\log x)^l/l!).$$

The application of the lemma will be to cases in which $N \leq (\log x)^2$ and $l \leq (\log x)^{1/3}$, so that the upper bound given in Lemma 4 comes to $O_\varepsilon(x^{1+\varepsilon})$, or what is good enough for our purposes, to $O(x^{3/2})$.

To prove Lemma 4, we first note that from Lemma 3,

$$A_{l+1}(y) \leq 4^{l+2} y^{2+\lambda/4}. \tag{41}$$

Thus

$$2^{l+1} \sum_{\substack{x\exp(-16(l+1)^2)<k_1k_2\cdots k_l\leq x \\ k_i>N \text{ for } 1\leq i\leq l}} A_{l+1}(x/k_1k_2\cdots k_l) \tag{42}$$
$$\leq (16)^{l+1}x^2 \sum_{\text{(same range)}} k_1^{-2}\cdots k_l^{-2}\exp\left(\frac{\lambda}{4}\left(\log x - \sum_1^l \log k_i\right)\right)$$
$$\leq (16)^{l+1}x^{2+\lambda/4} \sum_{\text{all } k_i>N,\ \prod_1^l k_i > x\exp(-16(l+1)^2)} \prod_1^l k_i^{-(2+\lambda/4)}.$$

The sum in the right side of (42) above is itself

$$\leq \iint\cdots\int_R \prod_{i=1}^l t_i^{-(2+\lambda/4)}\,dt_l\cdots dt_1,$$

where $R = \{(t_1, t_2, \ldots, t_l)\colon t_i \geq N \text{ for } 1 \leq i \leq l \text{ and } \prod_1^l t_i \geq xe^{-16l^2}\}$.

On setting $s_i = \log t_i$, $1 \leq i \leq l$, this integral becomes

$$\iint\cdots\int_{R_l'} \exp\left(-\beta\sum_{i=1}^l s_i\right)\,ds_l\cdots ds_1,$$

where $\beta = 1 + \lambda/4$, and where

$$R'_l = \left\{ (s_1, \dots, s_l) : s_i \geq \log N \text{ for } 1 \leq i \leq l \right.$$
$$\left. \text{and } \sum_1^l s_i \geq \log x - 16l^2 \right\}.$$

Seen as an iterated integral, the innermost integral is a function of $s_1, s_2, \dots, s_{l-1}$ and is

$$\int_{\max(\log x - 16l^2 - s_1 - s_2 - \cdots - s_{l-1}, \log N)}^{\infty} \exp(-\beta(s_1 + s_2 + \cdots + s_{l-1}))e^{-\beta s_l}\, ds_l$$
$$\leq \min\{x^{-\beta}e^{16\beta l^2}, N^{-\beta}e^{-\beta(s_1+s_2+\cdots+s_{l-1})}\}.$$

Thus the original multiple integral is

$$\leq x^{-\beta}e^{16\beta l^2}\operatorname{Vol}\left(\left\{(s_1, s_2, \dots, s_{l-1}) : \log x - 16l^2 - \log N \right.\right.$$
$$\left.\left. > \sum_1^{l-1} s_i \text{ and all } s_i < \log N\right\}\right)$$
$$+ N^{-\beta}\iint\cdots\int_{R'_{l-1}} e^{-\beta(s_1+s_2+\cdots+s_{l-1})}\, ds_{l-1}, \cdots ds_1,$$

where $R'_{l-1} := \{(s_1, s_2, \dots, s_{l-1}) : s_i \geq \log N \text{ and } \sum_1^{l-1} s_i \geq \log x - 16l^2 - \log N\}$. The first term above is just $x^{-\beta}\exp(16\beta l^2)(\log x)^l/l!$, while the second term is of the same form as the original integral. Hence, we proceed by induction. Let

$$F(l, z) := \iint\cdots\int_{R(l,z)} \exp\left(-\beta\sum_{i=1}^{l} s_i\right) ds_l \cdots ds_1,$$

where $R(l, z) := \{(s_1, \dots, s_l): s_i \geq \log N \text{ for } 1 \leq i \leq l \text{ and } \sum_{i=1}^{l} s_i \geq z\}$. In this terminology, we have shown above that

$$F(l, z) \leq e^{-\beta z}(z + 16l^2)^l/l! + N^{-\beta}F(l-1, z - \log N). \tag{43}$$

Now $F(1, z) = \int_{\max(\log N, z)}^{\infty} e^{-\beta s}\, ds = \beta^{-1}\min(N^{-\beta}, e^{-\beta z})$, and in particular if $z > \log N$ then $F(1, z) = \beta^{-1}e^{-\beta z}$. Now from this and from (43), if $z > l\log N$ then

$$F(l, z) \leq e^{-\beta z}\sum_{j=2}^{l}(z + 16j^2)^j N^{\beta(j-L)}/j!. \tag{44}$$

