

Pacific Journal of Mathematics

**ANY BLASCHKE MANIFOLD OF THE HOMOTOPY TYPE OF
 CP^n HAS THE RIGHT VOLUME**

CHUNG-TAO YANG

ANY BLASCHKE MANIFOLD OF THE HOMOTOPY TYPE OF CP^n HAS THE RIGHT VOLUME

C. T. YANG

Dedicated to Professor S. S. Chern

The aim of this paper is to prove the result stated in the title.

By a *Blaschke manifold* [1, p. 135], we mean a connected closed Riemannian manifold which has the property that the cut locus of each of its points, when viewed in the tangent space, is a round sphere of a constant radius. It is well known that in any Blaschke manifold, all geodesics are smoothly simply closed and have the same length. The canonical examples of a Blaschke manifold are the unit n -sphere S^n , the real, complex, quaternionic projective n -spaces RP^n , CP^n , HP^n and the Cayley projective plane CaP^2 with their standard Riemannian metric. These Blaschke manifolds will be referred to as the *canonical Blaschke manifolds*. For general informations on Blaschke manifolds, see [1].

The Blaschke conjecture says that *any Blaschke manifold, up to a constant factor, is isometric to a canonical Blaschke manifold*. This conjecture looks plausible, because it has been shown in [3, 7] that any Blaschke manifold either is diffeomorphic to S^n or RP^n , or is of the homotopy type of CP^n , or is a 1-connected closed manifold having the integral cohomology ring of HP^n or CaP^2 . However, so far it has been proved only for spheres and real projective spaces [2, 6, 8, 9].

One crucial step in the proof of the Blaschke conjecture for spheres is to show that any Blaschke manifold diffeomorphic to S^n has the right volume. Hence we formulate the weak Blaschke conjecture [10] which says that *any Blaschke manifold has the right volume*.

Let M be a d -dimensional Blaschke manifold, UM the space of unit tangent vectors of M and CM the space of oriented closed geodesics in M . Then UM and CM are oriented connected smooth manifolds and there is a natural oriented smooth circle bundle $\pi: UM \rightarrow CM$. In [8], it is shown that, if e denotes the Euler class of this

circle bundle, then

$$i(M) = \frac{1}{2} \langle e^{d-1}, [CM] \rangle$$

(i.e., one half of the value of e^{d-1} at the fundamental homology class $[CM]$ of CM) is an integer, called the *Weinstein integer* of M , and that, if ℓ denotes the length of closed geodesics in M , then

$$\text{vol } M = \left(\frac{\ell}{2\pi} \right)^d i(M) \text{vol } S^d.$$

Because of these results, the weak Blaschke conjecture means that any Blaschke manifold has the right Weinstein integer. Since the Weinstein integer of a Blaschke manifold depends only on the ring structure of the integral cohomology ring of its geodesic space, the weak Blaschke conjecture is essentially a topological problem rather than a geometrical problem.

The purpose of this paper is to prove the weak Blaschke conjecture for complex projective spaces. In fact, we are going to prove the following

THEOREM. *If M is a Blaschke manifold of the homotopy type of the complex projective n -space $\mathbf{C}P^n$, $n \geq 1$, then the Weinstein integer of M is equal to that of $\mathbf{C}P^n$, i.e., $\binom{2n-1}{n-1}$. In other words, if ℓ denotes the length of closed geodesics in M and S^{2n} denotes the unit $2n$ -sphere, then*

$$\text{vol } M = \left(\frac{\ell}{2\pi} \right)^{2n} \binom{2n-1}{n-1} \text{vol } S^{2n}.$$

In particular, if closed geodesics in M are of the same length as those in $\mathbf{C}P^n$, then

$$\text{vol } M = \text{vol } \mathbf{C}P^n.$$

However, we are not able to prove results for complex projective spaces analogous to those for spheres as seen in [2, 6]. If one succeeds in doing so, then the Blaschke conjecture for complex projective spaces follows.

Let \mathbf{R}^k be the euclidean k -space of coordinates x_1, \dots, x_k , let D^k be the unit closed k -disk in \mathbf{R}^k given by $x_1^2 + \dots + x_k^2 \leq 1$, and let S^{k-1} be the unit $(k-1)$ -sphere in \mathbf{R}^k given by $x_1^2 + \dots + x_k^2 = 1$.

For the sake of convenience, we regard \mathbf{R}^k as a subspace of \mathbf{R}^{k+1} by identifying every $(x_1, \dots, x_k) \in \mathbf{R}^k$ with $(x_1, \dots, x_k, 0) \in \mathbf{R}^{k+1}$. Let \mathbf{R}^k be naturally oriented, let D^k have the same orientation as \mathbf{R}^k and let S^{k-1} be oriented so that $\partial D^k = S^{k-1}$.

If k is even, say $k = 2n + 2$, we may regard \mathbf{R}^{2n+2} as the unitary $(n + 1)$ -space \mathbf{C}^{n+1} by identifying every $(x_1, x_2, \dots, x_{2n+1}, x_{2n+2}) \in \mathbf{R}^{2n+2}$ with $(x_1 + \sqrt{-1}x_2, \dots, x_{2n+1} + \sqrt{-1}x_{2n+2}) \in \mathbf{C}^{n+1}$. Then there is a natural free orthogonal action of S^1 on S^{2n+1} . The orbit space S^{2n+1}/S^1 is the complex projective n -space which we denote by CP^n . Since the projection of S^{2n+1} into CP^n is an oriented S^1 bundle, there is a natural orientation on CP^n . Since $S^{2n+1} \subset S^{2n+3}$, $CP^n \subset CP^{n+1}$.

