

Pacific Journal of Mathematics

**ON REGULAR COVERINGS OF 3-MANIFOLDS BY
HOMOLOGY 3-SPHERES**

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ON REGULAR COVERINGS OF 3-MANIFOLDS BY HOMOLOGY 3-SPHERES

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We study homology 3-spheres \widetilde{M} that admit fixed point free actions by a finite group G . If G also admits a fixed point free orthogonal action on S^3 and if certain projective $Z[G]$ -modules satisfy a cancellation property we show that the regular covering $\widetilde{M} \rightarrow \widetilde{M}/G$ is induced from a standard regular covering $S^3 \rightarrow S^3/G$ by means of a map $f: \widetilde{M}/G \rightarrow S^3/G$ whose degree is relatively prime to the order of G (Theorem 1). We also completely characterize those regular coverings $\widetilde{M} \rightarrow M$ where M is Seifert fibered (§4). Finally, starting with any given regular covering $\widetilde{M}_0 \rightarrow M_0$ with group of covering transformations G , M_0 irreducible, and \widetilde{M}_0 a homology 3-sphere, we show how to construct another regular covering $\widetilde{M} \rightarrow M$ with \widetilde{M} a homology 3-sphere and the same group G of covering transformations, with M sufficiently large, M and M_0 not homotopy equivalent, and a degree 1 map $f: M \rightarrow M_0$ that induces the regular covering $\widetilde{M} \rightarrow M$ from the regular covering $\widetilde{M}_0 \rightarrow M_0$.

1. Introduction. It is a classical result that the finite groups that admit a fixed point free orthogonal action on the 3-sphere S^3 are exactly the groups of the following four classes (see [ST] or [Mil]):

(I) The binary polyhedral groups, that is, the binary dihedral groups

$$Q_{4n} = \{x, y; x^2 = (xy)^2 = y^n\}, \quad n \geq 2;$$

the binary tetrahedral group

$$T_{24} = \{x, y; x^2 = (xy)^3 = y^3, x^4 = 1\};$$

the binary octahedral group

$$O_{48} = \{x, y; x^2 = (xy)^3 = y^4, x^4 = 1\};$$

the binary icosahedral group

$$I_{120} = \{x, y; x^2 = (xy)^3 = y^5, x^4 = 1\}.$$

(II) The groups

$$D(2^k, 2l+1) = \{x, y; x^{2^k} = 1, y^{2l+1} = 1, xyx^{-1} = y^{-1}\},$$

$$k \geq 3, l \geq 1.$$

(III) The groups

$$T(8, 3^k) = \{x, y, z; x^2 = (xy)^2 = y^2, z^{3^k} = 1, \\ zxz^{-1} = y, zyz^{-1} = xy\}, \quad k \geq 2.$$

(IV) Cyclic groups \mathbb{Z}_m and direct products $\mathbb{Z}_m \times G$, where G is any group in classes (I), (II) or (III), with order relatively prime to m .

Except for the cyclic groups, the groups G are uniquely determined up to conjugacy in $O(4)$. The orbit manifold S^3/G is a spherical space form. If G is not cyclic then S^3/G is uniquely determined up to isometry. We refer to the natural covering $q: S^3 \rightarrow S^3/G$ and the action of G on S^3 as being standard. Each space form S^3/G admits a Seifert fibration.

A homology 3-sphere is a 3-manifold with the same homology as a 3-sphere. It was shown in [Mi1] and [L] that if the finite group G acts fixed point freely on some homology 3-sphere, then it must belong to one of the classes (I), ..., (IV) or to the following class of groups:

(V) The groups

$$Q(8n, k, l) = \{x, y, z; x^2 = (xy)^2 = y^{2n}, z^{kl} = 1, \\ xzx^{-1} = z^r, yzy^{-1} = z^{-1}\}$$

where n, k, l are relatively prime odd integers, $r \equiv -1 \pmod{k}$ and $r \equiv 1 \pmod{l}$ or direct products $\mathbb{Z}_m \times Q(8n, k, l)$ where m and the order $8nkl$ of the group $Q(8n, k, l)$ are relatively prime.

Some of the groups in (V) act fixed point freely on some homology 3-spheres and some cannot act fixed point freely on any homology 3-sphere (see [DM], p. 278). It is a conjecture that the groups in (V) cannot act fixed point freely on S^3 (see [Th]).

