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Dedicated to Professor S. S. Chern

The aim of this paper is to prove the result stated in the title.

By a Blaschke manifold [1, p. 135], we mean a connected closed Riemannian manifold which has the property that the cut locus of each of its points, when viewed in the tangent space, is a round sphere of a constant radius. It is well known that in any Blaschke manifold, all geodesics are smoothly simply closed and have the same length. The canonical examples of a Blaschke manifold are the unit $n$-sphere $S^n$, the real, complex, quaternionic projective $n$-spaces $RP^n$, $CP^n$, $HP^n$ and the Cayley projective plane $CaP^2$ with their standard Riemannian metric. These Blaschke manifolds will be referred to as the canonical Blaschke manifolds. For general informations on Blaschke manifolds, see [1].

The Blaschke conjecture says that any Blaschke manifold, up to a constant factor, is isometric to a canonical Blaschke manifold. This conjecture looks plausible, because it has been shown in [3, 7] that any Blaschke manifold either is diffeomorphic to $S^n$ or $RP^n$, or is of the homotopy type of $CP^n$, or is a 1-connected closed manifold having the integral cohomology ring of $HP^n$ or $CaP^2$. However, so far it has been proved only for spheres and real projective spaces [2, 6, 8, 9].

One crucial step in the proof of the Blaschke conjecture for spheres is to show that any Blaschke manifold diffeomorphic to $S^n$ has the right volume. Hence we formulate the weak Blaschke conjecture [10] which says that any Blaschke manifold has the right volume.

Let $M$ be a $d$-dimensional Blaschke manifold, $UM$ the space of unit tangent vectors of $M$ and $CM$ the space of oriented closed geodesics in $M$. Then $UM$ and $CM$ are oriented connected smooth manifolds and there is a natural oriented smooth circle bundle $\pi: UM \to CM$. In [8], it is shown that, if $e$ denotes the Euler class of this
circle bundle, then
\[ i(M) = \frac{1}{2} \langle e^{d-1}, [CM] \rangle \]
(i.e., one half of the value of \( e^{d-1} \) at the fundamental homology class \([CM]\) of \( CM \)) is an integer, called the Weinstein integer of \( M \), and that, if \( \ell \) denotes the length of closed geodesics in \( M \), then
\[ \text{vol } M = \left( \frac{\ell}{2\pi} \right)^d i(M) \text{ vol } S^d. \]

Because of these results, the weak Blaschke conjecture means that any Blaschke manifold has the right Weinstein integer. Since the Weinstein integer of a Blaschke manifold depends only on the ring structure of the integral cohomology ring of its geodesic space, the weak Blaschke conjecture is essentially a topological problem rather than a geometrical problem.

The purpose of this paper is to prove the weak Blaschke conjecture for complex projective spaces. In fact, we are going to prove the following

**Theorem.** If \( M \) is a Blaschke manifold of the homotopy type of the complex projective \( n \)-space \( \mathbb{C}P^n \), \( n \geq 1 \), then the Weinstein integer of \( M \) is equal to that of \( \mathbb{C}P^n \), i.e., \( \left( \frac{2n-1}{n-1} \right) \). In other words, if \( \ell \) denotes the length of closed geodesics in \( M \) and \( S^{2n} \) denotes the unit 2n-sphere, then
\[ \text{vol } M = \left( \frac{\ell}{2\pi} \right)^{2n} \left( \frac{2n-1}{n-1} \right) \text{ vol } S^{2n}. \]

In particular, if closed geodesics in \( M \) are of the same length as those in \( \mathbb{C}P^n \), then
\[ \text{vol } M = \text{vol } \mathbb{C}P^n. \]

However, we are not able to prove results for complex projective spaces analogous to those for spheres as seen in [2, 6]. If one succeeds in doing so, then the Blaschke conjecture for complex projective spaces follows.

Let \( \mathbb{R}^k \) be the euclidean \( k \)-space of coordinates \( x_1, \ldots, x_k \), let \( D^k \) be the unit closed \( k \)-disk in \( \mathbb{R}^k \) given by \( x_1^2 + \cdots + x_k^2 \leq 1 \), and let \( S^{k-1} \) be the unit \((k-1)\)-sphere in \( \mathbb{R}^k \) given by \( x_1^2 + \cdots + x_k^2 = 1 \).
For the sake of convenience, we regard $R^k$ as a subspace of $R^{k+1}$ by identifying every $(x_1, \ldots, x_k) \in R^k$ with $(x_1, \ldots, x_k, 0) \in R^{k+1}$. Let $R^k$ be naturally oriented, let $D^k$ have the same orientation as $R^k$ and let $S^{k-1}$ be oriented so that $\partial D^k = S^{k-1}$.

If $k$ is even, say $k = 2n + 2$, we may regard $R^{2n+2}$ as the unitary $(n+1)$-space $C^{n+1}$ by identifying every $(x_1, x_2, \ldots, x_{2n+1}, x_{2n+2}) \in R^{2n+2}$ with $(x_1 + \sqrt{-1}x_2, \ldots, x_{2n+1} + \sqrt{-1}x_{2n+2}) \in C^{n+1}$. Then there is a natural free orthogonal action of $S^1$ on $S^{2n+1}$. The orbit space $S^{2n+1}/S^1$ is the complex projective $n$-space which we denote by $CP^n$. Since the projection of $S^{2n+1}$ into $CP^n$ is an oriented $S^1$ bundle, there is a natural orientation on $CP^n$. Since $S^{2n+1} \subset S^{2n+3}$, $CP^n \subset CP^{n+1}$.