Throughout this paper, integers are used as coefficients in both homology and cohomology. For any oriented closed manifold Y , $[Y]$ denotes the fundamental homology class on Y . It is clear that, if g is the generator of $H^2(CP^1) = H^2(CP^n)$ with $g \cap [CP^1] = 1$, then $g^n \cap [CP^n] = 1$.

Hereafter, M always denotes a Blaschke manifold of the homotopy type of CP^n , $n \geq 1$. Since the case $n = 1$ has been determined [4], we assume below that $n > 1$.

Let g be a generator of $H^2(M)$ and let M be so oriented that $g^n \cap [M] = 1$. Let UM be the closed smooth $(4n - 1)$ -manifold consisting of all unit tangent vectors of M , and let CM be the closed smooth $(4n - 2)$ -manifold consisting of all oriented closed geodesics in M . Then

(1) UM and CM are 1-connected and there is a natural oriented smooth S^{2n-1} bundle $\tau: UM \rightarrow M$ and a natural oriented smooth S^1 bundle $\pi: UM \rightarrow CM$ such that for any $u \in UM$, u is the unit tangent vector of πu at τu .

Since M is oriented, it follows from (1) that there is a natural orientation on UM and then a natural orientation on CM .

As a consequence of (1), we have

(2) The Gysin sequences of the oriented sphere bundles $\tau: UM \rightarrow M$ and $\pi: UM \rightarrow CM$, namely

$$\begin{aligned} \dots \rightarrow H^{k-2n}(M) \xrightarrow{\sim e(\tau)} H^k(M) \xrightarrow{\tau^*} H^k(UM) \rightarrow H^{k-2n+1}(M) \rightarrow \dots, \\ \dots \rightarrow H^{k-2}(CM) \xrightarrow{\sim e} H^k(CM) \xrightarrow{\pi^*} H^k(UM) \rightarrow H^{k-1}(CM) \rightarrow \dots \end{aligned}$$

are exact, where $e(\tau)$ and e are the respective Euler classes of the oriented sphere bundles.

Since $e(\tau) \cap [M]$ is the Euler characteristic of M which is equal to $n + 1$, it follows from (2) that

$$(3) \quad \begin{aligned} H^k(UM) &= \begin{cases} \mathbf{Z} & \text{for } k = 2i \text{ or } 4n - 1 - 2i, \\ & i = 0, \dots, n - 1, \\ \mathbf{Z}_{n+1} & \text{for } k = 2n, \\ 0 & \text{otherwise,} \end{cases} \\ H^k(CM) &= \begin{cases} (i + 1)\mathbf{Z} & \text{for } k = 2i \text{ or } 4n - 2 - 2i, \\ & i = 0, \dots, n - 1, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where \mathbf{Z} denotes the group of integers, \mathbf{Z}_{n+1} denotes the group of integers modulo $n + 1$ and $(i + 1)\mathbf{Z}$ denotes the direct sum of $i + 1$ copies of \mathbf{Z} . If a is an element of $H^2(CM)$ with $\pi^*a = \tau^*g$, then for any $i = 1, \dots, n$, $(\pi^*a)^i$ is a generator of $H^{2i}(UM)$ and for any $i = 1, \dots, n - 1$, $\{a^i, a^{i-1}e, \dots, ae^{i-1}, e^i\}$ is a basis of $H^{2i}(CM)$. Moreover, $H^{2n}(CM)$ is generated by $\{a^n, a^{n-1}e, \dots, ae^{n-1}, e^n\}$ and hence the cohomology ring $H^*(CM)$ is generated by $\{a, e\}$.

REMARK 1. The element $a \in H^2(CM)$ in (3) can be replaced by and only by $a + ke$ with $k \in \mathbf{Z}$. For our purpose, we shall pick a special a as specified in (5).

(4) The involution $\lambda: UM \rightarrow UM$ defined by $\lambda(u) = -u$, is orientation-preserving and it induces an involution $\lambda: CM \rightarrow CM$ such that $\lambda\pi = \pi\lambda$. Moreover, $\lambda: CM \rightarrow CM$ is orientation-reversing.

Proof. It is a consequence of the following facts. First, for any $x \in M$, $\lambda(\tau^{-1}x) = \tau^{-1}x$ and $\tau: \tau^{-1}x \rightarrow \tau^{-1}x$ is orientation-preserving. Second, for any $\gamma \in CM$, $\lambda(\pi^{-1}\gamma) = \pi^{-1}(-\gamma)$ and $\lambda: \pi^{-1}\gamma \rightarrow \pi^{-1}(-\gamma)$ is orientation-reversing.

(5) The element $a \in H^2(CM)$ in (3) can be uniquely chosen such that

$$e = a - b, \quad b = \lambda^*a.$$

Proof. Let γ be an oriented closed geodesic in M and let p and q be two points of γ which divide γ into two arcs of equal length. It is known that the union of all the closed geodesics in M which pass through p and q is a smooth 2-sphere K , and that K can be oriented

so that $g \cap [K] = 1$. Let D and D' be the oriented closed 2-disks in K such that they have the same orientation as K and $\partial D = \gamma = -\partial D'$.