Since $z > \log N > 2l$ under the assumption $N \geq 8$ in Lemma 4, the sum in (44) is dominated by the last term $(z + 16l^2)^l/l!$, so that

$$F(l, z) \ll e^{-\beta z}((z + 16l^2)^l/l!). \tag{45}$$

We apply (45) with $z = \log x = 16l^2$ to obtain, for $x > N^l e^{16l^2}$,

$$\iint \cdots \int_R \prod_{i=1}^{l} t_i^{-(2+\lambda/4)}\, dt_l \cdots dt_1 \ll x^{-\beta} \exp(16l^2\beta)((\log x)^l/l!). \tag{46}$$

In view of (42) and the following inequalty, this proves Lemma 4.

For x sufficiently large, though, if $l \leq (\log x)^{1/3}$ and $N \leq (\log x)^2$, then

$$(16)^l x \exp(16\beta l^2)(N^\beta + (\log x)^l/l!) < x^{3/2}. \tag{47}$$

Thus for large x the second term in (40) is negligible, even in comparison to the potential error in the first term of (40). The main term of that, putting aside for now the contribution from the "O" in $(1 + O(l^2/(\log x - \sum_1^l \log k_i)))$, is

$$\sum_{\substack{k_1 k_2 \cdots k_l \leq x \exp(-16(l+1)^2) \\ k_i > N \text{ for } 1 \leq i \leq l}} k_1^{-2} k_2^{-2} \cdots k_l^{-2} \left(\log x - \sum_1^l \log(k_i)\right)^l .$$

But this is less than

$$\int_{\substack{t_1 t_2 \cdots t_l \leq x \exp(-16l^2) \\ t_i \geq N \text{ for } 1 \leq i \leq l}} \int \cdots \int (t_1^{-2} t_2^{-2} \cdots t_l^{-2})(\log x)^l\, dt_l \cdots dt_2\, dt_1$$
$$\leq (\log x/N)^l.$$

The error term just put aside is likewise

$$\ll l^2 \int\int \cdots \int (t_1^{-2} \cdots t_l^{-2})(\log x)^{l-1}\, dt_l \cdots dt_1 \ll (l^2/\log x)(\log x/N)^l.$$

Thus for x sufficiently large, $l \leq (\log x)^{1/3}$ and $N \leq (\log x)^2$,

$$W_l(x, N) \leq (1 + O(l^2/\log x))(\log x/N)^l (2^l \lambda^{l+1}/l!) x^2 + O(x^{3/2}). \tag{48}$$

Next we obtain a similar lower bound for $W_l(x, N)$. From (17) and Lemma 1, we have

$$(49)\qquad W_l(x, N) \geq (x^2 2^l \lambda^{l+1}/l!) \sum_{\substack{k_1 k_2 \cdots k_l \leq x \exp(-16(l+1)^2)(1+1/N)^{-2l} \\ k_i > N \text{ for } 1 \leq i \leq l}} k_1^{-2} k_2^{-2} \cdots k_l^{-2} \cdot \left(\log x - \sum_{i=1}^{l} \log((1 + 1/N)^2 k_i) \right)^l + \text{two error terms.}$$

Let $x' = x(1 + 1/N)^{-2l}$, and let $S_l := \{K = (k_1, k_2, \dots, k_l) : k_1 k_2 \cdots k_l \leq x' \exp(-16(l + 1)^2)$ and $k_i > N$ for $1 \leq i \leq l\}$. The first of the above-mentioned error terms stems from the factor $1 + O(k^2/\log y)$ in Lemma 1. For $K \in S_l$, this factor, applied to each of the contributions to the sum in (49), is $1 + O((l + 1)^2/\log x)$ so that the whole sum is also perturbed by only a factor of $(1 + O((l + 1)^2/\log x)$ due to that source of error. The other term in (49) is the contribution to $\sum\sum \cdots \sum A_{l+1}(x'/k_1 k_2 \cdots k_l)$ due to $K = (k_1, k_2, \dots, k_l)$ for which $k_i > N$, $1 \leq i \leq l$, but $\prod_{i=1}^{l} k_i > x' \exp(-16(l + 1)^2)$.