Throughout this paper, integers are used as coefficients in both homology and cohomology. For any oriented closed manifold $Y$, $[Y]$ denotes the fundamental homology class on $Y$. It is clear that, if $g$ is the generator of $H^2(CP^1) = H^2(CP^n)$ with $g \cap [CP^1] = 1$, then $g^n \cap [CP^n] = 1$.

Hereafter, $M$ always denotes a Blaschke manifold of the homotopy type of $CP^n$, $n \geq 1$. Since the case $n = 1$ has been determined [4], we assume below that $n > 1$.

Let $g$ be a generator of $H^2(M)$ and let $M$ be so oriented that $g^n \cap [M] = 1$. Let $UM$ be the closed smooth $(4n - 1)$-manifold consisting of all unit tangent vectors of $M$, and let $CM$ be the closed smooth $(4n - 2)$-manifold consisting of all oriented closed geodesics in $M$. Then

(1) $UM$ and $CM$ are 1-connected and there is a natural oriented smooth $S^{2n-1}$ bundle $\tau: UM \to M$ and a natural oriented smooth $S^1$ bundle $\pi: UM \to CM$ such that for any $u \in UM$, $u$ is the unit tangent vector of $\pi u$ at $\tau u$.

Since $M$ is oriented, it follows from (1) that there is a natural orientation on $UM$ and then a natural orientation on $CM$.

As a consequence of (1), we have

(2) The Gysin sequences of the oriented sphere bundles $\tau: UM \to M$ and $\pi: UM \to CM$, namely

$$
\cdots \to H^{k-2n}(M) \xrightarrow{e(\tau)} H^k(M) \xrightarrow{\tau^*} H^k(UM) \to H^{k-2n+1}(M) \to \cdots,
\cdots \to H^{k-2}(CM) \xrightarrow{e} H^k(CM) \xrightarrow{\pi^*} H^k(UM) \to H^{k-1}(CM) \to \cdots
$$

are exact, where $e(\tau)$ and $e$ are the respective Euler classes of the oriented sphere bundles.
Since \( e(\tau) \cap [M] \) is the Euler characteristic of \( M \) which is equal to \( n + 1 \), it follows from (2) that

\[
H^k(UM) = \begin{cases} 
\mathbb{Z} & \text{for } k = 2i \text{ or } 4n - 1 - 2i, \\
\mathbb{Z}_{n+1} & \text{for } k = 2n, \\
0 & \text{otherwise,}
\end{cases}
\]

\[
H^k(CM) = \begin{cases} 
\mathbb{Z} & \text{for } k = 2i \text{ or } 4n - 2 - 2i, \\
(i + 1)\mathbb{Z} & \text{for } k = 2i \text{ or } 4n - 2 - 2i, \\
0 & \text{otherwise,}
\end{cases}
\]

where \( \mathbb{Z} \) denotes the group of integers, \( \mathbb{Z}_{n+1} \) denotes the group of integers modulo \( n + 1 \) and \( (i + 1)\mathbb{Z} \) denotes the direct sum of \( i + 1 \) copies of \( \mathbb{Z} \). If \( a \) is an element of \( H^2(CM) \) with \( \pi^*a = \tau^*g \), then for any \( i = 1, \ldots, n \), \( (\pi^*a)^i \) is a generator of \( H^{2i}(UM) \) and for any \( i = 1, \ldots, n - 1 \), \( \{a^i, a^{i-1}e, \ldots, ae^{i-1}, e^i\} \) is a basis of \( H^{2i}(CM) \). Moreover, \( H^{2n}(CM) \) is generated by \( \{a^n, a^{n-1}e, \ldots, ae^{n-1}, e^n\} \) and hence the cohomology ring \( H^*(CM) \) is generated by \( \{a, e\} \).

**Remark 1.** The element \( a \in H^2(CM) \) in (3) can be replaced by and only by \( a + ke \) with \( k \in \mathbb{Z} \). For our purpose, we shall pick a special \( a \) as specified in (5).

(4) The involution \( \lambda: UM \rightarrow UM \) defined by \( \lambda(u) = -u \), is orientation-preserving and it induces an involution \( \lambda: CM \rightarrow CM \) such that \( \lambda\pi = \pi\lambda \). Moreover, \( \lambda: CM \rightarrow CM \) is orientation-reversing.

**Proof.** It is a consequence of the following facts. First, for any \( x \in M \), \( \lambda(\tau^{-1}x) = \tau^{-1}x \) and \( \tau: \tau^{-1}x \rightarrow \tau^{-1}x \) is orientation-preserving. Second, for any \( \gamma \in CM \), \( \lambda(\pi^{-1}\gamma) = \pi^{-1}(-\gamma) \) and \( \lambda: \pi^{-1}\gamma \rightarrow \pi^{-1}(-\gamma) \) is orientation-reversing.

(5) The element \( a \in H^2(CM) \) in (3) can be uniquely chosen such that

\[
e = a - b, \quad b = \lambda^*a.
\]

**Proof.** Let \( \gamma \) be an oriented closed geodesic in \( M \) and let \( p \) and \( q \) be two points of \( \gamma \) which divide \( \gamma \) into two arcs of equal length. It is known that the union of all the closed geodesics in \( M \) which pass through \( p \) and \( q \) is a smooth 2-sphere \( K \), and that \( K \) can be oriented.
so that \( g \cap [K] = 1 \). Let \( D \) and \( D' \) be the oriented closed 2-disks in \( K \) such that they have the same orientation as \( K \) and \( \partial D = \gamma = -\partial D' \).