Since $\tau: UM \rightarrow M$ is an S^{2n-1} bundle with $2n-1 \geq 3$, there is a map $f: K \rightarrow UM$ such that for any $x \in K$, $\tau f(x) = x$, and for any $x \in \gamma$, $\pi f(x) = \gamma$. Then we have maps

$$\pi f: K \rightarrow CM, \quad \pi(f|D): D/\partial D \rightarrow CM, \quad \pi(f|D'): D'/\partial D' \rightarrow CM$$

which represent three elements of $H_2(CM)$, say \bar{e} , \bar{a} , \bar{b} . It is not hard to see that \bar{e} , \bar{a} , \bar{b} are unique and

$$\bar{e} = \bar{a} + \bar{b}.$$

Now we assert that

$$\bar{b} = \lambda_* \bar{a}.$$

Let

$$h: D \times [0, \pi] \rightarrow K$$

be the homotopy such that (i) for any $x \in D$, $h(x, 0) = x$, and (ii) if ξ is a geodesic segment from p to q contained in D , then for any $\theta \in [0, \pi]$, $h(\xi \times \{\theta\})$ is a geodesic segment from p to q such that ξ and $h(\xi \times \{\theta\})$ intersect at an angle θ at p and $h: \xi \times \{\theta\} \rightarrow h(\xi \times \{\theta\})$ is isometric. Intuitively speaking, h is the homotopy such that $h(D \times \{\theta\})$ is the closed 2-disk in K obtained by rotating D an angle θ around p and q . Therefore $h(D \times \{0\}) = D$, $h(D \times \{\pi\}) = D'$ and for any $\theta \in [0, \pi]$, $h(\partial D \times \{\theta\})$ is an oriented closed geodesic in M containing p and q such that $h(\partial D \times \{0\}) = \gamma$ and $h(\partial D \times \{\pi\}) = \lambda\gamma$. Hence we have a map

$$H': \partial(D \times [0, \pi]) \rightarrow UM$$

such that (i) for any $x \in D$, $H'(x, 0) = \lambda f(x) = \lambda f h(x, 0)$ and $H'(x, \pi) = f h(x, \pi)$ and (ii) for any $(x, \theta) \in \partial D \times [0, \pi]$, $H'(x, \theta)$ is the unit tangent vector of $\lambda h(\partial D \times \{\theta\})$ at $h(x, \theta)$. Clearly for any $(x, \theta) \in \partial(D \times [0, \pi])$, $\tau H'(x, \theta) = h(x, \theta)$. Since $\pi: UM \rightarrow M$ is an S^{2n-1} bundle with $2n-1 \geq 3$, H' can be extended to a map

$$H: D \times [0, \pi] \rightarrow UM$$

such that for any $(x, \theta) \in D \times [0, \pi]$, $\tau H(x, \theta) = h(x, \theta)$. The homotopy H induces a homotopy

$$\pi H: D/\partial D \times [0, \pi] \rightarrow CM$$

which is a homotopy between $\lambda\pi(f|D)$ and $\pi(f|D')$. Hence $\lambda_* \bar{a} = \bar{b}$.

Let $e, a \in H^2(CM)$ be the elements as seen in (2) and (3). Then

$$\begin{aligned} e \cap \bar{e} &= \pi^* e \cap \pi_*^{-1} \bar{e} = 0, \\ a \cap \bar{e} &= \pi^* a \cap \pi_*^{-1} \bar{e} = \tau^* g \cap \tau_*^{-1} [K] = g \cap [K] = 1. \end{aligned}$$

Moreover, we see from the Gysin homology and cohomology sequences of $\pi: UM \rightarrow CM$ that

$$e \cap \bar{a} = 1.$$

As noted in Remark 1, a can be replaced by and only by $a + ke$, where $k \in \mathbf{Z}$. Hence we can uniquely choose a such that

$$a \cap \bar{a} = 1.$$

Let

$$b = a - e.$$

It is easy to verify that

$$\begin{aligned} a \cap \bar{a} &= 1, & a \cap \bar{b} &= 0, \\ b \cap \bar{a} &= 0, & b \cap \bar{b} &= 1, \end{aligned}$$

which means that $\{a, b\}$ is the basis of $H^2(CM)$ dual to the basis $\{\bar{a}, \bar{b}\}$ of $H_2(CM)$. Since $\lambda_* \bar{a} = \bar{b}$, it follows that $\lambda^* a = b$. Hence the proof is completed.

REMARK 2. The choice of $a \in H^2(CM)$ given in (5) is a key step of the proof of our theorem. In fact, we shall prove later that in $H^*(CM)$,

$$a^{n+1} = 0.$$

If this is shown, then our theorem can be proved as follows. Since $a^{n+1} = 0$, $b^{n+1} = \lambda^* a^{n+1} = 0$ so that

$$\begin{aligned} e^{2n-1} &= (a - b)^{2n-1} \\ &= (-1)^{n-1} \binom{2n-1}{n-1} a^n b^{n-1} + (-1)^n \binom{2n-1}{n} a^{n-1} b^n. \end{aligned}$$

By (4), $a^{n-1} b^n = -a^n b^{n-1}$ and then

$$e^{2n-1} = (-1)^{n-1} 2 \binom{2n-1}{n-1} a^n b^{n-1}.$$

By Poincaré duality, there is an element $(a^n)^* \in H^{2n-2}(CM)$ such that $a^n (a^n)^* \cap [CM] = 1$. Since $a^{n+1} = 0$, we may let $(a^n)^* = r b^{n-1}$, where $r \in \mathbf{Z}$. Therefore

$$1 = a^n (a^n)^* \cap [CM] = (a^n b^{n-1}) \cap [CM]$$

so that $a^n b^{n-1} \cap [CM] = r = \pm 1$. Hence the Weinstein integer of M is

$$i(M) = \frac{1}{2} e^{2n-1} \cap [CM] = \binom{2n-1}{n-1}.$$

REMARK 3. If M is merely a Riemannian $2n$ -manifold, $n > 1$, which is of the homotopy type of CP^n and in which all geodesics are smoothly closed and have the same length, (1), (2), (3) and (4) remain valid. Hence the stronger assumption that M is a Blaschke manifold of the homotopy type of CP^n , $n > 1$, is used for the first time in the proof of (5).

