

# Pacific Journal of Mathematics

**MULTI-TUPLE HULLS**

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## MULTI-TUPLE HULLS

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**We study two general families of hulls related to vector-valued functions and their interrelations with multi-tuple Shilov boundaries of uniform algebras.**

**1. Introduction.** The classical hulls—polynomial, rational, holomorphic,  $A$ -convex etc. are tightly and naturally connected with functional approximations and interpolations. Recently several new families of hulls have come into appearance. Namely Basener [2] has used a generalization of the family of polynomial hulls in his study of  $q$ -holomorphic functions. Recently Slodkowski [8] has used a generalization of rational hulls in his investigation of analytic perturbation of Taylor spectrum, and Corach and Suárez [3] have introduced a general family of rational hulls in order to evaluate the topological stable rank of some algebras. In [11] there were investigated the properties of two families of hulls, the so called  *$n$ -tuple rational  $A$ -convex hulls* and  *$n$ -tuple  $A$ -convex hulls*.

The multi-tuple Shilov boundaries of commutative Banach algebras have proved to be essential tools in the investigation of multi-dimensional analytic structures in algebra spectra. Results concerning relationships between these boundaries and the analyticity in algebra spectra have appeared often during the last fifteen years (e.g. Basener [1], Kumagai [5], Sibony [6], Tonev ([11], [12]) etc.). Various properties of multi-dimensional Shilov boundaries have been investigated by Basener [1], [2], Sibony [6], Slodkowski ([7], [8]), Tonev ([10], [11]) and others.

In this paper we establish a unified approach to the above mentioned families of hulls, study their properties and investigate their interrelations with the vector valued functions and multi-tuple Shilov boundaries of uniform algebras.

**2. Multi-tuple rational  $A$ -convex hulls.** Let  $A$  be a *uniform algebra* over  $\mathbb{C}$  with unit. That is  $A$  is a separating closed subalgebra of the space of all continuous complex valued functions on some compact Hausdorff space  $X$  which contains the constants and the norm of

$f \in A$  is the maximum of  $|f(x)|$  on  $X$ . As usual  $\text{sp } A$  denotes the maximal ideal space of  $A$  and  $\hat{f}$  denotes the Gelfand extension of a given function  $f$  of  $A$ . The Shilov boundary  $\partial A$  of  $A$  is the smallest closed subset of  $\text{sp } A$  on which the Gelfand extensions of all functions of  $A$  assume the maximums of their absolute values. Throughout this paper we shall assume that  $\text{sp } A$  is identified with the set  $X$  and that the Gelfand extensions  $\hat{f}(m)$  are identified with the algebra elements  $f(m)$ .  $A^n$  will denote the set of all  $n$ -tuples of functions from  $A$ .

**DEFINITION 1.** The  $n$ -tuple rational  $A$ -convex hull  $r_n(E)$  of a subset  $E$  of  $\text{sp } A$  is the biggest among all closed subsets  $K$  of  $\text{sp } A$  for which the equality

$$(1) \quad \min_{x \in K} \|F(x)\| = \min_{x \in E} \|F(x)\|$$

holds for every  $n$ -tuple  $F = (f_1, \dots, f_n)$  of functions in  $A$ .  $E$  is called  $n$ -tuple rationally  $A$ -convex if  $r_n(E) = E$ .

Obviously  $r_n(E)$  is a closed subset of  $\text{sp } A$ . One can see that  $r_n(E) = \{m \in \text{sp } A : \|S(m)\| \geq \min_{m \in E} \|S(m)\| \text{ for every } S \subset A \text{ with } \#S \leq n\}$ . Naturally, the last inequality is essential for regular subsets  $S$  of  $A$  only. As a corollary from this observation we get that  $E \subset \dots \subset r_{n+1}(E) \subset r_n(E) \subset \dots \subset r_1(E) \subset \text{sp } A$ .

The next proposition gives a useful characterization of the hulls  $r_n(E)$ .

**PROPOSITION 1.** The  $n$ -tuple rational  $A$ -convex hull of a subset  $E$  of  $\text{sp } A$  coincides with the set  $r_n(E) = \{m \in \text{sp } A : F(m) \in F(E) \text{ for all } F \in A^n\}$ .

*Proof.* Denote for a while the set  $\{m \in \text{sp } A : F(m) \in F(E) \text{ for all } F \in A^n\}$  by  $K$ . Let  $m_0 \in r_n(E)$  and let  $F \in A^n$  be such that  $F(m_0) = 0$ . By (1)  $F$  vanishes within  $E$  so that  $0 = F(m_0) \in F(E)$ . Hence  $K \supset r_n(E)$ . Assuming conversely that  $K \setminus r_n(E) \neq \emptyset$ , for any point  $m_0 \in K \setminus r_n(E)$  we have  $\|F(m_0)\| < \min_{m \in E} \|F(m)\|$  for some  $F \in A^n$ . Hence  $H(m_0) = 0$  but  $0 \notin H(E)$  for  $H = F - F(m_0) \in A^n$  in contradiction with  $m_0 \in K$ . The proposition is proved.

The  $n$ -tuple rational  $A$ -convex hull is a multi-tuple version of the rational  $A$ -convex hull  $r(E) = \{m \in \text{sp } A : f(m) \in f(E) \text{ for all } f \in A\}$  of any subset  $E$  of the spectrum of a uniform algebra  $A$ —namely, as it is easy to check, the 1-tuple rational  $A$ -convex hull  $r_1(E)$  coincides with  $r(E)$ .

From Proposition 1 it follows that the  $n$ -tuple rational  $A$ -convex hulls coincide with the sets  $\bigcap_{F \in A^n} F^{-1} \circ F(E)$ , i.e. with the *generalized rational hulls* introduced by Corach and Suárez in [3] and utilized by them in their recent investigations on topological stable ranks of commutative Banach algebras.

