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ALBERT JEU-LIANG SHEU

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In order to study how the C^* -algebra $C(S_\mu U(3))$ of twisted $SU(3)$ groups introduced by Woronowicz is related to the deformation quantization of the Lie-Poisson $SU(3)$, we need to understand the algebraic structure of $C(S_\mu U(3))$ better. In this paper, we shall use Bragiel's result about the irreducible representations of $C(S_\mu U(3))$ and the theory of groupoid C^* -algebras to give an explicit description of the C^* -algebra structure of $C(S_\mu U(3))$, which indicates that $C(S_\mu U(3))$ is some kind of foliation C^* -algebra of the singular symplectic foliation of the Lie-Poisson group $SU(3)$.

In recent years, there has been a rapid growth of interest in the theory of quantum groups [D]. In particular, S. L. Woronowicz has developed a C^* -algebraic theory of quantum groups, which has motivated a lot of research [B, Po, Ro, S, Va-So, Wo1, Wo2].

In [S], the explicit knowledge of the C^* -algebra structure of $C(S_\mu U(2))$ [Wo1, S] has helped us to find a deformation quantization [BFFLS, Ri1, Ri2, Ri3] of the Lie-Poisson $SU(2)$ [D, Lu-We], which is in a sense compatible with the quantization of the group structure of $SU(2)$ by the "twisted groups" $S_\mu U(2)$. On the other hand, although both $C(S_\mu U(2))$ and $C(S_\mu U(3))$ [Wo1, Wo2] are defined as universal C^* -algebras of certain generators and relations, the algebraic structure of the latter seems to be much more complicated than that of the former. In [B], Bragiel classified the irreducible representations of the C^* -algebra $C(S_\mu U(3))$ of the twisted $SU(3)$ groups (with $0 < \mu < 1$) and showed that $C(S_\mu U(3))$ is a type-I C^* -algebra [Pe]. In this paper, enlightened by the ideas in [M-Re, Cu-M], we shall use Bragiel's result and the theory of groupoid C^* -algebras [Re] to give an explicit description of the C^* -algebra structure of $C(S_\mu U(3))$, which indicates that $C(S_\mu U(3))$ is some kind of foliation C^* -algebra of the singular symplectic foliation of the Lie-Poisson group $SU(3)$ [Co, We, Lu-We].

We shall use freely the concepts and properties of the theory of groupoid C^* -algebras throughout this paper. A good reference for this is [Re]. First let us fix notations. Let \mathbb{T} be the unit circle in \mathbb{C} and \mathbb{T}^2 be the two-torus embedded in \mathbb{C}^2 . We shall denote by ϕ and

ψ the two canonical coordinate functions of \mathbb{T}^2 with values in \mathbb{T} . For any groupoid \mathfrak{G} , we denote by $\mathfrak{G}|P$ the reduction of \mathfrak{G} by the subset P of the unit space of \mathfrak{G} [Re]. If a locally compact group G acts on a space X by an action τ , we shall denote by $X \times_\tau G$ the corresponding transformation group groupoid.

We define $\mathfrak{G} := \overline{\mathbb{Z}}^3 \times_\alpha \mathbb{Z}^5 | \overline{\mathbb{Z}}_{\geq}^3$, where $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty\}$, the subscript \geq denotes the nonnegative part, and \mathbb{Z}^5 acts on $\overline{\mathbb{Z}}^3$ by translation determined by the first three components, i.e. $\alpha(\mu)(\nu) = \nu - (\mu_1, \mu_2, \mu_3)$ for $\mu \in \mathbb{Z}^5$ and $\nu \in \overline{\mathbb{Z}}^3$. Since the last two copies of \mathbb{Z} act trivially on $\overline{\mathbb{Z}}^3$, we have $C^*(\mathfrak{G}) \cong C^*(\mathfrak{G}_0) \otimes C^*(\mathbb{Z}^2) \cong C^*(\mathfrak{G}_0) \otimes C(\mathbb{T}^2)$, where $\mathfrak{G}_0 := \overline{\mathbb{Z}}^3 \times_\tau \mathbb{Z}^3 | \overline{\mathbb{Z}}_{\geq}^3$ and τ is the action by translation. We assume that under the above isomorphism, the standard basis elements e_4 and e_5 of \mathbb{Z}^5 correspond to the conjugates $\overline{\phi}$ and $\overline{\psi}$ of the canonical coordinate functions on \mathbb{T}^2 (instead of ϕ and ψ in order to be more compatible with the notations used in [B] for the later discussion). Recall that the regular representation ρ_3 of $C^*(\mathfrak{G}_0)$ on the open dense invariant subset \mathbb{Z}_{\geq}^3 is faithful [M-Re], and hence $C^*(\mathfrak{G})$ can be faithfully represented on the Hilbert space $l^2(\mathbb{Z}_{\geq}^3) \otimes L^2(\mathbb{T}^2)$ through $\tilde{\rho}_3 := \rho_3 \otimes m$ where m is the representation of $C(\mathbb{T}^2)$ by multiplication operators on $L^2(\mathbb{T}^2)$.