(6) Let

$$\tau': W_1 \rightarrow M, \quad \pi': W_2 \rightarrow CM$$

be the smooth D^{2n} bundle and D^2 bundle associated with $\tau: UM \rightarrow M$ and $\pi: UM \rightarrow CM$ respectively. Then W_1 and W_2 are 1-connected compact smooth $4n$ -manifolds with boundary UM and there is a 1-connected closed smooth $4n$ -manifold W obtained by pasting together W_1 and W_2 along their common boundary UM via the identity diffeomorphism. Moreover, there is a natural involution $\lambda: W \rightarrow W$ such that $\lambda|UM$ and $\lambda|CM$ coincide with those given in (4) and it has M as its fixed point set.

We let W_1 be oriented so that $\partial W_1 = UM$, and let W have the same orientation as W_1 .

The inclusion map of CM into W induces an isomorphism of $H^2(W)$ onto $H^2(CM)$. If we use the isomorphism to identify $H^2(W)$ with $H^2(CM)$, then

$$H^k(W) = \begin{cases} (i+1)\mathbf{Z} & \text{for } k = 2i \text{ or } 4n - 2i, i = 0, \dots, n, \\ 0 & \text{otherwise,} \end{cases}$$

and for any $i = 1, \dots, n$, $\{a^i, a^{i-1}e, \dots, ae^{i-1}, e^i\}$ is a basis of $H^{2i}(W)$ and so is $\{a^i, a^{i-1}b, \dots, ab^{i-1}, b^i\}$, where

$$b = \lambda^* a, \quad e = a - b.$$

Moreover, the cohomology ring $H^*(W)$ is generated by $\{a, e\}$ as well as by $\{a, b\}$.

Proof. The computation of $H^k(W)$ is a consequence of (3) and the Mayer-Vietoris sequence of $(W; W_1, W_2)$ and the rest is rather clear.

REMARK 4. For the special case $M = \mathbb{C}P^n$, closed geodesics in M are of length π and there is a λ -invariant homeomorphism f of W onto $\mathbb{C}P^n \times \mathbb{C}P^n$ given as follows.

Whenever $u \in UM$, there is a totally geodesic smooth 2-sphere K_u in M which is the union of the geodesic segments from τu to $\exp(\pi/2)u$, where \exp is the exponential map. W_1 is obtained from $[0, 1] \times UM$ by identifying every $(0, u) \in [0, 1] \times UM$ with τu . For (r, u) in W_1 , we let

$$f(r, u) = (\exp(r\pi/8)u, \exp(-r\pi/8)u).$$

W_2 is obtained from $[0, 1] \times UM$ by identifying every $(0, u) \in [0, 1] \times UM$ with πu . For any $(r, u) \in [0, 1] \times UM$, there is a unique $u_r \in UM$ such that u_r is tangent to K_u at τu and the angle from u to u_r is $(1 - r)\pi/2$ using the orientation on K_u . For (r, u) in W_2 , we let

$$f(r, u) = (\exp(2 - r)(\pi/8)u_r, \exp(-2 + r)(\pi/8)u_r).$$

Notice that if πu is the equator of K_u and $f(0, u) = (x, y)$, then x is the north pole of K_u and y is the south pole of K_u .

Let us use f to identify W with $\mathbb{C}P^n \times \mathbb{C}P^n$. Then $p: W \rightarrow M$ defined by $p(x, y) = x$ is a trivial fibre bundle of fibre $\mathbb{C}P^n$ and $p: CM \rightarrow M$ is a non-trivial fibre bundle of fibre $\mathbb{C}P^{n-1}$. Hence it is preferable to consider $H^*(W)$ rather than $H^*(CM)$.

For the general case, we are not able to construct the fibration $p: W \rightarrow M$. However, we can still prove that $H^*(W)$ is isomorphic to $H^*(\mathbb{C}P^n \times \mathbb{C}P^n)$ as for the special case $M = \mathbb{C}P^n$. This is what we are going to do from now on.

(7) The fixed point set M of $\lambda: W \rightarrow W$ is a closed smooth $2n$ -manifold such that

$$a^n \cap [M] = 1, \quad e \cap [M] = 0.$$

Moreover, there is a smooth imbedding

$$\phi: \mathbb{C}P^n \rightarrow W$$

such that

- (i) $a^n \cap \phi_*[\mathbb{C}P^n] = 1, \quad b \cap \phi_*[\mathbb{C}P^n] = 0,$
- (ii) M and $\phi(\mathbb{C}P^n)$ intersect transversally at a single point and
- (iii) $\phi(\mathbb{C}P^n)$ and $\lambda\phi(\mathbb{C}P^n)$ intersect transversally at an odd number of points.

Proof. Since the homomorphism of $H^2(W)$ into $H^2(M)$ induced by the inclusion map of M into W maps a into g , we infer that $a^n \cap [M] = g^n \cap [M] = 1$. Since M is the fixed point set of $\lambda: W \rightarrow W$ and λ is orientation-preserving, it follows that

$$b \cap [M] = \lambda^* a \cap [M] = a \cap \lambda_* [M] = a \cap [M].$$

Hence $e \cap [M] = (a - b) \cap [M] = 0$.

Let $\phi': \mathbf{C}P^1 \rightarrow CM$ be a smooth imbedding homotopic to the imbedding of $\pi(f|D)$ of $D/\partial D (= \mathbf{C}P^1)$ into CM given in the proof of (5). Then

$$a \cap \phi'_* [\mathbf{C}P^1] = 1, \quad b \cap \phi'_* [\mathbf{C}P^1] = 0.$$

Since for any $k = 3, \dots, 2n - 2$, $\pi_k(CM) = \pi_k(UM) = \pi_k(M) = 0$ and since $\dim CM > 2 \dim \mathbf{C}P^{n-1}$, ϕ' can be extended to a smooth imbedding $\phi'': \mathbf{C}P^{n-1} \rightarrow CM$.