In this paper we will study those 3-manifolds M which admit regular coverings by a homology 3-sphere \widetilde{M} . Thus if G denotes the group of covering transformations, then G belongs to one of the classes (I), ..., (V). If G is in (I), ..., (IV) then G has a fixed point free orthogonal action on S^3 . We address the following

Problem. Find conditions under which there is a degree 1 map $f: M \rightarrow S^3/G$ so that the covering $p: \widetilde{M} \rightarrow M$ is induced from the standard covering $q: S^3 \rightarrow S^3/G$ by the map $f: M \rightarrow S^3/G$.

If G is a cyclic group \mathbb{Z}_n then any regular covering $p: \widetilde{M} \rightarrow M$, with \widetilde{M} a homology 3-sphere and group of covering transformations \mathbb{Z}_n , can be induced from a standard covering $q: S^3 \rightarrow S^3/\mathbb{Z}_n$ by a degree 1 map $f: M \rightarrow S^3/\mathbb{Z}_n$ onto the lens space S^3/\mathbb{Z}_n which is determined uniquely up to homotopy equivalence. See [LS2].

Suppose G is a finite group. Let $N = \sum_{x \in G} x$ denote the norm element in the integral group ring $Z[G]$. For any integer r the left ideal generated by r and N is denoted by (r, N) . If r is relatively prime to the order of G , then the ideal (r, N) is a finitely generated projective $Z[G]$ -module (see [SW1]). We say that G has the weak cancellation property if (r, N) is free whenever it is stably free. (G has the cancellation property, if $Z[G] \oplus P \cong Z[G] \oplus Q$ implies that $P \cong Q$ for finitely generated $Z[G]$ -modules P, Q).

Amongst the groups in (I), ..., (IV) the following are known to have the weak cancellation property:

(1) All cyclic groups. (In fact finite abelian groups have the cancellation property [SE].)

(2) The groups T_{24} , O_{48} and I_{120} .

(3) The groups Q_{2^k} , Q_{4p} with p an odd prime. See [SW2].

The augmentation ideal of $Z[G]$ is denoted by $A[G]$. A (G, m) -complex is a finite connected m -dimensional CW-complex X with $\pi_1(X) \cong G$ and whose universal covering space is $(m - 1)$ -connected.

In §3 we prove

THEOREM 1. *Let $p: \widetilde{M} \rightarrow M$ be a regular covering of the 3-manifold M by the homology 3-sphere \widetilde{M} . Assume that the group G of covering transformations has the weak cancellation property, where G is in one of the classes (I), (II), (III), or (IV). If $q: S^3 \rightarrow S^3/G$ is a standard covering then*

(1) *The mapping cones C_p and C_q are homotopy equivalent $(G, 4)$ -complexes with $\pi_4 \cong A[G]$.*

(2) *There is a map $f: M \rightarrow S^3/G$ with degree relatively prime to the order of G and with $f_*\pi_1(M) = \pi_1(S^3/G) = G$ such that the regular covering $p: \widetilde{M} \rightarrow M$ is induced from the standard regular covering $q: S^3 \rightarrow S^3/G$ by the map $f: M \rightarrow S^3/G$.*

If the manifold M admits a Seifert fibration then either $\widetilde{M} = S^3$, $M = S^3/G$, and $p: \widetilde{M} \rightarrow M$ is standard, or the group of covering transformations of $p: \widetilde{M} \rightarrow M$ is cyclic and is a transformation group of the Seifert fibration of \widetilde{M} induced by $p: \widetilde{M} \rightarrow M$. See Theorem (4.1). We give explicit descriptions of Seifert fibered homology 3-spheres with fixed point free cyclic group actions.

THEOREM 2. *Let $p_0: \widetilde{M}_0 \rightarrow M_0$ be a regular covering of the irreducible 3-manifold M_0 by a homology 3-sphere \widetilde{M}_0 with group of*

covering transformations G . Then there is a sufficiently large 3-manifold M containing an incompressible torus, M and M_0 not homotopy equivalent, a regular covering $p: \widetilde{M} \rightarrow M$ of M by a homology 3-sphere \widetilde{M} with the same group G of covering transformations and a degree 1 map $f: M \rightarrow M_0$ such that the regular covering $p: \widetilde{M} \rightarrow M$ is induced from the regular covering $p_0: \widetilde{M}_0 \rightarrow M_0$ by $f: M \rightarrow M_0$.

Starting with a standard covering $q: S^3 \rightarrow S^3/G$ we can thus construct an abundance of sufficiently large 3-manifolds containing incompressible tori, that admit regular coverings by homology 3-spheres and with group of covering transformations G .