For x sufficiently large, if $N \leq \log^2 x$ and $l \leq \log^{1/3} x$, then the hypotheses of Lemma 4 are satisfied, so that this error term is $O((16)^l x' e^{16 l^2 \beta}((\log x')^l/l!))$ and thus $O(x^{3/2})$ as before. Hence, for such x, N and l,

$$(50)\quad W_l(x, N) \geq (2^l (x')^2 \lambda^{l+1}/l!)(1 + O((l + 1)^2/\log x)P + O(x^{3/2}),$$

where

$$P = \sum_{\substack{k_1 k_2 \cdots k_l \leq x' \exp(-16(l+1)^2) \\ k_i > N \text{ for } 1 \leq i \leq l}} k_1^{-2} k_2^{-2} \cdots k_l^{-2} \left(\log(x') - \sum_{i=1}^{l} \log k_i \right)^l .$$

Now we need a lower bound for P. Clearly,

$$P \geq \int_R t_1^{-2} t_2^{-2} \cdots t_l^{-2} \left(\log(x') - \sum_{i=1}^{l} \log t_i \right)^l dt_l \cdots dt_1,$$

where $R = \{(t_1, t_2, \dots t_l) : t_1 t_2 \cdots t_l \leq x' e^{-16(l+1)^2}$ and $t_i \geq N + 1$ for $1 \leq i \leq l\}$. After a change of variables $(u_i = \log t_i - \log(N + 1)$,

$1 \le i \le l$) this integral becomes

$$(N+1)^{-l} \int_U e^{-(u_1+u_2+\cdots+u_l)} \cdot \left(\log(x') - l\log(N+1) - \sum_{i=1}^{l} u_i \right)^l du_l \cdots du_1 ,$$

where $U = \{(u_1, u_2, \dots, u_l)\colon \sum_{i=1}^{l} u_i \le \log(x') - 16(l+1)^2 - l\log(N+1)$ and $u_i \ge 0$ for $1 \le i \le l\}$. This, though, is just

$$(N+1)^{-l} \int_0^L (u^{l-1}/(l-1)!)e^{-u}(M-u)^l\, du ,$$

where $L = \log(x') = l\log(N+1) - 16(l+1)^2$ and $M = \log(x') - l\log(N+1)$. Thus

$$P \ge ((N+1)^{-l}/(l-1)!) \left\{ \int_0^\infty - \int_L^\infty (u^{l-1}e^{-u}(M-u)^l\, du \right\}. \tag{51}$$

The $-\int_L^\infty$ contribution here is quite small. In fact, for large x, for $N \le \log^2 x$ and for $l \le (\log x)^{1/3}$,

$$\int_L^\infty u^{l-1}(M-u)^l e^{-u}\, du \le \int_L^\infty u^{2l-1}e^{-u}\, du \le 2^{2l-1}e^{-L},$$

this last because $(1+1/u)^{2l-1}e^{-1} < \frac{1}{2}$ throughout the interval of integration. But in view of the constraints on l and N, $2^{2l-1}e^{-L} \le x^{-3/4}$ for large x. The main term in our lower bound for P is

$$\begin{aligned} &((N+1)^{-l}/(l-1)!) \int_0^\infty u^{l-1}e^{-u}(M-u)^l\, du \\ &= ((N+1)^{-l}/(l-1)!) \sum_{j=0}^{l} (-1)^j \binom{l}{j} M^{l-j} \int_0^\infty u^{l-1+j}e^{-u}\, du \\ &= (N+1)^{-l}M^l \sum_{j=0}^{l} (-M)^{-j} l(l-1+j)!/((j!)(l-j)!). \end{aligned}$$

In view of the constraints on l and N, we have $M \ge (1-\varepsilon)\log x$ (ε may be taken as small as we please by choosing a large enough lower bound for x). Thus the last sum above is dominated by its first term, and it simplifies to $(1 + O(l^2/\log x))$. Thus

$$P = (1 + O(l^2/\log x))(N+1)^{-l}M^l. \tag{52}$$

From (50), we now conclude that

$$(53)\ W_l(x, N) \geq (2^l(x')^2\lambda^{l+1}/l!)(1 + O((l+1)^2/\log x))(N+1)^{-l} \cdot (\log(x') - l\log(N+1))^l + O(x^{3/2}).$$

Since $x' = (1 + 1/N)^{-2l}x$, this boils down to

$$(54)\qquad W_l(x, N) \geq (2^l x^2 \lambda^{l+1}(\log x)^l/N^l(l!))(1+1/N)^{-3l} \times (1 + O(l^2 \log\log x/\log x)) + O(x^{3/2}),$$

for $1 \leq l \leq (\log x)^{1/3}$, $N \leq (\log x)^2$ and x sufficiently large. Together with (48), and under the same constraints, this gives

$$(55)\ W_l(x, N) = \frac{x^2 2^l \lambda^{l+1}}{(l!)}\left(\frac{\log x}{N}\right)^l \times \exp\left(O\left(\frac{l^2 \log\log x}{\log x}\right) + O\left(\frac{l}{N}\right)\right) + O(x^{3/2}).$$