Since \( \tau: UM \to M \) is an \( S^{2n-1} \) bundle with \( 2n - 1 \geq 3 \), there is a map \( f: K \to UM \) such that for any \( x \in K \), \( \tau f(x) = x \), and for any \( x \in \gamma \), \( \pi f(x) = \gamma \). Then we have maps
\[
\pi f: K \to CM, \quad \pi(f|D): D/\partial D \to CM, \quad \pi(f|D'): D'/\partial D' \to CM
\]
which represent three elements of \( H_2(CM) \), say \( \bar{e}, \bar{a}, \bar{b} \). It is not hard to see that \( \bar{e}, \bar{a}, \bar{b} \) are unique and
\[
\bar{e} = \bar{a} + \bar{b}.
\]

Now we assert that
\[
\bar{b} = \lambda_* \bar{a}.
\]
Let
\[
h: D \times [0, \pi] \to K
\]
be the homotopy such that (i) for any \( x \in D \), \( h(x, 0) = x \), and (ii) if \( \xi \) is a geodesic segment from \( p \) to \( q \) contained in \( D \), then for any \( \theta \in [0, \pi] \), \( h(\xi \times \{\theta\}) \) is a geodesic segment from \( p \) to \( q \) such that \( \xi \) and \( h(\xi \times \{\theta\}) \) intersect at an angle \( \theta \) at \( p \) and \( h: \xi \times \{\theta\} \to h(\xi \times \{\theta\}) \) is isometric. Intuitively speaking, \( h \) is the homotopy such that \( h(D \times \{\theta\}) \) is the closed 2-disk in \( K \) obtained by rotating \( D \) an angle \( \theta \) around \( p \) and \( q \). Therefore \( h(D \times \{0\}) = D \), \( h(D \times \{\pi\}) = D' \) and for any \( \theta \in [0, \pi] \), \( h(\partial D \times \{\theta\}) \) is an oriented closed geodesic in \( M \) containing \( p \) and \( q \) such that \( h(\partial D \times \{0\}) = \gamma \) and \( h(\partial D \times \{\pi\}) = \gamma \). Hence we have a map
\[
H': \partial(D \times [0, \pi]) \to UM
\]
such that (i) for any \( x \in D \), \( H'(x, 0) = \lambda f(x) = \lambda f h(x, 0) \) and \( H'(x, \pi) = f h(x, \pi) \) and (ii) for any \( (x, \theta) \in \partial D \times [0, \pi] \), \( H'(x, \theta) \) is the unit tangent vector of \( \lambda h(\partial D \times \{\theta\}) \) at \( h(x, \theta) \). Clearly for any \( (x, \theta) \in \partial(D \times [0, \pi]) \), \( \tau H'(x, \theta) = h(x, \theta) \). Since \( \pi: UM \to M \) is an \( S^{2n-1} \) bundle with \( 2n - 1 \geq 3 \), \( H' \) can be extended to a map
\[
H: D \times [0, \pi] \to UM
\]
such that for any \( (x, \theta) \in D \times [0, \pi] \), \( \tau H(x, \theta) = h(x, \theta) \). The homotopy \( H \) induces a homotopy
\[
\pi H: D/\partial D \times [0, \pi] \to CM
\]
which is a homotopy between \( \lambda \pi(f|D) \) and \( \pi(f|D') \). Hence \( \lambda_* \bar{a} = \bar{b} \).
Let \( e, a \in H^2(CM) \) be the elements as seen in (2) and (3). Then
\[
e \cap \bar{e} = \pi^* e \cap \pi_*^{-1} \bar{e} = 0,
\]
\[
a \cap \bar{e} = \pi^* a \cap \pi_*^{-1} \bar{e} = \tau^* g \cap \tau_*^{-1} [K] = g \cap [K] = 1.
\]
Moreover, we see from the Gysin homology and cohomology sequences of \( \pi: UM \to CM \) that
\[
e \cap \bar{a} = 1.
\]
As noted in Remark 1, \( a \) can be replaced by and only by \( a + ke \), where \( k \in \mathbb{Z} \). Hence we can uniquely choose \( a \) such that
\[
a \cap \bar{a} = 1.
\]
Let
\[
b = a - e.
\]
It is easy to verify that
\[
a \cap \bar{a} = 1, \quad a \cap \bar{b} = 0, \\
b \cap \bar{a} = 0, \quad b \cap \bar{b} = 1,
\]
which means that \( \{a, b\} \) is the basis of \( H^2(CM) \) dual to the basis \( \{\bar{a}, \bar{b}\} \) of \( H_2(CM) \). Since \( \lambda_* \bar{a} = \bar{b} \), it follows that \( \lambda^* a = b \). Hence the proof is completed.