Also starting from a fixed point free action of one of the groups $Q(8n, k, l)$ on some homology 3-sphere \widetilde{M}_0 we can thus produce examples of sufficiently large homology 3-spheres \widetilde{M} containing incompressible tori and admitting fixed point free actions by $Q(8n, k, l)$.

The case of $G = I_{120}$, the binary icosahedral group (this is the case of nontrivial regular coverings of homology 3-spheres by homology 3-spheres) was considered in [LS1].

2. Preliminaries. Throughout this paper we work in the PL category. A PL homeomorphism we simply call an isomorphism. Our reference for 3-manifold concepts is [He1].

A 3-manifold M is irreducible if each 2-sphere in M bounds a 3-cell in M . Note that if a 3-manifold is regularly covered by a homology 3-sphere, then it is necessarily orientable (the covering transformations must preserve the orientation by the Lefschetz fixed point theorem).

A surface is a connected compact 2-manifold. A surface F in a 3-manifold M is proper if $F \cap \partial M = \partial F$, and it is incompressible in M if it is not a 2-sphere or a 2-cell and if for each 2-cell $D \subset M$ with $D \cap F = \partial D$ there is a 2-cell $D_0 \subset F$ such that $\partial D_0 = \partial D$. An orientable connected closed 3-manifold is sufficiently large if it is irreducible and contains a 2-sided incompressible closed surface.

In [LS2] the following proposition was proved.

PROPOSITION (2.1). *Let W be a compact 3-manifold with ∂W a torus. Suppose there is a 1-sphere $S^1 \subset \partial W$ such that $H_1(W) = Z[S^1]$. Then there is a connected proper 2-sided surface $F \subset W$ such that ∂F is a 1-sphere in ∂W and ∂F intersects S^1 transversally in exactly one point.*

The proof of Theorem 1 will be based on the following.

PROPOSITION (2.2). *Suppose G is a finite group with periodic cohomology and with minimal free period k . If G has the weak cancellation property then all (G, M) -complexes with $m = lk$ and $\pi_m(X) \cong A[G]$ as π_1 -modules are homotopy equivalent.*

Proof. This follows from results in [Dy], see also [LS1]. If X is a (G, m) -complex with $\pi_m(X) \cong A[G]$, then its algebraic m -type is $(G, A[G], r)$ where $r = r(X) \in H^{m+1}(G, \pi_m(X)) = \mathbb{Z}_{|G|}$ is the k -invariant, $|G|$ the order of G . It is a unit in $\mathbb{Z}_{|G|}$. Therefore (r, N) is a projective ideal. It must be stably free (see Theorem 3.5 of [Dy]—the condition $m \geq 3$ is only needed for one of the directions in this theorem). By hypothesis (r, N) is actually free. According to Corollary (8.4) of [Dy] this means that there is only one isomorphism class of algebraic m -types, and therefore only one homotopy type of (G, m) -complexes with $\pi_m = A[G]$. \square

3. Proof of Theorem 1. In the following let G be any group from one of the classes (I), (II), (III) or (IV), $p: \widetilde{M} \rightarrow M$ a regular covering with G as group of covering transformations, \widetilde{M} a homology 3-sphere, and $q: S^3 \rightarrow S^3/G$ the regular covering corresponding to any fixed point free orthogonal action of G on S^3 .

If X is a space and $f: X \rightarrow Y$ is a map let CX , SX and C_f denote the unreduced cone, suspension, and mapping cone respectively.

Define W to be the space $W = G \times C\widetilde{M} / (g_1, \tilde{x}, 0) = (g_2, \tilde{x}, 0)$. See Figure 1.

Note that W is 3-connected since collapsing one of the cones to a point gives a homotopy equivalence

$$W = \underbrace{S\widetilde{M} \vee \cdots \vee S\widetilde{M}}_{|G|-1 \text{ copies}} = \underbrace{S^4 \vee \cdots \vee S^4}_{|G|-1 \text{ copies}}.$$

Also note that there is a natural G -action on W defined by $G \times W \rightarrow W$, $h \cdot (g, \tilde{x}, t) = (hg, h(\tilde{x}), t)$ and that

$$W/G = C\widetilde{M} / \{(\tilde{x}, 0) = (g(\tilde{x}), 0)\} = C_p.$$

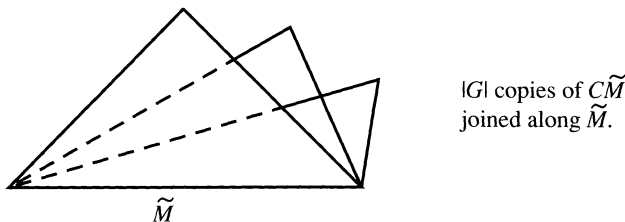


FIGURE 1

Since this action is fixed point free this implies that W is the universal covering space of C_p .