In [B], the irreducible representations of $C(S_\mu U(3))$ are classified into six 2-parameter families (with parameters in \mathbb{T}^2) of irreducible representations $\pi_3, \pi_{21}, \pi_{22}, \pi_{11}, \pi_{12}$ and π_0 (listed here in the same order as in [B]) on Hilbert spaces $l^2(\mathbb{Z}_{\geq}^3), l^2(\mathbb{Z}_{\geq}^2), l^2(\mathbb{Z}_{\geq}^2), l^2(\mathbb{Z}_{\geq}^1), l^2(\mathbb{Z}_{\geq}^1)$, and $l^2(\mathbb{Z}_{\geq}^0) = \mathbb{C}$, respectively. The 2-parameter family of irreducible representations π (on a Hilbert space \mathcal{H}_π) in the above list determine a representation $\tilde{\pi}$ of $C(S_\mu U(3))$ on $\mathcal{H}_\pi \otimes L^2(\mathbb{T}^2)$. Since $\pi_3(u_{ij})$'s and $\pi_3(u_{ij}^*)$'s are (finite) linear combinations of weighted (multivariable) shifts on $l^2(\mathbb{Z}_{\geq}^3)$ with weight functions extendable to $\overline{\mathbb{Z}}_{\geq}^3$ continuously, and since the weight functions involved in each $\pi_3(u_{ij})$ or $\pi_3(u_{ij}^*)$ are products of the canonical functions $\phi, \psi, \overline{\phi}$ and $\overline{\psi}$ on \mathbb{T}^2 and functions on $\overline{\mathbb{Z}}_{\geq}^3$ independent of the parameters in \mathbb{T}^2 , it is easy to identify the 2-parameter family $\tilde{\pi}_3(u_{ij})$ or $\tilde{\pi}_3(u_{ij}^*)$ with an element in $C_c(\mathfrak{G}) \subseteq C^*(\mathfrak{G})$ (which is faithfully represented on $l^2(\mathbb{Z}_{\geq}^3) \otimes L^2(\mathbb{T}^2)$) for each u_{ij} . For example, with $C_c(\overline{\mathbb{Z}}_{\geq}^3)$ and \mathbb{Z}^5 canonically embedded in $C_c(\mathfrak{G})$, we have

$$\begin{aligned} \tilde{\pi}_3(u_{11}^*) &= e_1 f_{11}, & \tilde{\pi}_3(u_{12}^*) &= e_2 f_{12}, \\ \tilde{\pi}_3(u_{13}^*) &= e_5 f_{13}, & \tilde{\pi}_3(u_{21}^*) &= e_3 f_{21}, \\ \tilde{\pi}_3(u_{31}^*) &= e_4 f_{31}, \end{aligned}$$

where, for $(N, M, L) \in \overline{\mathbb{Z}}_{\geq}^3$,

$$\begin{aligned} f_{11}(N, M, L) &= (1 - \mu^{2(N+1)})^{1/2}, \\ f_{12}(N, M, L) &= \mu^{N+1}(1 - \mu^{2(M+1)})^{1/2}, \\ f_{13}(N, M, L) &= \mu^{2+N+M}, \\ f_{21}(N, M, L) &= \mu^N(1 - \mu^{2(L+1)})^{1/2}, \\ f_{31}(N, M, L) &= \mu^{N+L}. \end{aligned}$$

Note that for $0 < \mu < 1$, the above expressions have canonical meaning even when N, M or L is ∞ . Thus we can factor the homomorphism $\tilde{\pi}_3$ through $C^*(\mathfrak{G})$, i.e. there exists a homomorphism

$$\eta : C(S_\mu U(3)) \rightarrow C^*(\mathfrak{G})$$

such that $\tilde{\pi}_3 = \tilde{\rho}_3 \circ \eta$. We shall see later that η is in fact injective since all the representations $\tilde{\pi}$ of $C(S_\mu U(3))$ mentioned above can be factored through η .

Let us consider the following invariant subsets of the unit space of \mathfrak{G} ,

$$\begin{aligned} X_3 &= \{(N, M, L) | N, M, L \in \mathbb{Z}_{\geq}\} = \mathbb{Z}_{\geq}^3, \\ X_{21} &= \{(N, M, L) | N, M \in \mathbb{Z}_{\geq} \text{ and } L = \infty\} \cong \mathbb{Z}_{\geq}^2, \\ X_{22} &= \{(N, M, L) | N, L \in \mathbb{Z}_{\geq} \text{ and } M = \infty\} \cong \mathbb{Z}_{\geq}^2, \\ X_{11} &= \{(N, M, L) | N \in \mathbb{Z}_{\geq} \text{ and } M = L = \infty\} \cong \mathbb{Z}_{\geq}, \\ X_{12} &= \{(N, M, L) | M \in \mathbb{Z}_{\geq} \text{ and } N = L = \infty\} \cong \mathbb{Z}_{\geq} \end{aligned}$$