**Remark 2.** The choice of \( a \in H^2(CM) \) given in (5) is a key step of the proof of our theorem. In fact, we shall prove later that in \( H^*(CM) \),
\[
a^{n+1} = 0.
\]
If this is shown, then our theorem can be proved as follows. Since \( a^{n+1} = 0, b^{n+1} = \lambda^* a^{n+1} = 0 \) so that
\[
e^{2n-1} = (a - b)^{2n-1}
\]
\[
= (-1)^{n-1} \binom{2n-1}{n-1} a^n b^{n-1} + (-1)^n \binom{2n-1}{n} a^{n-1} b^n.
\]
By (4), \( a^{n-1} b^n = -a^n b^{n-1} \) and then
\[
e^{2n-1} = (-1)^{n-1} 2 \binom{2n-1}{n-1} a^n b^{n-1}.
\]
By Poincaré duality, there is an element \( (a^n)^* \in H^{2n-2}(CM) \) such that \( a^n(a^n)^* \cap [CM] = 1 \). Since \( a^{n+1} = 0 \), we may let \( (a^n)^* = rb^{n-1} \), where \( r \in \mathbb{Z} \). Therefore
\[
1 = a^n(a^n)^* \cap [CM] = (a^n b^{n-1} \cap [CM]).
\]
so that $a^n b^{n-1} \cap [CM] = r = \pm 1$. Hence the Weinstein integer of $M$ is

$$i(M) = \frac{1}{2} e^{2n-1} \cap [CM] = \binom{2n-1}{n-1}.$$  

**Remark 3.** If $M$ is merely a Riemannian $2n$-manifold, $n > 1$, which is of the homotopy type of $CP^n$ and in which all geodesics are smoothly closed and have the same length, (1), (2), (3) and (4) remain valid. Hence the stronger assumption that $M$ is a Blaschke manifold of the homotopy type of $CP^n$, $n > 1$, is used for the first time in the proof of (5).

(6) Let

$$\tau': W_1 \to M, \quad \pi': W_2 \to CM$$

be the smooth $D^{2n}$ bundle and $D^2$ bundle associated with $\tau: UM \to M$ and $\pi: UM \to CM$ respectively. Then $W_1$ and $W_2$ are $1$-connected compact smooth $4n$-manifolds with boundary $UM$ and there is a $1$-connected closed smooth $4n$-manifold $W$ obtained by pasting together $W_1$ and $W_2$ along their common boundary $UM$ via the identity diffeomorphism. Moreover, there is a natural involution $\lambda: W \to W$ such that $\lambda|UM$ and $\lambda|CM$ coincide with those given in (4) and it has $M$ as its fixed point set.

We let $W_1$ be oriented so that $\partial W_1 = UM$, and let $W$ have the same orientation as $W_1$.

The inclusion map of $CM$ into $W$ induces an isomorphism of $H^2(W)$ onto $H^2(CM)$. If we use the isomorphism to identify $H^2(W)$ with $H^2(CM)$, then

$$H^k(W) = \begin{cases} (i + 1)\mathbb{Z} & \text{for } k = 2i \text{ or } 4n - 2i, \ i = 0, \ldots, n, \\ 0 & \text{otherwise}, \end{cases}$$

and for any $i = 1, \ldots, n$, $\{a^i, a^{i-1}e, \ldots, ae^{i-1}, e^i\}$ is a basis of $H^{2i}(W)$ and so is $\{a^i, a^{i-1}b, \ldots, ab^{i-1}, b^i\}$, where

$$b = \lambda^* a, \quad e = a - b.$$  

Moreover, the cohomology ring $H^*(W)$ is generated by $\{a, e\}$ as well as by $\{a, b\}$.

**Proof.** The computation of $H^k(W)$ is a consequence of (3) and the Mayer-Vietoris sequence of $(W; W_1, W_2)$ and the rest is rather clear.
REMARK 4. For the special case $M = \mathbb{C}P^n$, closed geodesics in $M$ are of length $\pi$ and there is a $\lambda$-invariant homeomorphism $f$ of $W$ onto $\mathbb{C}P^n \times \mathbb{C}P^n$ given as follows.

Whenever $u \in UM$, there is a totally geodesic smooth 2-sphere $K_u$ in $M$ which is the union of the geodesic segments from $\tau u$ to $\exp(\pi/2)u$, where $\exp$ is the exponential map. $W_1$ is obtained from $[0, 1] \times UM$ by identifying every $(0, u) \in [0, 1] \times UM$ with $\tau u$. For $(r, u)$ in $W_1$, we let

$$f(r, u) = (\exp(r\pi/8)u, \exp(-r\pi/8)u).$$

$W_2$ is obtained from $[0, 1] \times UM$ by identifying every $(0, u) \in [0, 1] \times UM$ with $\pi u$. For any $(r, u) \in [0, 1] \times UM$, there is a unique $u_r \in UM$ such that $u_r$ is tangent to $K_u$ at $\tau u$ and the angle from $u$ to $u_r$ is $(1-r)\pi/2$ using the orientation on $K_u$. For $(r, u)$ in $W_2$, we let

$$f(r, u) = (\exp(2-r)(\pi/8)u_r, \exp(-2+r)(\pi/8)u_r).$$

Notice that if $\pi u$ is the equator of $K_u$ and $f(0, u) = (x, y)$, then $x$ is the north pole of $K_u$ and $y$ is the south pole of $K_u$.

Let us use $f$ to identify $W$ with $\mathbb{C}P^n \times \mathbb{C}P^n$. Then $p: W \to M$ defined by $p(x, y) = x$ is a trivial fibre bundle of fibre $\mathbb{C}P^n$ and $p: CM \to M$ is a non-trivial fibre bundle of fibre $\mathbb{C}P^{n-1}$. Hence it is preferable to consider $H^*(W)$ rather than $H^*(CM)$.