Since  $F(m) \in F(E)$  if and only if the function  $H = F - F(m)$  vanishes on  $E$ , Proposition 1 implies the following

**PROPOSITION 2.** *The  $n$ -tuple rational  $A$ -convex hull  $r_n(E)$  of a closed subset  $E$  of  $\text{sp } A$  coincides with the set of these points  $m$  in  $\text{sp } A$  such that every  $n$ -tuple  $F \in A^n$  with  $F(m) = 0$  vanishes on  $E$ .*

In general the  $n$ -tuple rational  $A$ -convex hull of a subset of  $\text{sp } A$  does not coincide with the algebra spectrum. For instance if  $A$  is the disc-algebra  $A(\Delta)$ , then  $r_1(S^1) = r_1(\partial A) = S^1 \neq \bar{\Delta} = \text{sp } A$ , as one can see by applying, say, (1) to the identity function in  $\mathbb{C}^1$ . In this respect the following corollary from Proposition 2 is of some interest.

**COROLLARY 1 [3].**  *$r_n(E) = \text{sp } A$  if and only if every  $n$ -tuple  $F(m)$  over  $A$  which does not vanish on  $E$  is regular.*

**EXAMPLE 1.** If  $n \geq 2$ , then the 1-tuple rational  $A(B^n(1))$ -convex hull of the unit sphere  $S^n(1)$  in  $\mathbb{C}^n$  is the unit ball  $B^n(1)$ .

Indeed, as known, if  $n \geq 2$  any holomorphic function vanishes on  $S^n(1)$  whenever it vanishes inside  $B^n(1)$ .

**EXAMPLE 2.** The  $n$ -tuple rational convex hulls  $\rho_n(E)$ .

Let  $\Lambda$  be an arbitrary set and let  $\mathbb{C}^\Lambda$  be the Cartesian product of  $\Lambda$  copies of the complex plane, equipped by the natural topology. Given a compact subset  $E$  in  $\mathbb{C}^\Lambda$  let  $R(E)$  be the closure in  $C(E)$  of all rational functions  $p/q$  in  $\mathbb{C}^\Lambda$  with non-vanishing on  $E$  denominators  $q$ . Since these rational functions are dense in  $R(E)$ , the  $n$ -tuple rational  $R(E)$ -convex hull of  $E$  is the biggest among all compact subsets  $N$  of  $\mathbb{C}^\Lambda$ , such that  $(r_1, \dots, r_n)(N) = (r_1, \dots, r_n)(E)$  for every  $n$ -tuple  $(r_1, \dots, r_n)$  of rational functions  $r_j = p_j/q_j$  in  $\mathbb{C}^\Lambda$  with  $q_j \neq 0$  on  $N$ . We shall refer to this hull as  *$n$ -tuple rational convex hull* of  $E$  and shall denote it by  $\rho_n(E)$ .

Being the  $n$ -tuple rational  $R(E)$ -convex hull of  $E$ ,  $\rho_n(E)$  is the biggest among all sets  $N$  in  $\mathbb{C}^\Lambda$  for which the inequality

$$\inf_{z \in N} \|(r_1(z), \dots, r_n(z))\| = \min_{z \in E} \|(r_1(z), \dots, r_n(z))\|$$

holds for every  $n$ -tuple  $(r_1, \dots, r_n)$  of rational functions in  $\mathbb{C}^\Lambda$  with non-vanishing on  $E$  denominators.

A subset  $E \subset \mathbb{C}^\Lambda$  is  $n$ -tuple rationally convex if  $\rho_n(E) = E$ . The  $n$ -tuple rational convex hulls  $\rho_n(E)$  are natural generalizations of the usual rational convex hulls  $r(E) = \{z \in \mathbb{C}^\Lambda : |r(z)| \leq \max_{z \in E} |r(z)|\}$  for every rational function  $r(z)$  that is bounded on  $E$ —namely by Corollary 1 one can observe that the 1-tuple rational convex hull  $\rho_1(E)$  of a set  $E \subset \mathbb{C}^\Lambda$  coincides with  $r(E)$ .

By applying the relation from Proposition 1 to the identity mapping in  $\mathbb{C}^n$  we get

**PROPOSITION 3.** *Every compact set  $E$  in  $\mathbb{C}^n$  is  $k$ -tuple rationally convex for any  $k \geq n$ .*

As we shall see below,  $\rho_n(E)$  coincides with the  $n$ -tuple rational  $P(E)$ -convex hull of  $E$ , i.e. the  $n$ -tuple rational  $A$ -convex hulls of all compact subsets  $E$  in  $\mathbb{C}^\Lambda$  are equal for both algebras  $A = P(E)$  and  $A = R(E)$ .

**PROPOSITION 4.** *The  $n$ -tuple rational hull  $\rho_n(E)$  of every compact subset  $E$  in  $\mathbb{C}^\Lambda$  is equal to its  $n$ -tuple rational  $P(E)$ -convex hull.*

*Proof.* Denote for a while the  $n$ -tuple rational  $A$ -convex hull of a set by  $r_n^A(E)$ . Clearly  $r_n^{P(E)}(E) \supset r_n^{R(E)}(E) = \rho_n(E)$ , because  $P(E) \subset R(E)$ . If we assume that  $r_n^{P(E)}(E) \setminus \rho_n(E) \neq \emptyset$ , by Proposition 3 we can find a point  $m_0$  in  $r_n^{P(E)}(E)$  and an  $n$ -tuple  $F \in R^n(E)$  such that  $F(m_0) = 0$ , but  $\|F(m)\| \neq 0$  on  $E$ . If  $F = (f_1, \dots, f_n)$ ,  $f_j = p_j/q_j$ , where  $p_j, q_j$  are polynomials,  $q_j \neq 0$  on  $E$ , then  $(p_1(m_0), \dots, p_n(m_0)) = 0$  but the  $n$ -tuple  $(p_1, \dots, p_n)$  does not vanish on  $E$ . Proposition 3 indicates that this contradicts the choice of the point  $m_0 \in r_n^{P(E)}(E)$ .

In the case  $n = 1$  Proposition 4 restricts to the well known equality between the rational hull and the  $P(E)$ -convex hull of a set  $E \in \text{sp } A$ , i.e.  $r(E) = h^{P(E)}(E)$  (e.g. [4]), which by the way motivates the names “ $n$ -tuple rational hull” and “ $n$ -tuple rational  $A$ -convex hull” given to the sets  $\rho_n(E)$  and  $r_n^A(E)$  respectively.