and $X_0 = \{(\infty, \infty, \infty)\}$. We define $X_i = X_{i1} \cup X_{i2}$ for $i = 1, 2$, and σ_i (resp. σ_{in}) to be the quotient map from $C^*(\mathfrak{G}|\overline{X}_{i+1})$ to $C^*(\mathfrak{G}|\overline{X}_i)$ (resp. $C^*(\mathfrak{G}|\overline{X}_{in})$) for $i = 0, 1, 2$, (resp. $i = 1, 2$ and $n = 1, 2$) where \overline{X}_i is the closure of X_i in the unit space of \mathfrak{G} . Since $\tilde{\pi}_3(u_{ij})\tilde{\pi}_3(u_{ij}^*) = f_{ij}^2$ for the u_{ij} 's listed above and they separate points in $\mathbb{Z}_{\geq} \times \overline{\mathbb{Z}}_{\geq}^2$, i.e. points (N, M, L) with $N < \infty$, it is easy to check that $C_c(X_3) = C_c(\mathbb{Z}_{\geq}^3) \subseteq \text{Im}(\eta)$ (by considering the level sets of these f_{ij} 's). Now since those weights f_{ij} are non-vanishing on \mathbb{Z}_{\geq}^3 and $C_c(\mathbb{Z}_{\geq}^3) \subseteq \text{Im}(\eta)$, the convolution algebra $C_c(\mathbb{Z}_{\geq}^3 \times_{\alpha} \mathbb{Z}^5)$ and hence $C^*(\mathbb{Z}_{\geq}^3 \times_{\alpha} \mathbb{Z}^5) \cong C(\mathbb{T}^2) \otimes \mathcal{K}$ are contained in the C^* -algebra generated by (the weighted shifts) $\eta(u_{ij}^*)$ of the u_{ij}^* 's listed above and hence in $\text{Im}(\eta)$ where \mathcal{K} is the algebra of compact operators (on $l^2(\mathbb{Z}_{\geq}^3)$ here).

Now we consider the diagonal homomorphism $(\sigma_{21}, \sigma_{22})$ from $C^*(\mathfrak{G})$ to $C^*(\mathfrak{G}|\overline{X}_{21}) \oplus C^*(\mathfrak{G}|\overline{X}_{22})$. It is easy to see that $\mathfrak{G}|\overline{X}_{2n} \cong \overline{\mathbb{Z}}^2 \times_{\alpha(2, n)} \mathbb{Z}^5 | \overline{\mathbb{Z}}_{\geq}^2$ where \mathbb{Z}^5 acts on $\overline{\mathbb{Z}}^2$ through the action $\alpha(2, n)$ in the way that 2 components (depending on n) of \mathbb{Z}^5 act on $\overline{\mathbb{Z}}^2$ by

translation while the other 3 components act trivially. More precisely, $\alpha(2, 1)(\mu) \cdot \nu = \nu - (\mu_1, \mu_2)$ and $\alpha(2, 2)(\mu) \cdot \nu = \nu - (\mu_1, \mu_3)$ for $\mu \in \mathbb{Z}^5$ and $\nu \in \mathbb{Z}^2$. Thus

$$C^*(\mathfrak{G}|\bar{X}_{2n}) \cong C^*(\bar{\mathbb{Z}}^2 \times_{\tau} \mathbb{Z}^2 | \bar{\mathbb{Z}}_{\geq 2}^2) \otimes C^*(\mathbb{Z}^3) \cong C^*(\bar{\mathbb{Z}}^2 \times_{\tau} \mathbb{Z}^2 | \bar{\mathbb{Z}}_{\geq 2}^2) \otimes C(\mathbb{T}^3),$$

where the canonical generators of \mathbb{Z}^3 are e_3, e_4, e_5 when $n = 1$, and e_2, e_4, e_5 when $n = 2$. It is straightforward to check that $(\sigma_{21} \circ \eta)(u_{ij})$'s ($1 \leq i, j \leq 3$) are supported in $\bar{\mathbb{Z}}^2 \times_{\alpha(2,1)} \mathbb{Z}^4 | \bar{\mathbb{Z}}_{\geq 2}^2$ where \mathbb{Z}^4 is generated by e_1, e_2, e_3 and e_5 in \mathbb{Z}^5 , while $(\sigma_{22} \circ \eta)(u_{ij})$'s are supported in $\bar{\mathbb{Z}}^2 \times_{\alpha(2,2)} \mathbb{Z}^4 | \bar{\mathbb{Z}}_{\geq 2}^2$ with \mathbb{Z}^4 generated by e_1, e_2, e_3 and e_4 in \mathbb{Z}^5 . Furthermore, from the weight functions f_{ij} listed above, it is easy to check that $C_c(X_2) \subseteq \text{Im}(\sigma_2 \circ \eta)$ and hence