For the general case, we are not able to construct the fibration $p: W \to M$. However, we can still prove that $H^*(W)$ is isomorphic to $H^*(\mathbb{C}P^n \times \mathbb{C}P^n)$ as for the special case $M = \mathbb{C}P^n$. This is what we are going to do from now on.

(7) The fixed point set $M$ of $\lambda: W \to W$ is a closed smooth $2n$-manifold such that

$$a^n \cap [M] = 1, \quad e \cap [M] = 0.$$

Moreover, there is a smooth imbedding

$$\phi: \mathbb{C}P^n \to W$$

such that

(i) $a^n \cap \phi_*[\mathbb{C}P^n] = 1$, $b \cap \phi_*[\mathbb{C}P^n] = 0$,

(ii) $M$ and $\phi(\mathbb{C}P^n)$ intersect transversally at a single point and

(iii) $\phi(\mathbb{C}P^n)$ and $\lambda \phi(\mathbb{C}P^n)$ intersect transversally at an odd number of points.
Proof. Since the homomorphism of $H^2(W)$ into $H^2(M)$ induced by the inclusion map of $M$ into $W$ maps $a$ into $g$, we infer that $a^n \cap [M] = g^n \cap [M] = 1$. Since $M$ is the fixed point set of $\lambda: W \to W$ and $\lambda$ is orientation-preserving, it follows that

$$b \cap [M] = \lambda^* a \cap [M] = a \cap \lambda^* [M] = a \cap [M].$$

Hence $e \cap [M] = (a - b) \cap [M] = 0$.

Let $\phi': \mathbb{CP}^1 \to CM$ be a smooth imbedding homotopic to the imbedding of $\pi(f|D)$ of $D/\partial D (= \mathbb{CP}^1)$ into $CM$ given in the proof of (5). Then

$$a \cap \phi'[\mathbb{CP}^1] = 1, \quad b \cap \phi'[\mathbb{CP}^1] = 0.$$

Since for any $k = 3, \ldots , 2n - 2$, $\pi_k(CM) = \pi_k(U M) = \pi_k(M) = 0$ and since $\dim CM > 2 \dim \mathbb{CP}^{n-1}$, $\phi'$ can be extended to a smooth imbedding $\phi'' : \mathbb{CP}^{n-1} \to CM$.

Let $T$ be a closed tubular neighborhood of $\mathbb{CP}^{n-1}$ in $\mathbb{CP}^n$ and let $\pi' : W_2 \to CM$ be the $D^2$ bundle we had earlier. Then $\phi''$ can be extended to a smooth imbedding $\phi''' : T \to W_2$ such that

$$\phi'''(T) = \pi'^{-1} \phi''(CM).$$

Clearly $\phi'''(\partial T)$ is a smooth $(2n - 1)$-sphere in $UM$ at which $\phi'''(T)$ intersects $UM$ transversally. Since $\pi_{2n-1}(W_1) = \pi_{2n-1}(M) = 0$ and $\dim W = 2 \dim \mathbb{CP}^n > 4$, we infer that $\phi'''$ can be extended to a smooth imbedding $\phi : \mathbb{CP}^n \to W$ such that $\phi(CP^n - T) \subset W_1$. From the construction of $\phi$, we see that

$$a \cap \phi[\mathbb{CP}^1] = 1, \quad b \cap \phi[\mathbb{CP}^1] = 0.$$

Therefore for any $i = 2, \ldots , n$,

$$a \cap \phi[C\mathbb{P}^i] = \phi[C\mathbb{P}^{i-1}], \quad b \cap \phi[C\mathbb{P}^i] = 0.$$

Hence

$$a^n \cap \phi[\mathbb{CP}^n] = 1, \quad b \cap \phi[\mathbb{CP}^n] = 0.$$

Let $p : \widetilde{W} \to W$ be the smooth $S^1$ bundle of Euler class $e$. From its Gysin sequence, we see that

$$H^k(\widetilde{W}) = \begin{cases} \mathbb{Z} & \text{for } k = 2i \text{ or } 4n + 1 - 2i, \quad i = 0, \ldots , n; \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for any $i = 0, \ldots , n$, $(p^*a)^i$ is a generator of $H^{2i}(\widetilde{W})$. Since $e \cap [M] = 0$, $p^{-1}M$ is diffeomorphic to $S^1 \times M$ so that there is an oriented closed smooth submanifold $M'$ of $p^{-1}M$ such that
$p: M' \rightarrow M$ is an orientation-preserving diffeomorphism. Now

$$(p^* a)^n \cap [M'] = a^n \cap p_*[M'] = a^n \cap [M] = 1.$$ 

Hence $[M']$ is a generator of $H_{2n}(\tilde{W})$.

Since $e^n \cap \phi_*[CP^n] = a^n \cap \phi_*[CP^n] = 1$, $p^{-1}\phi(\text{CP}^n)$ is a $(2n + 1)$-sphere. From the Gysin sequence of $p: \tilde{W} \rightarrow W$, we see that $[p^{-1}\phi(\text{CP}^n)]$ is a generator of $H_{2n+1}(\tilde{W})$. Therefore, by Poincaré duality, $[M'] \cap [p^{-1}\phi(\text{CP}^n)] = \pm 1$. Hence $[M] \cap \phi_*[\text{CP}^n] = \pm 1$. That $[M] \cap \phi_*[\text{CP}^n] = 1$ is a consequence of the choice of the orientation of $W$. In fact, $\phi$ may be so chosen that the closed $2n$-disk $\phi(\text{CP}^n) \cap W_1$ intersects $M$ transversally at exactly one point.