Together with Proposition 2, Proposition 4 implies the following

**COROLLARY 2.** *The  $n$ -tuple rational convex hull  $\rho_n(E)$  of a compact subset  $E$  of  $\mathbb{C}^\Lambda$  coincides with the set of all points  $z \in \mathbb{C}^\Lambda$  such that for*

every  $n$ -tuple of polynomials  $(p_1, \dots, p_n)$  vanishing at  $\mathbf{z}$  the variety  $\{\mathbf{y} \in \mathbb{C}^\Lambda : p_j(\mathbf{y}) = 0, j = 1, \dots, n\}$  meets  $E$ .

The sets described in Corollary 2 are precisely the  $(n - 1)$ -th rational hulls which were introduced by Slodkowski in [8] and used by him in his recent investigation of analytic perturbation of Taylor spectrum of  $n$ -tuples of commuting operators. Note that, as shown in [3],  $\bigcap_{n \geq 1} r_n(E) = E$ .

Let  $V(f_1, \dots, f_n)$  be the vanishing set of a fixed  $n$ -tuple  $(f_1, \dots, f_n)$  over  $A$ , i.e.

$$V(f_1, \dots, f_n) = \{m \in \text{sp } A : f_1(m) = f_2(m) = \dots = f_n(m) = 0\}.$$

Denote by  $A_E$  the closure in  $C(E)$  of restrictions of all elements of  $A$  on a fixed closed subset  $E$  of  $\text{sp } A$ .

**THEOREM 1.** *Let  $k$  be a fixed integer,  $1 \leq k \leq n - 1$ . The  $n$ -tuple rational  $A$ -convex hull  $r_n(E)$  of a closed subset  $E$  of  $\text{sp } A$  coincides with the set of these points  $m \in \text{sp } A$  which belong to the  $k$ -tuple rational  $A_{V(S)}$ -convex hulls  $r_k(E \cap V(S))$  of the sets  $E \cap V(S)$  for every set  $S$  in  $A$  such that  $\#S \leq n - k$  and with  $m$  belonging to  $V(S)$ .*

*Proof.* Let  $K = \{m \in \text{sp } A : m \in r_k(E \cap V(S)) \text{ for every } S \subset A \text{ with } \#S \leq n - k \text{ and } m \in V(S)\}$ . Suppose that  $K \setminus r_n(E) \neq \emptyset$  and let  $m_0 \in K \setminus r_n(E)$ . By Definition 1 we can find an  $n$ -tuple  $F = (f_1, \dots, f_n)$  over  $A$  such that  $\|F(m_0)\| < \min_{m \in E} \|F(m)\|$ . By applying, if necessary, an orthogonal transformation in  $\mathbb{C}^n$ , we can suppose from the beginning that  $F(m_0) = (f_1(m_0), \dots, f_k(m_0), 0, \dots, 0)$ . Hence  $m_0 \in V(S)$  where  $S = (f_{k+1}, \dots, f_n)$ . For  $T = (f_1, \dots, f_k)$  we have  $\|T(m_0)\| = \|F(m_0)\| < \min_{m \in E} \|F(m)\| \leq \min_{E \cap V(S)} \|F(m)\| = \min_{E \cap V(S)} \|T(m)\| = \min_{r_k(E \cap V(S))} \|T(m)\|$ . Consequently  $m_0 \notin r_k(E \cap V(S))$  in contradiction with  $m_0 \in K$ . We conclude that  $K \subset r_n(E)$ .

Suppose conversely that  $r_n(E) \setminus K \neq \emptyset$  and let  $m_0 \in r_n(E) \setminus K$ . Let  $S$  be an  $(n - k)$ -tuple over  $A$  such that  $m_0 \in V(S) \setminus r_k(E \cap V(S))$ . Consequently there exists a  $k$ -tuple  $T \in A_{V(S)}^k$ , such that  $\|T(m_0)\| < r = \min_{E \cap V(S)} \|T(m)\|$ . Without loss of generality we can assume that  $T \in A^k$ . For every positive  $\varepsilon < r$  we can find a neighborhood  $V_\varepsilon$  of the set  $E \cap V(S)$  in  $E$  on which  $\|T(m)\| > r - \varepsilon$ . Hence for every  $m \in E$  we have

$$(2) \quad (C_\varepsilon^2 \|S(m)\|^2 + \|T(m)\|^2)^{1/2} > r - \varepsilon$$

for some positive constant  $C_\varepsilon$  large enough. Because  $(C_\varepsilon S, T) \in A^n$ , (2) holds also on  $r_n(E)$ . In particular at  $m_0 \in V(S)$  we have  $\|T(m_0)\| > r - \varepsilon$  and henceforth  $\|T(m_0)\| \geq r$  because of the liberty of the choice of  $\varepsilon$ . Since this contradicts with the initial inequality  $\|T(m_0)\| < r$ , we conclude that  $r_n(E) \subset K$ . The theorem is proved.

The case  $k = 1$  from Theorem 1 in particular says

**COROLLARY 3.** *The  $n$ -tuple rational  $A$ -convex hull  $r_n(E)$  of a closed set  $E$  in  $\text{sp} A$  coincides with the set of these points  $m \in \text{sp} A$  which belong to the rational  $A_{V(S)}$ -convex hulls  $r(E \cap V(S))$  of the sets  $E \cap V(S)$  for any set  $S$  in  $A$  whose cardinality does not exceed  $n - 1$  and such that  $m$  belongs to  $V(S)$ , i.e.*

$$r_n(E) = \{m \in \text{sp} A : m \in r(E \cap V(S)) \text{ for any } S \subset A \text{ with } \#S \leq n - 1, m \in V(S)\}.$$

**3. Multi-tuple  $A$ -convex hulls.** In this section we introduce another family of hulls in algebra spectra by putting some limitations on the  $n$ -tuples  $F$  from (1). Recall that an  $n$ -tuple  $F = (f_1, \dots, f_n) \in A^n$  is called *regular* if the functions  $f_1, \dots, f_n$  have no common zeros on  $\text{sp} A$ , i.e. if  $V(f_1, \dots, f_n) = \emptyset$ , or in other words, if the mapping  $F$  does not vanish on  $\text{sp} A$ .