$$C^*(\mathbb{Z}^2 \times_{\alpha(2,1)} \mathbb{Z}^4 | \mathbb{Z}_{\geq 2}^2) \oplus C^*(\mathbb{Z}^2 \times_{\alpha(2,2)} \mathbb{Z}^4 | \mathbb{Z}_{\geq 2}^2) \cong 2\mathcal{A} \otimes C(\mathbb{T}^2)$$

is contained in the C^* -algebra generated by $(\sigma_{21}, \sigma_{22})(\eta(u_{ij}^*))$ and hence in $\text{Im}((\sigma_{21}, \sigma_{22}) \circ \eta)$. Let ρ_2 be the faithful regular representation of $\bar{\mathbb{Z}}^2 \times_{\tau} \mathbb{Z}^2 | \bar{\mathbb{Z}}_{\geq 2}^2$ on $l^2(\mathbb{Z}_{\geq 2}^2)$ and $\tilde{\rho}_{2n} = \rho_2 \otimes m$ be the corresponding faithful representation of

$$C^*(\bar{\mathbb{Z}}^2 \times_{\alpha(2,n)} \mathbb{Z}^4 | \bar{\mathbb{Z}}_{\geq 2}^2) \cong C^*(\bar{\mathbb{Z}}^2 \times_{\tau} \mathbb{Z}^2 | \bar{\mathbb{Z}}_{\geq 2}^2) \otimes C(\mathbb{T}^2)$$

on $l^2(\mathbb{Z}_{\geq 2}^2) \otimes L^2(\mathbb{T}^2)$, where the isomorphism identifies e_3, e_5 with $\bar{\phi}, \bar{\psi}$ if $n = 1$, and identifies e_4, e_2 with $\bar{\phi}, \bar{\psi}$ if $n = 2$. Then it can be easily checked that

$$\tilde{\rho}_{2n}(\sigma_{2n}(\eta(u_{ij}))) = \tilde{\pi}_{2n}(u_{ij})$$

(note that in the above identification, the symbols N and M used in [B] need be interchanged when $n = 2$) and hence $\tilde{\pi}_{2n}$ factors through η . Let $\eta_{2n} := \sigma_{2n} \circ \eta$.

Now we consider $\sigma_{12} \circ \sigma_2$ and $\sigma_{11} \circ \sigma_2$. Since clearly $\sigma_{12} \circ \sigma_2$ factors through σ_{21} and $\sigma_{11} \circ \sigma_2$ factors through σ_{21} and σ_{22} , we may talk about $\sigma_{12} \circ \sigma_{21}$ ($= \sigma_{12} \circ \sigma_2$) and $\sigma_{11} \circ \sigma_{21} = \sigma_{11} \circ \sigma_{22}$ ($= \sigma_{11} \circ \sigma_2$) by abuse of language. Note that

$$C^*(\mathbb{Z}^2 \times_{\alpha(2,1)} \mathbb{Z}^4 | \mathbb{Z}_{\geq 2}^2) \oplus C^*(\mathbb{Z}^2 \times_{\alpha(2,2)} \mathbb{Z}^4 | \mathbb{Z}_{\geq 2}^2) \subseteq C^*(\mathfrak{G}|\bar{X}_2) \subseteq \ker(\sigma_{1n})$$

because $(\mathbb{Z}^2 \times_{\alpha(2,1)} \mathbb{Z}^4 | \mathbb{Z}_{\geq 2}^2) \cup (\mathbb{Z}^2 \times_{\alpha(2,2)} \mathbb{Z}^4 | \mathbb{Z}_{\geq 2}^2) \subseteq X_2$. It is again easy to see that $\mathfrak{G}|\bar{X}_{1n} \cong \bar{\mathbb{Z}} \times_{\alpha(1,n)} \mathbb{Z}^5 | \bar{\mathbb{Z}}_{\geq}$ where \mathbb{Z}^5 acts on $\bar{\mathbb{Z}}$ through the action $\alpha(1, n)$ in the way that one component (depending on n) of \mathbb{Z}^5 act on $\bar{\mathbb{Z}}$ by translation while the other 4 components act trivially.