Now from (13),

$$\Phi^{\pm}(x, \alpha) = \frac{1}{2}\sum_{l=0}^{\infty}(-1)^l W_l(x, N^{\pm}),$$

with $N^+ = [\alpha \log x]$ for Φ^+ and $N^- = [\alpha\log x] - 1$ for Φ^-. From (12), if we truncate this sum we get lower and upper bounds: if A is odd and $B = A + 1$, then

$$(56)\qquad \frac{1}{2}\sum_{l=0}^{A}(-1)^l W_l(X, N^-) < \Phi(x, \alpha) < \frac{1}{2}\sum_{l=0}^{B}(-1)^l W_l(x, N^+).$$

If we choose $B = [(\log x)^{1/3}]$, then for $\alpha > 4/\log\log x$ and $l \leq B$, the $l^2 \log\log x/\log x$ contribution to the error factor in (55) dominates that from l/N, and both are small, so that (55) boils down to

$$(57)\quad W_l(x, N^{\pm}) = \frac{x^2 2^l \lambda^{l+1}}{(l!)}\left(\frac{\log x}{N^{\pm}}\right)^l\left(1 + O\left(\frac{l^2\log\log x}{\log x}\right)\right) + O(x^{3/2})$$

for such l, N, and x. Thus in (56), the main terms are

$$\frac{1}{2}\sum_{l=0}^{A} x^2 2^l \lambda^{l+1}(-\log x/N^-)^l \Big/ l! \quad \text{and}$$

$$\frac{1}{2}\sum_{l=0}^{B} x^2 2^l \lambda^{l+1}(-\log x/N^+)^l \Big/ l!,$$

and the error factor in (57) perturbs these by at most

$$O\left(x^2\sum_{l=0}^{\infty}2^l\lambda^{l+1}(\log x/N^-)^l(l^2\log\log x/\log x)/l!\right).$$

But this is $O((x^2\log\log x/\log x)(z^2+1)e^z)$, where $z=2\lambda\log x/N^-$. Now for $\alpha\geq 4/\log\log x$, $N^-\geq(4\log x/\log\log x)-2$ so that $z\leq\frac{1}{2}\lambda\log\log x$, and

$$(x^2\log\log x/\log x)(z^2+1)e^z\ll x^2(\log\log x(1+\alpha^{-2})/\log x)e^{2\lambda/\alpha}.$$

This brings us to the main terms in (56). They are

$$\frac{1}{2}\lambda x^2\sum_{l=0}^{A}\left(\frac{2\lambda\log x}{N^-}\right)^l/l!,\quad\text{and}\quad\frac{1}{2}\lambda x^2\sum_{l=0}^{B}\left(\frac{2\lambda\log x}{N^+}\right)^l/l!$$

respectively. If we replace A and B with ∞ in these sums, the resulting change is $O((2\lambda/\alpha)^B/B!)$, and with $B=[\log x^{1/3}]$, that is $\ll 1/\log x$. Thus the main terms above are

$$\frac{1}{2}\lambda x^2(\exp(-2\lambda\log x/N^{\pm})+O(1/\log x)).$$

Replacing $(\log x/N^{\pm})$ with $1/\alpha$ here introduces an error factor of $\exp(O(1/\alpha^2\log x))$, so that the main terms boil down to

$$\frac{1}{2}\lambda x^2e^{-2\lambda/\alpha}(1+O((1+\alpha^{-2})/\log x)).$$

That is,

$$\begin{aligned}(58)\Phi(x,\alpha)&=\frac{1}{2}\lambda x^2e^{-2\lambda/\alpha}(1+O(1/\alpha^2\log x)+O(1/\log x))\\&\quad+O(x^2\log\log x(1+\alpha^{-2})/\log x)e^{2\lambda/\alpha}+O(x^{3/2})\\&=\frac{1}{2}\lambda x^2e^{-2\lambda/\alpha}(1+O(e^{4\lambda/\alpha}\log\log x(1+\alpha^{-2})(\log x)^{-1})),\end{aligned}$$

for all sufficiently large x and all α, $4/\log\log x\leq\alpha\leq(\log x)^2$. The condition $N\leq(\log x)^2$, which roughly coincides with $\alpha\leq\log x$, has been necessary in the workings of the main argument. But for $\alpha\geq\log x$, the claim made by Theorem 1 reduces to an assertion that $\Phi(x,\alpha)=\frac{1}{2}\lambda x^2(1+O(\log\log x/\log x))$. Now $\Phi(x,\alpha)$ is a nondecreasing function of α. But the upper bound part of this follows from Lemma 1, while the required lower bound follows from what we have proved above, on taking $\alpha=\log x$. Thus the theorem, while of no interest in this case, happens nonetheless to hold.

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