**DEFINITION 2.** The  $n$ -tuple  $A$ -convex hull  $h_n(E)$  of a closed subset  $E$  of  $\text{sp} A$  is the union of all closed subsets  $N$  of  $\text{sp} A$  which contain  $E$  and such that the inequality

$$(3) \quad \min_{m \in N} \|F(m)\| = \min_{m \in E} \|F(m)\|$$

holds for every non-vanishing on  $N$   $n$ -tuple  $F$  of functions from  $A$ .  $E$  is an  $n$ -tuple  $A$ -convex set if it coincides with its  $n$ -tuple  $A$ -convex hull  $h_n(E)$ .

The  $n$ -tuple  $A$ -convex hulls are closed subsets of  $\text{sp} A$ , since the closure  $[N]$  of a set  $N \in \text{sp} A$  that satisfies (3) also satisfies (3). In fact  $h_n(E)$  coincides with the union of all subsets  $N$  of  $\text{sp} A$  which satisfy (3).

The next proposition, which proof is analogical to that of Proposition 1, gives a useful characterization of the hulls  $h_n(E)$ .

**PROPOSITION 5.** *Let  $E$  be a closed subset of  $\text{sp} A$ . The  $n$ -tuple  $A$ -convex hull  $h_n(E)$  of  $E$  coincides with the biggest among all subsets*

$N$  of  $\text{sp } A$  which contain  $E$  and for which

$$(4) \quad bF(N) \subset F(E)$$

for every  $F \in A^n$ .

*Proof* (see also [11]). First we show that if  $N = h_n(E)$  then (3) holds for every non-vanishing on  $h_n(E)$  mapping  $F \in A^n$ . Clearly  $0 \notin F(E)$  for every such  $F \in A^n$ . If  $c = \min_{m \in E} \|F(m)\|$  then  $F(E) \subset \mathbb{C}^n \setminus B(c) = \{z \in \mathbb{C}^n : \|z\| \geq c\}$  and by the definition of  $h_n(E)$  we have that  $bF(h_n(E)) \subset \mathbb{C}^n \setminus B(c)$ . This implies  $F(h_n(E)) \subset \mathbb{C}^n \setminus B(c)$ . Consequently

$$\min_{m \in h_n(E)} \|F(m)\| \geq c = \min_{m \in E} \|F(m)\|.$$

Since the opposite inequality is obviously fulfilled, we conclude that (3) holds with  $N = h_n(E)$  for every  $F \in A^n$  with  $\|F(m)\| \neq 0$  on  $h_n(E)$ .

Let now  $N$  be a closed subset of  $\text{sp } A$  which satisfies (3) for every  $F \in A^n$  that does not vanish on  $N$ , and assume that (4) is false, i.e. that  $bF(N) \setminus F(E) \neq \emptyset$ . Let  $z_0$  be a point from  $bF(N) \setminus F(E)$  and let  $m_0 \in F^{-1}(z_0)$ . Obviously

$$(5) \quad \|H(m_0)\| < \min_{m \in E} \|H(m)\|$$

for the  $n$ -tuple  $H = F - z_0$ . We can find also a point  $z_1 \in \mathbb{C}^n \setminus F(N)$  close enough to  $z_0$ , such that (5) holds for the  $n$ -tuple  $H_1 = F - z_1$ . But this contradicts to (3) since obviously  $H_1(m)$  does not vanish on  $N$ . The proposition is proved.

As the following example shows, the  $n$ -tuple  $A$ -convex hulls  $h_n(E)$  are natural multi-tuple versions of the  $A$ -convex hulls  $h(E) = \{m \in \text{sp } A : |f(m)| \leq \max_{m \in E} |F(m)| \text{ for all } f \in A\}$  of closed sets  $E$  in  $\text{sp } A$ . Recall that  $h(E)$  consists of all linear multiplicative functionals of  $A$  that possess continuous extensions on  $A_E$ . A set  $E \in \text{sp } A$  is  $A$ -convex if  $h(E) = E$ , i.e. if  $E = \{m \in \text{sp } A : |f(m)| \leq \max_{m \in E} |f(m)| \text{ for all } f \in A\}$ , or, equivalently, if  $\text{sp } A_E = E$  (e.g. [4]). The vanishing set  $V(S)$  of any subset  $S$  of  $A$  is a simple example for an  $A$ -convex set.

**EXAMPLE 3.** The 1-tuple  $A$ -convex hull  $h_1(E)$  of each closed subset  $E$  of  $\text{sp } A$  coincides with its usual  $A$ -convex hull  $h(E)$ .

Indeed, since  $bf(h_1(E)) \subset f(E)$ , we have that  $\max_{m \in h_1(E)} |f(m)| \leq \max_{m \in E} |f(m)|$  for every function  $f \in A$  and consequently  $h_1(E) \subset h(E)$  by the definition of the  $A$ -convex hull  $h(E)$ . Assume that  $h(E)$  contains properly  $h_1(E)$ . Then there exists a function  $f$  from  $A$  such that  $bf(h(E)) \not\subset f(E)$ . Let  $m_0$  be a point from  $h(E)$ , such that  $f(m_0) \in bf(h(E)) \setminus f(E)$ . By choosing a point  $z_0$  from  $\mathbb{C} \setminus f(h(E))$  close enough to  $f(m_0)$ , we can construct a function  $g(z) = f(z) - z_0 \in A_{h(E)}^{-1}$  for which  $|g(m_0)| < \min_{m \in E} |g(m)|$ . Hence  $1/|g(m_0)| > \max_{m \in E} 1/|g(m)|$ . Since  $1/g \in A_{h(E)}$  and  $\text{sp} A_{h(E)} = h(E)$ , there exists a function  $g_1$  from  $A$  such that  $|g_1(m_0)| > \max_{m \in E} |g_1(m)|$ , i.e.  $m_0 \notin h(E)$  in contradiction with the choice of  $m_0$ .