More precisely, $\alpha(1, 1)(\mu) \cdot \nu = \nu - \mu_1$ and $\alpha(1, 2)(\mu) \cdot \nu = \nu - \mu_2$ for $\mu \in \mathbb{Z}^5$ and $\nu \in \mathbb{Z}$. Thus

$$C^*(\mathfrak{G}|\bar{X}_{1n}) \cong C^*(\bar{\mathbb{Z}} \times_{\tau} \mathbb{Z}|\bar{\mathbb{Z}}_{\geq}) \otimes C^*(\mathbb{Z}^4) \cong C^*(\bar{\mathbb{Z}} \times_{\tau} \mathbb{Z}|\bar{\mathbb{Z}}_{\geq}) \otimes C(\mathbb{T}^4),$$

where the canonical generators of \mathbb{Z}^4 are e_2, e_3, e_4, e_5 when $n = 1$, and e_1, e_3, e_4, e_5 when $n = 2$. It is straightforward to check that $(\sigma_{11} \circ \sigma_2 \circ \eta)(u_{ij})$'s ($1 \leq i, j \leq 3$) are supported in $\bar{\mathbb{Z}} \times_{\alpha(1,1)} \mathbb{Z}^3|\bar{\mathbb{Z}}_{\geq}$ where \mathbb{Z}^3 is generated by e_1, e_2 and e_3 in \mathbb{Z}^5 , while the $(\sigma_{12} \circ \sigma_2 \circ \eta)(u_{ij})$'s are supported in $\bar{\mathbb{Z}} \times_{\alpha(1,2)} \mathbb{Z}^4|\bar{\mathbb{Z}}_{\geq}$ with \mathbb{Z}^4 generated by e_1, e_2, e_3 and e_5 in \mathbb{Z}^5 . Let ρ_1 be the faithful regular representation of $\bar{\mathbb{Z}} \times_{\tau} \mathbb{Z}|\bar{\mathbb{Z}}_{\geq}$ on $l^2(\mathbb{Z}_{\geq})$ and $\tilde{\rho}_{11} = \rho_1 \otimes m$ be the corresponding faithful representation of

$$C^*(\bar{\mathbb{Z}} \times_{\alpha(1,1)} \mathbb{Z}^3|\bar{\mathbb{Z}}_{\geq}) \cong C^*(\bar{\mathbb{Z}} \times_{\tau} \mathbb{Z}|\bar{\mathbb{Z}}_{\geq}) \otimes C(\mathbb{T}^2)$$

on $l^2(\mathbb{Z}_{\geq}^2) \otimes L^2(\mathbb{T}^2)$, where the isomorphism identifies e_3 and e_2 with $\bar{\phi}$ and $\bar{\psi}$ respectively. Then it can be easily checked that

$$\tilde{\rho}_{11}((\sigma_{11} \circ \sigma_2 \circ \eta)(u_{ij})) = \tilde{\pi}_{11}(u_{ij})$$

and hence $\tilde{\pi}_{11}$ factors through η and $\eta_{11} := \sigma_{11} \circ \sigma_2 \circ \eta = \sigma_{11} \circ \eta_{21} = \sigma_{11} \circ \eta_{22}$. On the other hand, we have

$$C^*(\bar{\mathbb{Z}} \times_{\alpha(1,2)} \mathbb{Z}^4|\bar{\mathbb{Z}}_{\geq}) \cong C^*(\bar{\mathbb{Z}} \times_{\tau} \mathbb{Z}|\bar{\mathbb{Z}}_{\geq}) \otimes C(\mathbb{T}^3),$$

where the conjugates of the three canonical coordinate functions of \mathbb{T}^3 correspond to the generators e_1, e_3 and e_5 in \mathbb{Z}^5 . Composing the above identification with $\text{id} \otimes \kappa_{12}$, we get a homomorphism λ_{12} from $C^*(\bar{\mathbb{Z}} \times_{\alpha(1,2)} \mathbb{Z}^4|\bar{\mathbb{Z}}_{\geq})$ to $C^*(\bar{\mathbb{Z}} \times_{\tau} \mathbb{Z}|\bar{\mathbb{Z}}_{\geq}) \otimes C(\mathbb{T}^2)$, where κ_{12} is the homomorphism from $C(\mathbb{T}^3)$ to $C(\mathbb{T}^2)$ induced by the map from \mathbb{T}^2 to \mathbb{T}^3 sending $z \in \mathbb{T}^2$ to $(z_1, -z_1, z_2)$. Let $\tilde{\rho}_{12} = \rho_1 \otimes m$ be the faithful representation of $C^*(\bar{\mathbb{Z}} \times_{\tau} \mathbb{Z}|\bar{\mathbb{Z}}_{\geq}) \otimes C(\mathbb{T}^2) \supseteq \text{Im}(\eta_{12})$, where $\eta_{12} = \lambda_{12} \circ (\sigma_{12} \circ \sigma_2 \circ \eta) = \lambda_{12} \circ (\sigma_{12} \circ \sigma_{21} \circ \eta)$. (Here we use the convention that $f \circ g$ is meaningful whenever $\text{Im}(g) \subseteq \text{Dom}(f)$.) Then $\tilde{\rho}_{12} \circ \lambda_{12}$ defines a representation of $\text{Im}(\sigma_{12} \circ \sigma_2 \circ \eta)$ on $l^2(\mathbb{Z}_{\geq}) \otimes L^2(\mathbb{T}^2)$. It is straightforward to check that

$$(\tilde{\rho}_{12} \circ \lambda_{12})((\sigma_{12} \circ \sigma_2 \circ \eta)(u_{ij})) = \tilde{\pi}_{12}(u_{ij})$$