PROPOSITION (3.1). C_p is a $(G, 4)$ -complex with $\pi_4(C_p) \cong A[G]$ (as $Z[G]$ -modules).

Proof. The same argument as in Lemma 3.3 of [LS1] applies.

COROLLARY (3.2). Suppose the group G satisfies the weak cancellation property. Then C_p and C_g are homotopy equivalent.

Proof. This follows immediately from Propositions (3.1) and (2.2). The minimal free period of G is 2 if G is cyclic and nontrivial, and 4 otherwise. □

Note. If $L_{n,k}, L_{n,l}$ are lens spaces with fundamental groups \mathbb{Z}_n , and $p_k: S^3 \rightarrow L_{n,k}, p_l: S^3 \rightarrow L_{n,l}$ are the universal coverings, then C_{p_k} and C_{p_l} are homotopy equivalent. But $L_{n,k}$ and $L_{n,l}$ are homotopy equivalent if and only if $kl \equiv \pm m^2 \pmod n$ for some m (see e.g. [Co], p. 96).

PROPOSITION (3.3). If C_p and C_q are homotopy equivalent, then there is a map $f: M \rightarrow S^3/G$ so that:

- (1) $f_*\pi_1(M) = \pi_1(S^3/G) = G$
- (2) $p_*(\pi_1(\widetilde{M})) = \ker(f_*: \pi_1(M) \rightarrow G)$
- (3) the degree of f is relatively prime to the order $|G|$ of G .

Proof. Let $h: C_p \rightarrow C_q$ be a homotopy equivalence and let $i: M \rightarrow C_p$ and $j: S^3/G \rightarrow C_q$ be the inclusions. Note that C_q is obtained from S^3/G by attaching a 4-cell. By the cellular approximation theorem we therefore can alter h by a homotopy if necessary, so that $hi(M) \subset S^3/G$. Let $f = hi: M \rightarrow S^3/G$. Thus we have the commutative diagram

$$\begin{array}{ccc}
 \widetilde{M} & \xrightarrow{\tilde{f}} & S^3 \\
 \downarrow p & & \downarrow q \\
 M & \xrightarrow{f} & S^3/G \\
 \downarrow i & & \downarrow j \\
 C_p & \xrightarrow{h} & C_q
 \end{array}$$

Note that $i_*: \pi_1(M) \rightarrow \pi_1(C_p)$ is an epimorphism and $j_*: \pi_1(S^3/G) \rightarrow \pi_1(C_q)$ is an isomorphism. Therefore $f_*\pi_1(M) = \pi_1(S^3/G)$.

Since $p_*\pi_1(\widetilde{M}) \subset \ker(f_*: \pi_1(M) \rightarrow G)$, it follows that $p_*\pi_1(M) = \ker(f_*: \pi_1(M) \rightarrow G)$. Property (3) follows from the commutative diagram

$$\begin{array}{ccc} H_3(M) = \mathbb{Z} & \xrightarrow{f_*} & H_3(S^3/G) \\ i_* \downarrow & & j_* \downarrow \\ H_3(C_p) = \mathbb{Z}_{|G|} & \xrightarrow{h_*} & H_3(C_q) = \mathbb{Z}_{|G|} \end{array}$$

where i_* , j_* are epimorphisms and h_* is an isomorphism. \square

Corollary (3.2) and Proposition (3.3) prove Theorem 1.

REMARK. It should be noted that a map $f: M \rightarrow S^3/G$ inducing the regular covering $p: \widetilde{M} \rightarrow M$ from the covering $q: S^3 \rightarrow S^3/G$ can be constructed by elementary obstruction theory. The properties (1) and (2) are consequences of this construction. The map $f: M \rightarrow S^3/G$ and its lift $\tilde{f}: \widetilde{M} \rightarrow S^3$ will define by coning a map $h: C_p \rightarrow C_q$ with $h_*: \pi_1(C_p) \rightarrow \pi_1(C_q)$ an isomorphism. This map, however, will in general not be a homotopy equivalence (which was used to prove property (3)).