In particular we obtain that  $h(E)$  is the biggest among all closed subsets  $N$  of  $\text{sp} A$  such that  $bf(N) \subset f(E)$  for every  $f \in A$ .

**EXAMPLE 4.** The  $n$ -tuple polynomial convex hulls  $\pi_n(E)$ .

Given a compact subset  $E$  in  $\mathbb{C}^\Lambda$  denote by  $P(E)$  the closure in  $C(E)$  of the set of all polynomials in  $\mathbb{C}^\Lambda$ . Since the polynomials are dense in  $P(E)$ , the  $n$ -tuple  $P(E)$ -convex hull of  $E$  is the biggest among all closed subsets  $N$  of  $\mathbb{C}^\Lambda$  such that  $b(p_1, \dots, p_n)(N) \subset (p_1, \dots, p_n)(E)$  for every  $n$ -tuple  $(p_1, \dots, p_n)$  of polynomials in  $\mathbb{C}^\Lambda$ . We shall refer to this hull as  $n$ -tuple polynomial convex hull of  $E$  and shall denote it by  $\pi_n(E)$ . Clearly  $E \subset \rho_n(E) \subset \pi_n(E)$  for every compact subset  $E$  in  $\mathbb{C}^\Lambda$ .

Being the  $n$ -tuple  $P(E)$ -convex hull of  $E$ ,  $\pi_n(E)$  is the biggest among all sets  $N$  in  $\mathbb{C}^\Lambda$  which contain  $E$  and such that the inequality

$$\inf_{z \in N} \|(p_1(z), \dots, p_n(z))\| = \min_{z \in E} \|(p_1(z), \dots, p_n(z))\|$$

holds for every non-vanishing on  $n$   $n$ -tuple  $(p_1, \dots, p_n)$  of polynomials in  $\mathbb{C}^\Lambda$ .

A subset  $E \subset \mathbb{C}^\Lambda$  is  $n$ -tuple polynomially convex if it coincides with its  $n$ -tuple polynomial hull  $\pi_n(E)$ . By applying the inclusion

$$b(p_1, \dots, p_n)(\pi_n(E)) \subset (p_1, \dots, p_n)(E)$$

to the identity mapping in  $\mathbb{C}^n$  we get that  $b(\pi_n(E)) \subset E$  for every compact subset  $E$  in  $\mathbb{C}^n$ . This implies that the  $n$ -tuple polynomial convex hull  $\pi_n(E)$  of  $E$  is contained in the union of  $E$  and the union of all bounded components of its complement in  $\mathbb{C}^n$ . Because of its maximality property,  $\pi_n(E)$  actually coincides with this union. Therefore  $E$  is  $n$ -tuple polynomially convex if and only if its complement  $\mathbb{C}^n \setminus E$  does not possess bounded components.

The  $n$ -tuple polynomial convex hulls  $\pi_n(E)$  of subsets  $E$  of  $\mathbb{C}^\Lambda$  are natural generalizations of their usual polynomial hulls  $\widehat{E} = \{z \in \mathbb{C}^\Lambda : |p(z)| \leq \max_{z \in E} |p(z)| \text{ for every polynomial } p \text{ in } \mathbb{C}^\Lambda\}$ —as Example 3 shows,  $\pi_1(E) = \widehat{E}$ . In general hulls  $\pi_n(E)$  are different from the usual polynomial hulls  $\widehat{E}$ . For instance the 2-tuple polynomial hull  $\pi_2(E)$  of the set  $E = \{(z_1, z_2) \in \mathbb{C}^2 : 1 \leq |z_1| \leq 2, |z_2| = 0\} \subset \mathbb{C}^2$  is the set  $E$  itself, which does not coincide with the usual polynomial hull  $\widehat{E} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq 2, |z_2| = 0\}$ . Indeed, for any  $z_0 \notin E$  we can choose an  $\varepsilon > 0$  small enough, so that the regular pair of polynomials  $(z_1 - z_0, z_2 + \varepsilon)$  to attain the minimum of its norm near  $z_0$  and outside  $E$  at the same time. This means that  $z_0 \notin \pi_2(E)$  for any  $z_0 \notin E$ , i.e. that  $\pi_2(E) \subset E$  and hence  $\pi_2(E) = E$  since  $\pi_2 \supset E$  by Definition 2.

As we know,  $r_n(E) \subset h_n(E)$  for every closed subset  $E$  of  $\text{sp } A$ . The following corollary establishes a somewhat opposite inclusion.

**COROLLARY 4.** *Let  $E$  be a closed subset of the spectrum of a uniform algebra  $A$ . Then  $h_n(E) \subset r_{n-1}(E)$ .*

*Proof* (see also [11]). If  $m_0 \notin r_{n-1}(E)$ , then by Definition 1 we can find an  $(n - 1)$ -tuple  $F = (f_1, \dots, f_{n-1})$  of functions from  $A$  with  $\|F(m_0)\| < \min_{m \in E} \|F(m)\|$ . The regular  $n$ -tuple  $(f_1, \dots, f_n, 1) = (F, 1)$  therefore satisfies the inequality

$$\|F(m_0)\|^2 + 1 < \min_{m \in E} \|F(m)\|^2 + 1.$$

Definition 2 indicates that  $m_0 \notin h_n(E)$ . We conclude that  $h_n(E) \subset r_{n-1}(E)$ , as claimed.

Observe that  $F(h_n(E)) \subset \pi_n(F(E))$  for any  $F \in A^n$  because  $F(E) \supset bF(h_n(E))$  and hence  $h_n(E) \subset \bigcap_{F \in A^n} F^{-1}(\pi_n(F(E)))$ . In fact both sets are equal. Indeed, denote the latter set by  $K$  and take an  $F \in A^n$  with  $\|F(m)\| \neq 0$  on  $K$ . Clearly  $\|z\| \geq \min_{m \in E} \|F(m)\|$  for every  $z \in \pi_n(F(E))$  because  $F(m)$  does not vanish on the set  $E \subset K$ . Thus  $\|F(m_0)\| \geq \min_{m \in E} \|F(m)\|$  for every point  $m_0$  in  $F^{-1}(\pi_n(F(E)))$  and therefore for every point  $m_0$  in  $K$  as well. Hence  $h^n(E) \supset K$ , i.e.