(note that in [B], M is replaced by N) for all i, j . From the weight functions f_{ij} listed above, it is easy to check that $C_c(X_1) \subseteq \text{Im}(\sigma_1 \circ \sigma_2 \circ \eta)$. So by the formulas for $\pi_{1n}(u_{ij})$ in [B], it is not hard to see that

$$\begin{aligned} & C^*(\mathbb{Z} \times_{\alpha(1,1)} \mathbb{Z}^3|\mathbb{Z}_{\geq}) \oplus \lambda_{12}(C^*(\mathbb{Z} \times_{\alpha(1,2)} \mathbb{Z}^4|\mathbb{Z}_{\geq})) \\ & \cong 2C^*(\mathbb{Z} \times_{\tau} \mathbb{Z}|\mathbb{Z}_{\geq}) \otimes C(\mathbb{T}^2) \cong 2\mathcal{K} \otimes C(\mathbb{T}^2) \end{aligned}$$

is contained in the C^* -algebra generated by $(\eta_{11}, \eta_{12})(u_{ij}^*)$ and hence in $\text{Im}((\eta_{11}, \eta_{12}))$. Notice that

$$C^*(\mathbb{Z} \times_{\alpha(1,1)} \mathbb{Z}^3 | \mathbb{Z}_{\geq}) \oplus C^*(\mathbb{Z} \times_{\alpha(1,2)} \mathbb{Z}^4 | \mathbb{Z}_{\geq}) \subseteq C^*(\mathfrak{G} | X_1)$$

is contained in the kernel of σ_0 .

Now we consider $\sigma_0 \circ \sigma_1 \circ \sigma_2$. Since $\sigma_0 \circ \sigma_1 \circ \sigma_2$ clearly factors through $\sigma_{11} \circ \sigma_2$ and $\sigma_{12} \circ \sigma_2$, we may talk about $\sigma_0 \circ \sigma_{11} \circ \sigma_2 = \sigma_0 \circ \sigma_{12} \circ \sigma_2 = \sigma_0 \circ \sigma_1 \circ \sigma_2$ by abuse of language. Note that $C^*(\mathfrak{G} | X_0) = C^*(\mathbb{Z}^5) \cong C(\mathbb{T}^5)$ and that $(\sigma_0 \circ \sigma_1 \circ \sigma_2 \circ \eta)(u_{ij})$'s ($1 \leq i, j \leq 3$) are supported in \mathbb{Z}^3 generated by e_1, e_2 and e_3 in \mathbb{Z}^5 . Composing the identification $C^*(\mathbb{Z}^3) \cong C(\mathbb{T}^3)$ with κ_0 (where the generators e_1, e_2, e_3 are identified with the conjugates of the corresponding coordinate functions of \mathbb{T}^3), we get a homomorphism λ_0 from $C^*(\mathbb{Z}^3)$ to $C(\mathbb{T}^2)$, where κ_0 is the homomorphism from $C(\mathbb{T}^3)$ to $C(\mathbb{T}^2)$ induced by the map from \mathbb{T}^2 to \mathbb{T}^3 sending $z \in \mathbb{T}^2$ to $(z_1, z_2, -z_1)$. Let $\tilde{\rho}_0 := m$. Then $\tilde{\rho} \circ \lambda_0$ is a representation of $C^*(\mathbb{Z}^3)$ on $L^2(\mathbb{T}^2)$. It is straightforward to check that

$$(\tilde{\rho}_0 \circ \eta_0)(u_{ij}) = \tilde{\pi}_0(u_{ij})$$

for all i, j , where $\eta_0 = \lambda_0 \circ \sigma_0 \circ \sigma_1 \circ \sigma_2 \circ \eta$ is a homomorphism from $C(S_\mu U(3))$ to $C^*(\mathbb{Z}^2) \cong C(\mathbb{T}^2)$. Comparing the definitions of κ_{12} and κ_0 and relating the generators of their domains $C^*(\mathbb{Z}^3)$ to those of \mathbb{Z}^5 as we specified above, it is easy to check that η_0 factors through η_{11} and η_{12} , say $\eta_0 = \tilde{\omega}_0 \circ (\eta_{11}, \eta_{12})$ for some $\tilde{\omega}_0$ defined on $\text{Im}(\eta_{11}, \eta_{12})$. Note that $\ker(\tilde{\omega}_0)$ contains the subalgebra

$$C^*(\mathbb{Z} \times_{\alpha(1,1)} \mathbb{Z}^3 | \mathbb{Z}_{\geq}) \oplus \lambda_{12}(C^*(\mathbb{Z} \times_{\alpha(1,2)} \mathbb{Z}^4 | \mathbb{Z}_{\geq})) \cong 2\mathcal{K} \otimes C(\mathbb{T}^2).$$