Let T be a closed tubular neighborhood of $\mathbf{C}P^{n-1}$ in $\mathbf{C}P^n$ and let $\pi': W_2 \rightarrow CM$ be the D^2 bundle we had earlier. Then ϕ'' can be extended to a smooth imbedding $\phi''': T \rightarrow W_2$ such that

$$\phi'''(T) = \pi'^{-1} \phi''(\mathbf{C}P^{n-1}).$$

Clearly $\phi'''(\partial T)$ is a smooth $(2n - 1)$ -sphere in UM at which $\phi'''(T)$ intersects UM transversally. Since $\pi_{2n-1}(W_1) = \pi_{2n-1}(M) = 0$ and $\dim W = 2 \dim \mathbf{C}P^n > 4$, we infer that ϕ''' can be extended to a smooth imbedding $\phi: \mathbf{C}P^n \rightarrow W$ such that $\phi(\mathbf{C}P^n - T) \subset W_1$. From the construction of ϕ , we see that

$$a \cap \phi_* [\mathbf{C}P^1] = 1, \quad b \cap \phi_* [\mathbf{C}P^1] = 0.$$

Therefore for any $i = 2, \dots, n$,

$$a \cap \phi_* [\mathbf{C}P^i] = \phi_* [\mathbf{C}P^{i-1}], \quad b \cap \phi_* [\mathbf{C}P^i] = 0.$$

Hence

$$a^n \cap \phi_* [\mathbf{C}P^n] = 1, \quad b \cap \phi_* [\mathbf{C}P^n] = 0.$$

Let $p: \widetilde{W} \rightarrow W$ be the smooth S^1 bundle of Euler class e . From its Gysin sequence, we see that

$$H^k(\widetilde{W}) = \begin{cases} \mathbf{Z} & \text{for } k = 2i \text{ or } 4n + 1 - 2i, \quad i = 0, \dots, n; \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for any $i = 0, \dots, n$, $(p^*a)^i$ is a generator of $H^{2i}(\widetilde{W})$. Since $e \cap [M] = 0$, $p^{-1}M$ is diffeomorphic to $S^1 \times M$ so that there is an oriented closed smooth submanifold M' of $p^{-1}M$ such that

$p: M' \rightarrow M$ is an orientation-preserving diffeomorphism. Now

$$(p^*a)^n \cap [M'] = a^n \cap p_*[M'] = a^n \cap [M] = 1.$$

Hence $[M']$ is a generator of $H_{2n}(\widetilde{W})$.

Since $e^n \cap \phi_*[CP^n] = a^n \cap \phi_*[CP^n] = 1$, $p^{-1}\phi(CP^n)$ is a $(2n + 1)$ -sphere. From the Gysin sequence of $p: \widetilde{W} \rightarrow W$, we see that $[p^{-1}\phi(CP^n)]$ is a generator of $H_{2n+1}(\widetilde{W})$. Therefore, by Poincaré duality, $[M'] \cap [p^{-1}\phi(CP^n)] = \pm 1$. Hence $[M] \cap \phi_*[CP^n] = \pm 1$. That $[M] \cap \phi_*[CP^n] = 1$ is a consequence of the choice of the orientation of W . In fact, ϕ may be so chosen that the closed $2n$ -disk $\phi(CP^n) \cap W_1$ intersects M transversally at exactly one point.

Altering ϕ by a homotopy if it is necessary, we may assume that $\phi(CP^n)$ and $\lambda\phi(CP^n)$ intersect transversally at finitely many points. Besides the point $M \cap \phi(CP^n)$, other points in $\phi(CP^n) \cap \lambda\phi(CP^n)$ are in pairs. Hence $\phi_*[CP^n] \cap (\lambda\phi)_*[CP^n] = \text{odd integer}$.

Let N be an integer $> 4n$, let

$$\lambda: CP^N \times CP^N \rightarrow CP^N \times CP^N$$

be the involution defined by $\lambda(x, y) = (y, x)$ and let $\{a, b\}$ be the basis of $H^2(CP^N \times CP^N)$ such that

$$\begin{aligned} a \cap [CP^N \times CP^N] &= [CP^{N-1} \times CP^N], \\ b \cap [CP^N \times CP^N] &= [CP^N \times CP^{N-1}]. \end{aligned}$$

(8) There is a smooth imbedding

$$f: W \rightarrow CP^N \times CP^N$$

such that $f\lambda = \lambda f$, $f^*a = a$ and $f^*b = b$. Moreover, there is a natural isomorphism

$$H^{2n}(CP^N \times CP^N) \cong H_{2n}(CP^N \times CP^N)$$

which maps every $x \in H^{2n}(CP^N \times CP^N)$ into $x \cap f_*[W] \in H_{2n}(CP^N \times CP^N)$.

Proof. There is a smooth map $f': W \rightarrow CP^N$ such that f'^* maps the generator g of $H^2(CP^N)$ into a . Since $\dim CP^N > 2 \dim W$, f' can be approximated by a smooth imbedding homotopic to f' . (See [5].) Therefore we may assume that f' is a smooth imbedding. Hence $f: W \rightarrow CP^N \times CP^N$ defined by $f(x) = (f'x, \lambda f'x)$ is as desired.

By Poincaré duality, there is an isomorphism $H^{2n}(W) \cong H_{2n}(W)$ which maps every $x \in H^{2n}(W)$ into $x \cap [W] \in H_{2n}(W)$. Since

$$f^*: H^{2n}(\mathbb{C}P^N \times \mathbb{C}P^N) \rightarrow H^{2n}(W)$$

and

$$f_*: H_{2n}(W) \rightarrow H_{2n}(\mathbb{C}P^N \times \mathbb{C}P^N)$$

are isomorphisms, the second part of (8) follows.