**COROLLARY 5.**  $h_n(E) = \bigcap_{F \in A^n} F^{-1}(\pi_n(F(E)))$ .

The next theorem is an  $n$ -tuple  $A$ -convex analogue of Corollary 3. Its proof follows the same lines as the proof of the case  $k = 1$  of Theorem 1.

**THEOREM 2.** *The  $n$ -tuple  $A$ -convex hull  $h_n(E)$  of a closed set  $E$  in  $\text{sp } A$  coincides with the set of these points  $m \in \text{sp } A$  which belong to the  $A_{V(S)}$ -convex hulls  $h(E \cap V(S))$  of the sets  $E \cap V(S)$  for any set  $S$  in  $A$  whose cardinality does not exceed  $n-1$  and such that  $m$  belongs to  $V(S)$ , i.e.*

$$h_n(E) = \{m \in \text{sp } A : m \in h(E \cap V(S)) \text{ for any } S \subset A \\ \text{with } \#S \leq n-1, m \in V(S)\}.$$

*Proof.* Denote for a while the set  $\{m \in \text{sp } A : m \in h(E \cap V(S)) \text{ for any } S \subset A \text{ with } \#S \leq n-1, m \in V(S)\}$  by  $K$  and suppose that  $K \setminus h_n(E) \neq \emptyset$ . By Definition 2 we can find an  $n$ -tuple  $F = (f_1, \dots, f_n) \subset A^n$  which does not vanish on  $K$  and such that  $\|F(m_0)\| < \min_{m \in E} \|F(m)\|$  for some  $m_0 \in K \setminus h_n(E)$ .

Without loss of generality (applying, if necessary, an orthogonal transformation in  $\mathbb{C}^n$ ) we can assume from the beginning that  $F(m_0) = (f_1(m_0), 0, \dots, 0)$ . Hence  $m_0 \in V(S)$  for  $S = (f_2, \dots, f_n)$  and  $f_1$  does not vanish on  $K \cap V(S)$ . Then  $|f_1(m_0)| = \|F(m_0)\| < \min_{m \in E} \|F(m)\| \leq \min_{E \cap V(S)} \|F(m)\| = \min_{E \cap V(S)} |f_1(m)|$ , i.e.  $\max_{m \in E \cap V(S)} |g(m)| < |g(m_0)|$  for  $g = 1/f_1 \in A_{V(S)}$ . Consequently  $m_0 \notin h(E \cap V(S))$  in contradiction with  $m_0 \in K$ . We conclude that  $K \subset h_n(K)$ .

Suppose conversely that  $h_n(E) \setminus K \neq \emptyset$  and let  $m_0 \in h_n(E) \setminus K$ . Let  $S$  be an  $(n-1)$ -tuple over  $A$  such that  $m_0 \in V(S) \setminus h(E \cap V(S))$ . Consequently there exists a function  $f \in A_{V(S)}$ , such that  $|f(m_0)| > \max_{E \cap V(S)} |f(m)|$ . Without loss of generality we can assume that  $f$  does not vanish on  $\text{sp } A$ . For  $g = 1/f \in A^{-1}$  we have  $|g(m_0)| < r = \min_{E \cap V(S)} |g(m)|$ . For any positive  $\varepsilon < r$  we can find a neighborhood  $V_\varepsilon$  of the set  $E \cap V(S)$  in  $E$  on which  $|g(m)| > r - \varepsilon$ . Hence for every  $m \in E$  we have

$$(5) \quad \left( C_\varepsilon^2 \sum_{j=1}^{n-1} |f_j(m)|^2 + |g(m)|^2 \right)^{1/2} > r - \varepsilon$$

for some positive constant  $C_\varepsilon$  large enough. (5) holds also on  $h_n(E)$  because the  $n$ -tuple  $(C_\varepsilon f_1, \dots, C_\varepsilon f_{n-1}, g)$  does not vanish on  $h_n(E)$ . In particular at  $m_0$  we have  $|g(m_0)| > r - \varepsilon$  and henceforth  $\|g(m_0)\| \geq r$  because of the liberty of the choice of  $\varepsilon$ . Since this contradicts with the initial inequality  $|g(m_0)| < r$ , we conclude that  $h_n(E) \subset K$ . The theorem is proved.

The sets described in Theorem 2 are precisely the hulls which were considered by Basener in [2] and used by him in his study of  $q$ -holomorphic functions.

**THEOREM 3.** *Let  $E$  be a closed subset of  $\text{sp } A$ . Then  $E \subset \dots \subset h_{n+1}(E) \subset h_n(E) \subset \dots \subset h_1(E) = h(E) \subset \text{sp } A$  and  $\bigcap_{n \geq 1} h_n(E) = E$ .*

*Proof.* The first part of the statement is obvious because if (3) is fulfilled for every  $(n + 1)$ -tuple  $F$  of functions from  $A$  which does not vanish on  $K$ , then it is fulfilled also for every  $n$ -tuple which does not vanish on  $K$ .

The inclusion  $\bigcap_n h_n(E) \supset E$  is clear. If  $m_0 \notin E$  then for every  $m$  in  $E$  we can find a function  $f_m \in A^{-1}$  such that  $|f_m(m)| > 1$  and  $|f_m(m_0)| < 1$ . Let  $U_m$  be an open neighborhood of  $m$  such that  $|f_m(x)| > 1$  for any  $x \in U_m$ . By a compactness argument there exist finitely many points  $m_1, \dots, m_k$  in  $E$  such that  $E \subset U_{m_1} \cup \dots \cup U_{m_k}$ . By replacing each  $f_{m_j}$  by some of its power we can assume from the beginning that  $|f_{m_j}(m_0)| < 1/k$  and  $|f_{m_j}(x)| > 1$  on  $U_{m_j}$ . Hence for  $F = (f_{m_1}, \dots, f_{m_k}) \in A^k$  we have that  $\|F(m)\| \neq 0$  on  $\text{sp } A$ ,  $\|F(m)\| > 1$  on  $E$  and  $\|F(m_0)\| < 1$ . Hence  $m_0 \notin h_k(E)$  and moreover  $m_0 \notin \bigcap_n h_n(E)$ . We conclude that  $\bigcap_n h_n(E) \subset E$ , as required.