Now we summarize what we have so far. There are homomorphisms $\eta_3 = \eta, \eta_{21}, \eta_{22}, \eta_{11}, \eta_{12}$ and η_0 from $C(S_\mu U(3))$ to

$$\begin{aligned} C^*(\mathfrak{G}) &= C^*(\overline{\mathbb{Z}}^3 \times_\alpha \mathbb{Z}^5 | \overline{\mathbb{Z}}_{\geq}^3) = C(\overline{\mathbb{Z}}^3 \times_\tau \mathbb{Z}^3 | \overline{\mathbb{Z}}_{\geq}^3) \otimes C(\mathbb{T}^2), \\ C^*(\overline{\mathbb{Z}}^2 \times_\tau \mathbb{Z}^2 | \overline{\mathbb{Z}}_{\geq}^2) &\otimes C(\mathbb{T}^2), \quad C^*(\overline{\mathbb{Z}}^2 \times_\tau \mathbb{Z}^2 | \overline{\mathbb{Z}}_{\geq}^2) \otimes C(\mathbb{T}^2), \\ C^*(\overline{\mathbb{Z}} \times_\tau \mathbb{Z} | \overline{\mathbb{Z}}_{\geq}) &\otimes C(\mathbb{T}^2), \quad C^*(\overline{\mathbb{Z}} \times_\tau \mathbb{Z} | \overline{\mathbb{Z}}_{\geq}) \otimes C(\mathbb{T}^2) \quad \text{and} \quad C(\mathbb{T}^2), \end{aligned}$$

respectively, such that

(1) each η_i or η_{in} factors through η_j with $j > i$, where $\eta_i := (\eta_{i1}, \eta_{i2})$ if $i = 1, 2$. In fact, $\eta_{21} = \omega_{21} \circ \eta$, $\eta_{22} = \omega_{22} \circ \eta$, $\eta_{11} = \omega_{11} \circ \eta_{21}$, $\eta_{11} = \omega'_{11} \circ \eta_{22}$, $\eta_{12} = \omega_{12} \circ \eta_{21}$, $\eta_0 = \omega_0 \circ \eta_{11}$ and $\eta_0 = \omega'_0 \circ \eta_{12}$ for some ω 's defined on the range of the corresponding η 's.

(2) Let $\eta_i = \tilde{\omega}_i \circ \eta_{i+1}$ for a suitable homomorphism $\tilde{\omega}_i$ defined on $\text{Im}(\eta_{i+1})$. Then $\text{ker}(\tilde{\omega}_i)$ contains a copy of $C(\mathbb{T}^2) \otimes \mathcal{H}$ if $i = 2$, and contains two copies of $C(\mathbb{T}^2) \otimes \mathcal{H}$ if $i = 0$ or 1 . Furthermore, $\text{Im}(\eta_0) \cong C(\mathbb{T}^2)$. Note that $\text{Ker}(\eta_i) = \eta_{i+1}^{-1}(\text{Ker}(\tilde{\omega}_i))$.

(3) $\tilde{\pi}_i = \tilde{\rho}_i \circ \eta_i$ ($i = 0, 3$) and $\tilde{\pi}_{in} = \tilde{\rho}_{in} \circ \eta_{in}$ ($i = 1, 2$) for some faithful representations $\tilde{\rho}_i$ and $\tilde{\rho}_{in}$ on $\text{Im}(\eta_i)$ and $\text{Im}(\eta_{in})$ respectively. Since the irreducible representations of $C(S_\mu U(3))$ are classified by those 2-parameter families of $\pi_0, \pi_{11}, \pi_{12}, \pi_{21}, \pi_{22}$, and π_3 , the spectrum of $C(S_\mu U(3))$ is a disjoint union of 6 copies of \mathbb{T}^2 as a set. On the other hand, by (1)–(3), all these representations π_i 's (or π_{in} 's) factor through η_j (or η_{jn}) with $j > i$ and hence $\eta = \eta_3$ is faithful. Thus, the type I C^* -algebra $C(S_\mu U(3))$ has a composition sequence

$$0 \subseteq \mathcal{I}_3 = \text{Ker}(\eta_2) \subseteq \mathcal{I}_2 = \text{Ker}(\eta_1) \subseteq \mathcal{I}_1 = \text{Ker}(\eta_0) \subseteq \mathcal{I}_0 = C(S_\mu U(3))$$