Altering $\phi$ by a homotopy if it is necessary, we may assume that $\phi(\text{CP}^n)$ and $\lambda \phi(\text{CP}^n)$ intersect transversally at finitely many points. Besides the point $M \cap \phi(\text{CP}^n)$, other points in $\phi(\text{CP}^n) \cap \lambda \phi(\text{CP}^n)$ are in pairs. Hence $\phi_*[\text{CP}^n] \cap (\lambda \phi)_*[\text{CP}^n] = \text{odd integer}.$

Let $N$ be an integer $> 4n$, let

$$\lambda: \text{CP}^N \times \text{CP}^N \rightarrow \text{CP}^N \times \text{CP}^N$$

be the involution defined by $\lambda(x, y) = (y, x)$ and let $\{a, b\}$ be the basis of $H^2(\text{CP}^N \times \text{CP}^N)$ such that

$$a \cap [\text{CP}^N \times \text{CP}^N] = [\text{CP}^{N-1} \times \text{CP}^N],$$
$$b \cap [\text{CP}^N \times \text{CP}^N] = [\text{CP}^N \times \text{CP}^{N-1}].$$

(8) There is a smooth imbedding

$$f: W \rightarrow \text{CP}^N \times \text{CP}^N$$

such that $f\lambda = \lambda f$, $f^*a = a$ and $f^*b = b$. Moreover, there is a natural isomorphism

$$H^{2n}(\text{CP}^N \times \text{CP}^N) \cong H_{2n}(\text{CP}^N \times \text{CP}^N)$$

which maps every $x \in H^{2n}(\text{CP}^N \times \text{CP}^N)$ into $x \cap f_*[W] \in H_{2n}(\text{CP}^N \times \text{CP}^N)$.

Proof. There is a smooth map $f': W \rightarrow \text{CP}^N$ such that $f'^*$ maps the generator $g$ of $H^2(\text{CP}^N)$ into $a$. Since $\dim \text{CP}^N > 2 \dim W$, $f'$ can be approximated by a smooth imbedding homotopic to $f'$. (See [5].) Therefore we may assume that $f'$ is a smooth imbedding. Hence $f: W \rightarrow \text{CP}^N \times \text{CP}^n$ defined by $f(x) = (f'x, \lambda f'x)$ is as desired.
By Poincaré duality, there is an isomorphism $H^{2n}(W) \cong H_{2n}(W)$ which maps every $x \in H^{2n}(W)$ into $x \cap [W] \in H_{2n}(W)$. Since

$$f^*: H^{2n}(CP^N \times CP^N) \to H^{2n}(W)$$

and

$$f_*: H_{2n}(W) \to H_{2n}(CP^N \times CP^N)$$

are isomorphisms, the second part of (8) follows.

Now we consider an oriented $\lambda$-invariant connected closed smooth $4n$-submanifold $X$ of $CP^N \times CP^N$, $n \geq 1$, which has the following properties of $W$ (or rather of $fW$).

(a) Let $f: X \to CP^N \times CP^N$ be the inclusion map. Then for any $i = 0, \ldots, n$,

$$f_*: H_{2i}(X) \to H_{2i}(CP^N \times CP^N)$$

is surjective. Moreover, there is an isomorphism

$$H^{2n}(CP^N \times CP^N) \cong H_{2n}(CP^N \times CP^N)$$

which maps every $x \in H^{2n}(CP^N \times CP^N)$ into $x \cap [X] \in H_{2n}(CP^N \times CP^N)$.

(b) The fixed point set $M$ of $\lambda: X \to X$ is a closed smooth $2n$-manifold which can be so oriented that

$$a^n \cap [M] = 1, \quad e \cap [M] = 0.$$

(c) There is a smooth imbedding $\phi: CP^n \to X$ such that

$$a^n \cap \phi_*[CP^n] = 1, \quad b \cap \phi_*[CP^n] = 0.$$

(d) $[M] \cap \phi_*[CP^n] = 1$,

$$\phi_*[CP^n] \cap (\lambda \phi)_*[CP^n] = \text{odd integer}.$$

For any $k = 0, \ldots, 2n$, we let $P_k(a, b)$ be the group of homogeneous polynomials in variables $a$ and $b$ of degree $k$ with integral coefficients. Then for any $i = 0, \ldots, 2n$,

$$H^{2i}(CP^N \times CP^N) = P_i(a, b).$$

As a consequence of (a), (b), (c), (d) above, we have

(9) There are unique $p(a, b), q(a, b) \in P_n(a, b)$ such that

$$p(a, b) \cap [X] = [M], \quad q(a, b) \cap [X] = \phi_*[CP^n].$$

Moreover,

$$a^n p(a, b) \cap [X] = 1, \quad ep(a, b) \cap [X] = 0;$$

$$a^n q(a, b) \cap [X] = 1, \quad bq(a, b) \cap [X] = 0.$$
Furthermore,

\[ p(a, b)q(a, b) \cap [X] = 1, \]
\[ q(a, b)q(b, a) \cap [X] = \text{odd integers}. \]