Now we consider an oriented λ -invariant connected closed smooth $4n$ -submanifold X of $\mathbb{C}P^N \times \mathbb{C}P^N$, $n \geq 1$, which has the following properties of W (or rather of fW).

(a) Let $f: X \rightarrow \mathbb{C}P^N \times \mathbb{C}P^N$ be the inclusion map. Then for any $i = 0, \dots, n$,

$$f_*: H_{2i}(X) \rightarrow H_{2i}(\mathbb{C}P^N \times \mathbb{C}P^N)$$

is surjective. Moreover, there is an isomorphism

$$H^{2n}(\mathbb{C}P^N \times \mathbb{C}P^N) \cong H_{2n}(\mathbb{C}P^N \times \mathbb{C}P^N)$$

which maps every $x \in H^{2n}(\mathbb{C}P^N \times \mathbb{C}P^N)$ into $x \cap [X] \in H_{2n}(\mathbb{C}P^N \times \mathbb{C}P^N)$.

(b) The fixed point set M of $\lambda: X \rightarrow X$ is a closed smooth $2n$ -manifold which can be so oriented that

$$a^n \cap [M] = 1, \quad e \cap [M] = 0.$$

(c) There is a smooth imbedding $\phi: \mathbb{C}P^n \rightarrow X$ such that

$$a^n \cap \phi_*[\mathbb{C}P^n] = 1, \quad b \cap \phi_*[\mathbb{C}P^n] = 0.$$

(d) $[M] \cap \phi_*[\mathbb{C}P^n] = 1$,

$$\phi_*[\mathbb{C}P^n] \cap (\lambda\phi)_*[\mathbb{C}P^n] = \text{odd integer}.$$

For any $k = 0, \dots, 2n$, we let $P_k(a, b)$ be the group of homogeneous polynomials in variables a and b of degree k with integral coefficients. Then for any $i = 0, \dots, 2n$,

$$H^{2i}(\mathbb{C}P^N \times \mathbb{C}P^N) = P_i(a, b).$$

As a consequence of (a), (b), (c), (d) above, we have

(9) There are unique $p(a, b), q(a, b) \in P_n(a, b)$ such that

$$p(a, b) \cap [X] = [M], \quad q(a, b) \cap [X] = \phi_*[\mathbb{C}P^n].$$

Moreover,

$$\begin{aligned} a^n p(a, b) \cap [X] &= 1, & e p(a, b) \cap [X] &= 0; \\ a^n q(a, b) \cap [X] &= 1, & b q(a, b) \cap [X] &= 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} p(a, b)q(a, b) \cap [X] &= 1, \\ q(a, b)q(b, a) \cap [X] &= \text{odd integers}. \end{aligned}$$

(10) (i) For any $i = 0, \dots, n$, $a^i b^{n-i} p(a, b) \cap [X] = 1$.

(ii) $p(a, b) = p(b, a)$.

(iii) $p(1, 0) = p(0, 1) = q(1, 1) = 1$.

(iv) Let K be the subgroup of $H^{2n+2}(\mathbf{C}P^N \times \mathbf{C}P^N) = P_{n+1}(a, b)$ consisting of the elements x with $x \cap [X] = 0$ and let L be the subgroup of $P_{n+1}(a, b)$ generated by $\{a^n b, a^{n-1} b^2, \dots, a^2 b^{n-1}, ab^n\}$. Then

$$P_{n+1}(a, b) = K \oplus L,$$

$q(0, 1) = \pm 1$ and $\{aq(b, a), bq(a, b)\}$ is a basis of K .

(v) $aq(b, a) - bq(a, b) = q(0, 1)ep(a, b)$.

Proof.

(i) Since, by (9), $(a - b)p(a, b) \cap [X] = 0$, we have

$$ap(a, b) \cap [X] = bp(a, b) \cap [X].$$

Hence for any $i = 0, \dots, n$,

$$a^i b^{n-i} p(a, b) \cap [X] = a^n p(a, b) \cap [X]$$

which is equal to 1 by (9).

(ii) Since $\lambda^* a = b$, $\lambda^* b = a$ and $\lambda_* [X] = [X]$, it follows from (i) and (9) that

$$\begin{aligned} a^n p(b, a) \cap [X] &= b^n p(a, b) \cap [X] = 1, \\ ep(b, a) \cap [X] &= -ep(a, b) \cap [X] = 0. \end{aligned}$$

Hence, by (9), $p(b, a) = p(a, b)$.

(iii) By (9) and (ii),

$$\begin{aligned} 1 &= p(a, b)q(a, b) \cap [X] = p(1, 0)a^n q(a, b) \cap [X] \\ &= p(1, 0) = p(0, 1). \end{aligned}$$

Let $q(a, b) = \sum_{i=0}^n \beta_i a^i b^{n-i}$. Then, by (9) and (i),

$$\begin{aligned} 1 &= q(a, b)p(a, b) \cap [X] = \sum_{i=0}^n \beta_i a^i b^{n-i} p(a, b) \cap [X] \\ &= \sum_{i=0}^n \beta_i = q(1, 1). \end{aligned}$$

(iv) By (a),

$$a^n \cap [X], a^{n-1}b \cap [X], \dots, ab^{n-1} \cap [X], b^n \cap [X]$$

are linearly independent elements of $H_{2n}(\mathbb{C}P^N \times \mathbb{C}P^N)$. Therefore

$$a^{n-1} \cap [X], a^{n-2}b \cap [X], \dots, ab^{n-2} \cap [X], b^{n-1} \cap [X]$$

are linearly independent elements of $H_{2n+2}(\mathbb{C}P^N \times \mathbb{C}P^N)$ and hence K does not have more than two linearly independent elements.