such that $\mathcal{I}_3 = \text{Ker}(\tilde{\omega}_2)$, $\mathcal{I}_2/\mathcal{I}_3 \cong \text{Ker}(\tilde{\omega}_1)$, $\mathcal{I}_1/\mathcal{I}_2 \cong \text{Ker}(\tilde{\omega}_0)$ and $\mathcal{I}_0/\mathcal{I}_1 \cong \text{Im}(\eta_0) \cong C(\mathbb{T}^2)$. Note that $C(Y_{i+1}) \otimes \mathcal{H}(\mathcal{H}) \subseteq \text{Ker}(\tilde{\omega}_i) \subseteq \text{Im}(\eta_{i+1}) \subseteq C(Y_{i+1}) \otimes \mathcal{B}(\mathcal{H})$ (for some L^2 -space \mathcal{H}), where Y_k is homeomorphic to \mathbb{T}^2 if $k = 3$ or 0 , and to the disjoint union of 2 copies of \mathbb{T}^2 if $k = 2$ or 1 . If $C(Y_{i+1}) \otimes \mathcal{H}(\mathcal{H}) \neq \text{Ker}(\tilde{\omega}_i)$, then we have non-trivial irreducible representations of $\text{Ker}(\tilde{\omega}_i)/C(Y_{i+1}) \otimes \mathcal{H}(\mathcal{H})$ which will induce irreducible representations of $C(S_\mu U(3))$ not unitarily equivalent to any of the π 's found in [B]. So we have $C(Y_{i+1}) \otimes \mathcal{H}(\mathcal{H}) = \text{Ker}(\tilde{\omega}_i)$.

We summarize what we obtained about the structure of the C^* -algebra $C(S_\mu U(3))$ in the following theorem.

THEOREM. *The C^* -algebra $C(S_\mu U(3))$ of the twisted SU(3) group has the composition sequence*

$$\mathcal{I}_3 \subseteq \mathcal{I}_2 \subseteq \mathcal{I}_1 \subseteq \mathcal{I}_0 = C(S_\mu U(3))$$

such that

$$\mathcal{I}_0/\mathcal{I}_1 \cong C(\mathbb{T}^2), \quad \mathcal{I}_1/\mathcal{I}_2 \cong \mathcal{I}_2/\mathcal{I}_3 \cong 2C(\mathbb{T}^2) \otimes \mathcal{H}$$

and $\mathcal{I}_3 \cong C(\mathbb{T}^2) \otimes \mathcal{H}$.

We remark that the above decomposition of $C(S_\mu U(3))$ is compatible with the singular foliation of the Lie-Poisson SU(3) [Lu-We] by the symplectic leaves [We]. More precisely, there are six 2-parameter families (with parameters in \mathbb{T}^2) of symplectic leaves diffeomorphic to $\mathbb{C}^0, \mathbb{C}^1, \mathbb{C}^1, \mathbb{C}^2, \mathbb{C}^2$ and \mathbb{C}^3 , respectively as pointed out by

A. Weinstein in a private communication. With each leaf of positive dimension quantized by the Weyl quantization [Hö, Vo], it is likely that we can find a deformation quantization (in the sense of [Ri1]) of the Poisson $SU(3)$ as we did for the case of Poisson $SU(2)$ in [S]. In a sense as explained in [S], $C(S_\mu U(3))$ can be regarded as a foliation C^* -algebra of the (singular) symplectic foliation on $SU(3)$.

With some more effort to analyse the data obtained, we are able to describe the topology of the spectrum Y of $C(S_\mu U(3))$. In order to do so, we shall say that a copy of \mathbb{T}^2 approximates another copy of \mathbb{T}^2 in a topological space in type ... if any sequence in the first \mathbb{T}^2 converges to any element in the second \mathbb{T}^2 , and in type ____, —, > or =, if a sequence $z(n)$ in the first \mathbb{T}^2 converges to w in the second \mathbb{T}^2 if and only if $z(n)_2 \rightarrow w_2$, $z(n)_1 \rightarrow w_1$, $z(n)_1 z(n)_2 \rightarrow \bar{w}_2$ or $z(n)_1 z(n)_2 \rightarrow w_1 w_2$ respectively. Now clearly Y is a union of the above Y_k 's, and by a more detailed analysis of the factorizability among η 's than the one specified in (1), we can conclude that Y is a disjoint union of $Y_0, Y_{11}, Y_{12}, Y_{21}, Y_{22}$ and Y_3 (each homeomorphic to \mathbb{T}^2) such that (i) Y_3 is open dense in Y in the way that Y_3 approximates $Y_{21}, Y_{22}, Y_{11}, Y_{12}$ and Y_0 in type ____, —, ..., ..., and ... , respectively, (ii) Y_{21} and Y_{22} are disjoint open sets with dense union $Y_2 = Y_{21} \cup Y_{22}$ in $Y \setminus Y_3$ such that Y_{21} approximates Y_{11}, Y_{12} , and Y_0 in type —, = and ... respectively, and Y_{22} approximates $Y_{11} Y_{12}$ and Y_0 in type ____ > and ... respectively ($Y_{12} \cap \bar{Y}_{22} = \emptyset$), (iii) Y_{11} and Y_{12} are disjoint open sets with dense union $Y_1 = Y_{11} \cup Y_{12}$ in $Y \setminus (Y_3 \cup Y_2)$ such that Y_{11} and Y_{12} approximating Y_0 in type = and ____ respectively.

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