(10) (i) For any \( i = 0, \ldots, n \), \( a^ib^{n-i}p(a, b) \cap [X] = 1 \).
(ii) \( p(a, b) = p(b, a) \).
(iii) \( p(1, 0) = p(0, 1) = q(1, 1) = 1 \).
(iv) Let \( K \) be the subgroup of \( H^{2n+2}(CP^N \times CP^N) = P_{n+1}(a, b) \) consisting of the elements \( x \cap [X] = 0 \) and let \( L \) be the subgroup of \( P_{n+1}(a, b) \) generated by \( \{a^n b, a^{n-1}b^2, \ldots, a^2b^{n-1}, ab^n\} \). Then

\[ P_{n+1}(a, b) = K \oplus L, \]
\[ q(0, 1) = \pm 1 \] and \( \{aq(b, a), bq(a, b)\} \) is a basis of \( K \).
(v) \( aq(b, a) - bq(a, b) = q(0, 1)ep(a, b) \).

**Proof.**

(i) Since, by (9), \( (a - b)p(a, b) \cap [X] = 0 \), we have

\[ ap(a, b) \cap [X] = bp(a, b) \cap [X]. \]

Hence for any \( i = 0, \ldots, n \),

\[ a^ib^{n-i}p(a, b) \cap [X] = a^n p(a, b) \cap [X] \]

which is equal to 1 by (9).

(ii) Since \( \lambda^*a = b, \lambda^*b = a \) and \( \lambda_*[X] = [X] \), it follows from (i) and (9) that

\[ a^n p(b, a) \cap [X] = b^n p(a, b) \cap [X] = 1, \]
\[ ep(b, a) \cap [X] = -ep(a, b) \cap [X] = 0. \]

Hence, by (9), \( p(b, a) = p(a, b) \).

(iii) By (9) and (ii),

\[ 1 = p(a, b)q(a, b) \cap [X] = p(1, 0)a^n q(a, b) \cap [X] \]
\[ = p(1, 0) = p(0, 1). \]

Let \( q(a, b) = \sum_{i=0}^n \beta_i a^ib^{n-i} \). Then, by (9) and (i),

\[ 1 = q(a, b)p(a, b) \cap [X] = \sum_{i=0}^n \beta_i a^ib^{n-i} p(a, b) \cap [X] \]
\[ = \sum_{i=0}^n \beta_i = q(1, 1). \]
(iv) By (a),
\[ a^n \cap [X], a^{n-1}b \cap [X], \ldots, ab^{n-1} \cap [X], b^n \cap [X] \]
are linearly independent elements of \( H_{2n}(CP^N \times CP^N) \). Therefore
\[ a^{n-1} \cap [X], a^{n-2}b \cap [X], \ldots, ab^{n-2} \cap [X], b^{n-1} \cap [X] \]
are linearly independent elements of \( H_{2n+2}(CP^N \times CP^N) \) and hence \( K \) does not have more than two linearly independent elements.

By (9),
\[
q(0, 1) = q(0, 1)a^n q(a, b) \cap [X] = q(a, b)q(b, a) \cap [X] = \text{odd integers.}
\]

We infer that in \( P_{n+1}(a, b) \),
\[
aq(b, a), a^n b, a^{n-1}b^2, \ldots, a^2 b^{n-1}, ab^n, bq(a, b)
\]
are linearly independent. Therefore \( \{aq(b, a), bq(a, b)\} \) generates a subgroup of \( K \) of finite index.

Let \( \{r(a, b), s(a, b)\} \) be a basis of \( K \). Then
\[
\{r(a, b), a^n b, a^{n-1}b^2, \ldots, a^2 b^{n-1}, ab^n, s(a, b)\}
\]
is a basis of \( P_{n+1}(a, b) \) so that we may assume that
\[
r(1, 0) = 1, \quad r(0, 1) = 0, \quad s(1, 0) = 0, \quad s(0, 1) = 1.
\]
Therefore there are \( r_1(a, b), s_1(a, b) \in P_n(a, b) \) such that
\[
r(a, b) = ar_1(a, b), \quad s(a, b) = bs_1(a, b).
\]
From this result, it follows that
\[
aq(b, a) = q(0, 1)r(a, b) = q(0, 1)ar_1(a, b)
\]
so that
\[
q(b, a) = q(0, 1)r_1(a, b).
\]
Since, by (iii), \( q(1, 1) = 1 \), we infer that
\[
q(0, 1) = \pm 1.
\]
Hence
\[
aq(b, a) = \pm r(a, b), \quad bq(a, b) = \pm s(a, b)
\]
and consequently \( \{aq(b, a), bq(a, b)\} \) is a basis of \( K \).
(v) By (9), \( ep(a, b) \) is in \( K \) and by (iv), \{a(q(b, a), b(q(a, b))\} is a basis of \( K \). Then for some integers \( s \) and \( t \),

\[
ep(a, b) = saq(b, a) + tbq(a, b). 
\]

By setting \( a = 1 \) and \( b = 0 \), we obtain \( sq(0, 1) = 1 \) by (iii). Therefore \( s = q(0, 1) \). Similarly, \( t = -q(0, 1) \). Hence our assertion follows.

\[
(11) \quad p(a, b) = \sum_{i=0}^{n} a^{n-i}b^i \quad \text{and} \quad q(a, b) = b^n. 
\]

Proof. Assume first that \( n = 1 \). By [4], we may set \( M = CP^1 \).