By (9),

$$\begin{aligned} q(0, 1) &= q(0, 1)a^n q(a, b) \cap [X] \\ &= q(a, b)q(b, a) \cap [X] = \text{odd integers.} \end{aligned}$$

We infer that in $P_{n+1}(a, b)$,

$$aq(b, a), a^n b, a^{n-1}b^2, \dots, a^2b^{n-1}, ab^n, bq(a, b)$$

are linearly independent. Therefore $\{aq(b, a), bq(a, b)\}$ generates a subgroup of K of finite index.

Let $\{r(a, b), s(a, b)\}$ be a basis of K . Then

$$\{r(a, b), a^n b, a^{n-1}b^2, \dots, a^2b^{n-1}, ab^n, s(a, b)\}$$

is a basis of $P_{n+1}(a, b)$ so that we may assume that

$$r(1, 0) = 1, \quad r(0, 1) = 0, \quad s(1, 0) = 0, \quad s(0, 1) = 1.$$

Therefore there are $r_1(a, b), s_1(a, b) \in P_n(a, b)$ such that

$$r(a, b) = ar_1(a, b), \quad s(a, b) = bs_1(a, b).$$

From this result, it follows that

$$aq(b, a) = q(0, 1)r(a, b) = q(0, 1)ar_1(a, b)$$

so that

$$q(b, a) = q(0, 1)r_1(a, b).$$

Since, by (iii), $q(1, 1) = 1$, we infer that

$$q(0, 1) = \pm 1.$$

Hence

$$aq(b, a) = \pm r(a, b), \quad bq(a, b) = \pm s(a, b)$$

and consequently $\{aq(b, a), bq(a, b)\}$ is a basis of K .

(v) By (9), $ep(a, b)$ is in K and by (iv), $\{aq(b, a), bq(a, b)\}$ is a basis of K . Then for some integers s and t ,

$$ep(a, b) = saq(b, a) + tbq(a, b).$$

By setting $a = 1$ and $b = 0$, we obtain $sq(0, 1) = 1$ by (iii). Therefore $s = q(0, 1)$. Similarly, $t = -q(0, 1)$. Hence our assertion follows.

$$(11) \quad p(a, b) = \sum_{i=0}^n a^{n-i} b^i \quad \text{and} \quad q(a, b) = b^n.$$

Proof. Assume first that $n = 1$. By [4], we may set

$$M = CP^1.$$

As seen in Remark 4, which is valid for $n = 1$, we may let W be $CP^1 \times CP^1$ and let M be the diagonal set in $CP^1 \times CP^1$. As we have done earlier, we let $\{a, b\}$ be the basis of $H^2(CP^1 \times CP^1)$ such that

$$\begin{aligned} a \cap [CP^1 \times CP^1] &= [CP^0 \times CP^1], \\ b \cap [CP^1 \times CP^1] &= [CP^1 \times CP^0], \end{aligned}$$

and let $p(a, b)$ and $q(a, b)$ be the elements of $H^2(CP^1 \times CP^1)$ such that

$$p(a, b) \cap [W] = [M], \quad q(a, b) \cap [W] = [CP^1 \times CP^0].$$

It is not hard to see that

$$p(a, b) = a + b, \quad q(a, b) = b.$$

Hence (11) holds for $n = 1$.

Now we proceed by induction on n and assume that our assertion holds when n is replaced by $n - 1$, $n > 1$. Since

$$X \subset CP^N \times CP^N \subset CP^{N+1} \times CP^{N+1},$$

we can use a λ -equivariant isotopy to alter X so that the following hold.

(1) $\phi(CP^n)$ is contained in $CP^{N+1} \times CP^N$ and intersects $CP^N \times CP^{N+1}$ transversally at $\phi(CP^{n-1})$.

(2) M and X are transversal to $CP^N \times CP^{N+1}$.

(3) $X' = X \cap (CP^N \times CP^N)$ is a connected closed smooth $(4n - 4)$ -manifold invariant under λ .

Let X' be oriented so that

$$[X'] = ab \cap [X].$$

We claim that X' satisfies (a), (b), (c), (d) with $n - 1$ in place of n .

For any $i = 0, \dots, n - 2$,

$$\begin{aligned} f_*H_{2i}(X') &= ab \cap f_*H_{2i+4}(X) \\ &= ab \cap H_{2i+4}(\mathbf{C}P^N \times \mathbf{C}P^N) = H_{2i}(\mathbf{C}P^N \times \mathbf{C}P^N). \end{aligned}$$

By (10), (iv),

$$ab \cup f^*H^{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N) = f^*H^{2n+2}(\mathbf{C}P^N \times \mathbf{C}P^N).$$

Then

$$ab \cap f_*H_{2n+2}(X) = f_*H_{2n-2}(X) = H_{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N)$$

and hence

$$f_*H_{2n-2}(X') = f_*(ab \cap H_{2n+2}(X)) = H_{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N).$$

Since

$$\begin{aligned} f^*H^{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N) \cap [X'] &= f^*H^{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N) \cap (ab \cap [X]) \\ &= (ab \cup f^*H^{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N)) \cap [X] \\ &= f^*H^{2n+2}(\mathbf{C}P^N \times \mathbf{C}P^N) \cap [X] \\ &\cong f_*H_{2n-2}(X) = f_*H_{2n-2}(X'), \end{aligned}$$

it follows that there is an isomorphism of $H^{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N)$ onto $H_{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N)$ which maps every $x \in H^{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N)$ into $x \cap f_*[X'] \in H_{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N)$. The rest is rather obvious.