#### 4. Multi-tuple Shilov boundaries and multi-tuple hulls.

**DEFINITION 3** (Basener, Sibony). The  $n$ -tuple Shilov boundary  $\partial^{(n)}A$  of a commutative Banach algebra  $A$  is the following subset of  $\text{sp } A$ :

$$\partial^{(n)}A = \left[ \bigcup \{ \partial A_{V(f_1, \dots, f_{n-1})} : (f_1, \dots, f_{n-1}) \in A^{n-1} \} \right],$$

where  $A^0 = \{0\}$  (see [1], [6], also [13]).

It is easy to check that  $\partial A = \partial^{(1)}A \subset \partial^{(2)}A \subset \partial^{(3)}A \subset \dots \subset \partial^{(n)}A \subset \dots \subset \text{sp } A$ . We shall recall some of the basic properties of multi-tuple Shilov boundaries.

**THEOREM 4** ([10, Theorem 1]).  $\partial^{(n)}A$  is the smallest closed subset of  $\text{sp } A$  on which every regular  $n$ -tuple  $F = (f_1, \dots, f_n) \in A$  assumes the minimum of its norm

$$(6) \quad \|F(m)\| = \left( \sum_{j=1}^n \|\hat{f}_j(m)\|^2 \right)^{\frac{1}{2}}, \quad m \in \text{sp } A.$$

In other words Theorem 4 says that  $\partial^{(n)}A$  is the smallest closed subset of  $\text{sp } A$  on which every regular  $n$ -tuple  $F = (f_1, \dots, f_n) \in A$  assumes the minimum of its norm (6).

**THEOREM 5** ([10, Theorem 3]).  $\partial^{(n)}A$  is the smallest closed subset of  $\text{sp } A$  such that the inclusion

$$(7) \quad F(\partial^{(n)}A) \supset bF(\text{sp } A)$$

holds for every  $n$ -tuple  $F = (f_1, \dots, f_n) \in A^n$ .

The next proposition in particular says that, in the case of algebra  $A = A(B^n(1))$ ,  $r_k(S^n(1)) = B^n(1)$  for any  $k < n$  unlike the case  $k = n$  when, according to Proposition 3,  $S^n(1)$  is  $n$ -tuple rationally  $A$ -convex and hence  $r_n(S^n(1)) = S^n(1)$ .

**PROPOSITION 6.**  $r_k(\partial^{(n)}A) = \text{sp } A$  for each  $k < n$ ; if  $r_n(\partial^{(n)}A) = \text{sp } A$  then  $r_n(E) \neq \text{sp } A$  for every proper closed subset  $E$  of  $\partial^{(n)}A$ .

Indeed, as shown in [10],  $V(G) \cap \partial^{(n)}A \neq \emptyset$  for every irregular  $k$ -tuple  $G \in A^k$ ,  $1 \leq k \leq n - 1$ . From Proposition 2 we conclude that  $r_k(\partial^{(n)}A) = \text{sp } A$  for every  $k < n$ . If  $E \neq \partial^{(n)}A$  then there is an  $n$ -tuple  $F \in A^n$  with  $\min_{m \in E} \|F(m)\| > \min_{m \in \text{sp } A} \|F(m)\| = 0$  and consequently, by Definition 1,  $r_n(E) \neq r_n(\partial^{(n)}A) = \text{sp } A$ .

**PROPOSITION 7.**  $h_n(E) = \text{sp } A$  if and only if  $E$  contains the  $n$ -tuple Shilov boundary  $\partial^{(n)}A$  of  $A$ .

Indeed, from Definition 2 and Theorem 4 it follows that  $h_n(\partial^{(n)}A) = \text{sp } A$ . Theorem 4 shows that  $h_n(E) \neq \text{sp } A$  if  $E \setminus \partial^{(n)}A \neq \emptyset$ .

Given an  $n$ -tuple  $(f_1, \dots, f_n) \in A^n$  let  $\sigma(f_1, \dots, f_n)$  be the joint spectrum of the  $n$ -tuple  $(f_1, \dots, f_n)$ , i.e.

$$\sigma(f_1, \dots, f_n) = \{(f_1(m), \dots, f_n(m)) : m \in \text{sp } A\}.$$

**PROPOSITION 8.** The joint spectrum of every  $n$ -tuple  $F$  over  $A$  is contained in the  $n$ -tuple polynomial hull of the set  $F(\partial^{(n)}A)$ , i.e.

$$\sigma(F) = F(\text{sp } A) \subset \pi_n(F(\partial^{(n)}A)), \quad F \in A^n.$$

*Proof* (see also [11]). Since by the Theorem 4

$$\min_{m \in \partial^{(n)}A} \|G(m)\| = \min_{m \in \text{sp } A} \|G(m)\|$$

for every regular  $n$ -tuple  $G \in A^n$ , the equality

$$\begin{aligned} & \min_{m \in \partial^{(n)}A} \|(p_1 \circ F(m), \dots, p_n \circ F(m))\| \\ &= \min_{m \in \text{sp } A} \|(p_1 \circ F(m), \dots, p_n \circ F(m))\| \end{aligned}$$

and, equivalently,

$$\min_{m \in F(\partial^{(n)}A)} \|(p_1, \dots, p_n)(\mathbf{z})\| = \min_{m \in F(\text{sp } A)} \|(p_1, \dots, p_n)(\mathbf{z})\|$$

are fulfilled for every  $n$ -tuple of polynomials  $p_1, \dots, p_n$  in  $\mathbf{C}^n$  without joint zeros on  $\sigma(F) = F(\text{sp } A)$ . Definition 2 indicates that the sets  $F(\partial^{(n)}A)$  and  $\sigma(F)$  have equal  $n$ -tuple polynomial hulls and consequently  $\sigma(F) \subset \pi_n(F(\partial^{(n)}A))$ , as claimed.