As seen in Remark 4, which is valid for \( n = 1 \), we may let \( W \) be \( CP^1 \times CP^1 \) and let \( M \) be the diagonal set in \( CP^1 \times CP^1 \). As we have done earlier, we let \{a, b\} be the basis of \( H^2(CP^1 \times CP^1) \) such that

\[
a \cap [CP^1 \times CP^1] = [CP^0 \times CP^1], \\
b \cap [CP^1 \times CP^1] = [CP^1 \times CP^0], 
\]

and let \( p(a, b) \) and \( q(a, b) \) be the elements of \( H^2(CP^1 \times CP^1) \) such that

\[
p(a, b) \cap [W] = [M], \quad q(a, b) \cap [W] = [CP^1 \times CP^0]. 
\]

It is not hard to see that

\[
p(a, b) = a + b, \quad q(a, b) = b. 
\]

Hence (11) holds for \( n = 1 \).

Now we proceed by induction on \( n \) and assume that our assertion holds when \( n \) is replaced by \( n - 1 \), \( n > 1 \). Since

\[
X \subset CP^N \times CP^N \subset CP^{N+1} \times CP^{N+1},
\]

we can use a \( \lambda \)-equivariant isotopy to alter \( X \) so that the following hold.

1. \( \phi(CP^n) \) is contained in \( CP^{N+1} \times CP^N \) and intersects \( CP^N \times CP^{N+1} \) transversally at \( \phi(CP^{n-1}) \).
2. \( M \) and \( X \) are transversal to \( CP^N \times CP^{N+1} \).
3. \( X' = X \cap (CP^N \times CP^N) \) is a connected closed smooth \((4n-4)\)-manifold invariant under \( \lambda \).
Let \( X' \) be oriented so that
\[
[X'] = ab \cap [X].
\]
We claim that \( X' \) satisfies (a), (b), (c), (d) with \( n - 1 \) in place of \( n \).

For any \( i = 0, \ldots, n - 2 \),
\[
f_*H_{2i}(X') = ab \cap f_*H_{2i+4}(X)
= ab \cap H_{2i+4}(\mathbb{CP}^N \times \mathbb{CP}^N) = H_{2i}(\mathbb{CP}^N \times \mathbb{CP}^N).
\]
By (10), (iv),
\[
ab \cup f^*H^{2n-2}(\mathbb{CP}^N \times \mathbb{CP}^N) = f^*H^{2n+2}(\mathbb{CP}^N \times \mathbb{CP}^N).
\]
Then
\[
ab \cap f_*H_{2n+2}(X) = f_*H_{2n-2}(X) = H_{2n-2}(\mathbb{CP}^N \times \mathbb{CP}^N)
\]
and hence
\[
f_*H_{2n-2}(X') = f_*(ab \cap H_{2n+2}(X)) = H_{2n-2}(\mathbb{CP}^N \times \mathbb{CP}^N).
\]
Since
\[
f^*H^{2n-2}(\mathbb{CP}^N \times \mathbb{CP}^N) \cap [X']
= f^*H^{2n-2}(\mathbb{CP}^N \times \mathbb{CP}^N) \cap (ab \cap [X])
= (ab \cup f^*H^{2n-2}(\mathbb{CP}^N \times \mathbb{CP}^N)) \cap [X]
= f^*H^{2n+2}(\mathbb{CP}^N \times \mathbb{CP}^N) \cap [X]
\cong f_*H_{2n-2}(X) = f_*H_{2n-2}(X'),
\]
it follows that there is an isomorphism of \( H^{2n-2}(\mathbb{CP}^N \times \mathbb{CP}^N) \) onto \( H_{2n-2}(\mathbb{CP}^N \times \mathbb{CP}^N) \) which maps every \( x \in H^{2n-2}(\mathbb{CP}^N \times \mathbb{CP}^N) \) into \( x \cap f_*[X'] \in H_{2n-2}(\mathbb{CP}^N \times \mathbb{CP}^N) \). The rest is rather obvious.

By the induction hypothesis, \( q'(a, b) = b^{n-1} \) is the unique element of \( H^{2n-2}(\mathbb{CP}^N \times \mathbb{CP}^N) \) such that
\[
q'(a, b) \cap [X'] = \phi_*[\mathbb{CP}^{n-1}]
\]
so that
\[
ab^n \cap [X] = b^{n-1} \cap (ab \cap [X]) = \phi_*[\mathbb{CP}^{n-1}].
\]
Then
\[
(a(b^n - q(a, b)) \cap [X] = \phi_*[\mathbb{CP}^{n-1}] - a \cap \phi_*[\mathbb{CP}^n] = 0.
\]
Therefore, by (10), (iv),
\[
b^n - q(a, b) = kq(b, a)
\]
for some integer \( k \). Since, by (10), (iii), \( q(1, 1) = 1 \), it follows that
k = 0 and hence
\[ q(a, b) = b^n. \]
From this result and (10), (v), it is clear that
\[ p(a, b) = \sum_{i=0}^{n} a^{n-i} b^i \]
follows.

**Proof of our theorem.** In \( H^*(W) \),
\[ a^{n+1} = aq(b, a) = 0 \]
and then in \( H^*(CM) \),
\[ a^{n+1} = 0. \]
Hence our assertion follows as seen in Remark 2.

**References**


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