By the induction hypothesis, $q'(a, b) = b^{n-1}$ is the unique element of $H^{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N)$ such that

$$q'(a, b) \cap [X'] = \phi_*[\mathbf{C}P^{n-1}]$$

so that

$$ab^n \cap [X] = b^{n-1} \cap (ab \cap [X]) = \phi_*[\mathbf{C}P^{n-1}].$$

Then

$$a(b^n - q(a, b)) \cap [X] = \phi_*[\mathbf{C}P^{n-1}] - a \cap \phi_*[\mathbf{C}P^n] = 0.$$

Therefore, by (10), (iv),

$$b^n - q(a, b) = kq(b, a)$$

for some integer k . Since, by (10), (iii), $q(1, 1) = 1$, it follows that

$k = 0$ and hence

$$q(a, b) = b^n.$$

From this result and (10), (v), it is clear that

$$p(a, b) = \sum_{i=0}^n a^{n-i} b^i$$

follows.

Proof of our theorem. In $H^*(W)$,

$$a^{n+1} = aq(b, a) = 0$$

and then in $H^*(CM)$,

$$a^{n+1} = 0.$$

Hence our assertion follows as seen in Remark 2.

REFERENCES

- [1] A. Besse, *Manifolds All of Whose Geodesics are Closed*, Ergebnisse der Mathematik, No. 93, Springer-Verlag, Berlin, 1978.
- [2] A. Berger, *Blaschke's Conjecture for Spheres*, Appendix D in the book above by A. Besse.
- [3] R. Bott, *On manifolds all of whose geodesics are closed*, Ann. Math., **60** (1954), 375–382.
- [4] L. W. Green, *Aug Wiedersehensflächen*, Ann. math., **78** (1963), 289–299.
- [5] A. Haefliger, *Plongements différentiables de variétés dans variétés*, Comment. Math. Helv., **36** (1961), 47–81.
- [6] J. L. Kazdan, *An Inequality Arising in Geometry*, Appendix E in the book above by A. Besse.
- [7] H. Samelson, *On manifolds with many closed geodesics*, Portugal. Math., **22** (1963), 193–196.
- [8] A. Weinstein, *On the volume of manifolds all of whose geodesics are closed*, J. Differential Geom., **9** (1974), 513–517.
- [9] C. T. Yang, *Odd-dimensional Wiedersehen manifolds are spheres*, J. Differential Geom., **15** (1980), 91–96.
- [10] ———, *On the Blaschke conjecture*, Yau, Seminar on Differential Geometry, pp. 159–171.

Received March 30, 1988. The author was supported in part by the National Science Foundation when the paper was prepared.

UNIVERSITY OF PENNSYLVANIA
PHILADELPHIA, PA 19104

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

V. S. VARADARAJAN
(Managing Editor)
University of California
Los Angeles, CA 90024-1555-05

HERBERT CLEMENS
University of Utah
Salt Lake City, UT 84112

THOMAS ENRIGHT
University of California, San Diego
La Jolla, CA 92093

NICHOLAS ERCOLANI
University of Arizona
Tucson, AZ 85721

R. FINN
Stanford University
Stanford, CA 94305

VAUGHAN F. R. JONES
University of California
Berkeley, CA 94720

STEVEN KERCKHOFF
Stanford University
Stanford, CA 94305

C. C. MOORE
University of California
Berkeley, CA 94720

MARTIN SCHARLEMANN
University of California
Santa Barbara, CA 93106

HAROLD STARK
University of California, San Diego
La Jolla, CA 92093

ASSOCIATE EDITORS

R. ARENS

E. F. BECKENBACH
(1906-1982)

B. H. NEUMANN

F. WOLF
(1904-1989)

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA
UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the 1991 *Mathematics Subject Classification* scheme which can be found in the December index volumes of *Mathematical Reviews*. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024-1555-05.

There are page-charges associated with articles appearing in the *Pacific Journal of Mathematics*. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* (ISSN 0030-8730) is published monthly except for July and August. Regular subscription rate: \$190.00 a year (10 issues). Special rate: \$95.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to *Pacific Journal of Mathematics*, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The *Pacific Journal of Mathematics* at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) is published monthly except for July and August. Second-class postage paid at Carmel Valley, California 93924, and additional mailing offices. Postmaster: send address changes to *Pacific Journal of Mathematics*, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Copyright © 1991 by Pacific Journal of Mathematics

Pacific Journal of Mathematics

Vol. 151, No. 2 December, 1991

Michael G. Eastwood and A. M. Pilato , On the density of twistor elementary states	201
Brian E. Forrest , Arens regularity and discrete groups	217
Yu Li Fu , On Lipschitz stability for F.D.E	229
Douglas Austin Hensley , The largest digit in the continued fraction expansion of a rational number	237
Uwe Kaiser , Link homotopy in \mathbb{R}^3 and S^3	257
Ronald Leslie Lipsman , The Penney-Fujiwara Plancherel formula for abelian symmetric spaces and completely solvable homogeneous spaces	265
Florin G. Radulescu , Singularity of the radial subalgebra of $\mathcal{L}(F_N)$ and the Pukánszky invariant	297
Albert Jeu-Liang Sheu , The structure of twisted $SU(3)$ groups	307
Morwen Thistlethwaite , On the algebraic part of an alternating link	317
Thomas (Toma) V. Tonev , Multi-tuple hulls	335
Arno van den Essen , A note on Meisters and Olech's proof of the global asymptotic stability Jacobian conjecture	351
Hendrik J. van Maldeghem , A characterization of the finite Moufang hexagons by generalized homologies	357
Bun Wong , A note on homotopy complex surfaces with negative tangent bundles	369
Chung-Tao Yang , Any Blaschke manifold of the homotopy type of CP^n has the right volume	379