A well known theorem from the uniform algebra theory says that if an algebra is generated (linearly) by its subset  $\Lambda$ , then the range of its Shilov boundary  $\partial A$  via the *spectral mapping*  $\widehat{\Lambda} : \text{sp } A \rightarrow \mathbf{C}^\Lambda : m \mapsto \{f(m) : f \in \Lambda\}$  of  $\Lambda$  is the smallest closed subset of  $\mathbf{C}^\Lambda$  whose polynomial hull is equal to the polynomial hull of the set  $\widehat{\Lambda}(\text{sp } A)$ . In the next theorem we make use of the  $n$ -tuple  $A$ -convex boundaries in order to obtain an extension of this result for the multi-tuple Shilov boundaries  $\partial^{(n)}A$ . Namely

**THEOREM 6.** *Let  $S = \{b_\lambda\}_{\lambda \in \Lambda}$  be a set which generates linearly a uniform algebra  $A$ . Then the range  $\widehat{\Lambda}(\partial^{(n)}A)$  of the  $n$ -tuple Shilov boundary via  $\widehat{\Lambda}$  is the smallest among all compact subsets  $E$  in  $\mathbf{C}^\Lambda$  whose  $n$ -tuple polynomial hulls  $\pi_n(E)$  are equal to the  $n$ -tuple polynomial hull  $\pi_n(\widehat{\Lambda}(\text{sp } A))$  of the range of  $\widehat{\Lambda}$ .*

*Proof* (see also [11]). Without loss of generality we can assume that  $E$  is a subset of  $\widehat{\Lambda}(\text{sp } A)$  and consequently that  $E = \widehat{\Lambda}(K)$  for some compact set  $K \in \text{sp } A$ . The  $n$ -tuple polynomial hulls  $\pi_n(E) = \pi_n(\widehat{\Lambda}(K))$  and  $\pi_n(\widehat{\Lambda}(\text{sp } A))$  are equal if and only if  $\min_{\mathbf{z} \in \widehat{\Lambda}(K)} \|P(\mathbf{z})\| = \min_{\mathbf{z} \in \widehat{\Lambda}(\text{sp } A)} \|P(\mathbf{z})\|$  for every  $n$ -tuple  $P = (p_1, \dots, p_n)$  of polynomials in  $\mathbf{C}^\Lambda$  with  $\|P(\mathbf{z})\| \neq 0$  on  $\pi_n(\widehat{\Lambda}(\text{sp } A))$ . Equivalently,  $\pi_n(\widehat{\Lambda}(K)) = \pi_n(\widehat{\Lambda}(\text{sp } A))$  if and only if  $\min_{m \in K} \|P \circ \widehat{\Lambda}(m)\| = \min_{m \in \text{sp } A} \|P \circ \widehat{\Lambda}(m)\|$  for any  $n$ -tuple of type  $P \circ \widehat{\Lambda} \in A^n$  which does not vanish on the set  $\widehat{\Lambda}^{-1}(h_n(\widehat{\Lambda}(\text{sp } A))) = \text{sp } A$ . Since the set of functions  $p_j \circ \widehat{\Lambda}(m)$  is dense in  $A$ ,  $\pi_n(\widehat{\Lambda}(K)) = \pi_n(\widehat{\Lambda}(\text{sp } A))$  if and only if  $\min_{m \in K} \|F(m)\| = \min_{m \in \text{sp } A} \|F(m)\|$  for every regular  $n$ -tuple  $F \in A^n$ . By Theorem

4 the  $n$ -tuple Shilov boundary  $\partial^{(n)}A$  is the smallest closed subset of  $\text{sp}A$  with the last property; and therefore  $\widehat{\Lambda}(\partial^{(n)}A)$  is the smallest closed subset of  $\mathbb{C}^\Lambda$  whose  $n$ -tuple polynomial hull is the same as the  $n$ -tuple polynomial hull of the set  $\widehat{\Lambda}(\text{sp}A)$ , as claimed.

**5. Remarks.** Recall that *function space* is called any linear subspace of the space  $C(X)$ , where  $X$  is a compact Hausdorff space, which is closed under the *uniform norm*  $\|f(x)\| = \max_{x \in X} |f(x)|$ , contains the constants and separates the points of  $X$ . It can be shown that for every function space  $B$  over  $X$  there exists a smallest set  $\text{Sh}_B(X)$  among all closed subsets  $E$  of  $X$  such that

$$\min_{x \in X} |f(x)| = \min_{x \in E} |f(x)|$$

for every *nonvanishing* on  $X$  function  $f$  in  $B$ . In general  $\text{Sh}_B(X)$  does not coincide with the usual Shilov boundary  $\partial B$  of  $B$  which, by definition is the smallest among all closed subsets  $E$  of  $X$  such that

$$\max_{x \in X} |f(x)| = \max_{x \in E} |f(x)|$$

for every function  $f$  in  $B$ . The  $n$ -tuple Shilov boundary of a function space  $B$  is the set

$$\text{Sh}_{B^n}(X) = \left[ \bigcup \text{Sh}_B(V(F)) : F = (f_1, \dots, f_{n-1}) \in B^{n-1} \right],$$

where  $V(F) = (f_1, \dots, f_{n-1})^{-1}(\mathbf{0})$ .

It is a matter of a simple verification to check that all the above results hold not only for uniform algebras but for function spaces as well with the Shilov boundary  $\partial A$  replaced with the boundary  $\text{Sh}_B(X)$  and the  $n$ -tuple Shilov boundary  $\partial^{(n)}A$  replaced with the  $n$ -tuple Shilov boundary  $\text{Sh}_{B^n}(X)$ .

The author is grateful to the referée for his or her useful suggestions.

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Received June 4, 1990 and in revised form January 17, 1991.

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The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) is published monthly except for July and August. Second-class postage paid at Carmel Valley, California 93924, and additional mailing offices. Postmaster: send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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# Pacific Journal of Mathematics

Vol. 151, No. 2      December, 1991

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