The composite $g \cdot f$, where $g: S^3/G \rightarrow S^3/G$ is any self map inducing an isomorphism on π_1 and having degree relatively prime to $|G|$, will also satisfy the conclusions of Proposition (3.3), and this will change $\deg f$ into the product $(\deg g) \cdot (\deg f)$. We can also alter $\deg f$ to any representative in its congruence class modulo $|G|$. (To see this let $B^3 \subset M$ be any 3-cell. Then collapsing ∂B^3 to a point gives a map $c: M \rightarrow M \vee S^2$. Consider the composite

$$M \xrightarrow{l} M \vee S^3 \xrightarrow{f \vee g} S^3/G \vee S^3/G \xrightarrow{\nabla} S^3/G$$

where $g: S^3 \rightarrow S^3/G$ has degree $k|G|$ and ∇ is the folding map. Then $\nabla(f \vee g)l$ will satisfy the conclusions of Proposition (3.3), where now the degree is $\deg f + k|G|$.)

4. Regular coverings of Seifert fibered 3-manifolds by homology 3-spheres. We have the following uniqueness result.

THEOREM (4.1). *Let M be a 3-manifold that admits a Seifert fibration, let $p: \widetilde{M} \rightarrow M$ be a non-trivial regular covering by a homology 3-sphere, and let G be the group of covering transformations. Then one of the following holds:*

- (1) *Either $\widetilde{M} = S^3$, $M = S^3/G$, and $p: S^3 \rightarrow S^3/G$ is standard.*
- (2) *Or G is cyclic and it is a transformation group of the Seifert*

fibration of \widetilde{M} induced by $p: \widetilde{M} \rightarrow M$ from the Seifert fibration on M .

Proof. \widetilde{M} is given the Seifert fibration induced by the regular covering $p: \widetilde{M} \rightarrow M$ from the Seifert fibration on M . Then G maps fibers onto fibers.

If $S_0^1 \subset M$ is a regular fiber, then the components of $p^{-1}(S_0^1)$ are all regular fibers.

Claim. If $S^1 \subset M$ is a singular fiber, then either $p^{-1}(S^1)$ is connected, or all components of $p^{-1}(S^1)$ are regular fibers.

Proof of Claim. Suppose $p^{-1}(S^1)$ is not connected and there is a singular fiber $\widetilde{S}^1 \subset p^{-1}(S^1)$. Since G acts transitively on the components of $p^{-1}(S^1)$ all components of $p^{-1}(S^1)$ are singular fibers and they have the same Seifert invariants. This contradicts the assumption that \widetilde{M} is a homology 3-sphere (see Satz 12 of [S]).

Case 1. All fibers of \widetilde{M} are regular.

Then by the remark preceding Satz 12 of [S] \widetilde{M} must be the 3-sphere. Hence $p: \widetilde{M} \rightarrow M$ is standard.

Case 2. M has a singular fiber \widetilde{S}^1 .

Then G acts without fixed points on \widetilde{S}^1 and therefore must be cyclic.

It remains to prove that in Case 2, G leaves the fibers setwise fixed. Let $S_1, \dots, S_s, S_{s+1}, \dots, S_{s+t}$ denote the singular fibers in M where $p^{-1}(S_1), \dots, p^{-1}(S_s)$ are the singular fibers in \widetilde{M} and the components of $p^{-1}(S_{s+i}), i = 1, \dots, t$, are all regular fibers in \widetilde{M} . Let n_i be the number of components in $p^{-1}(S_{s+i}), i = 1, \dots, t$.

The Seifert surface of \widetilde{M} is a 2-sphere (see [S] p. 207). Suppose the Seifert surface of M is a surface with Euler characteristic $2 - d$, $d \geq 0$. Let $V_1, \dots, V_{s+t} \subset M$ be disjoint fibered solid tori with centers in the fibers S_1, \dots, S_{s+t} . Then $p|: \widetilde{M} - p^{-1}(V_1 \cup \dots \cup V_{s+t}) \rightarrow M - (V_1 \cup \dots \cup V_{s+t})$ is a regular covering projection. The Seifert surface of $M - p^{-1}(V_1 \cup \dots \cup V_{s+t})$ is a 2-sphere with $s + n_1 + \dots + n_t$ holes, and the Seifert surface of $M - (V_1 \cup \dots \cup V_{s+t})$ has $s + t$ holes. The covering projection $p|$ induces a covering projection of, the Seifert surfaces.

Suppose it has k sheets. Then we have the following formula for the Euler characteristics of the Seifert surfaces:

$$2 - (s + n_1 + \cdots + n_t) = k(2 - d - s - t).$$

Note that $n_i \leq k$, $i = 1, \dots, t$. Therefore, $(s + d - 2)k \leq s - 2$. Necessarily, $d = 0$. If $s \leq 2$, \widetilde{M} must be a 3-sphere and $p: \widetilde{M} \rightarrow M$ is standard.

If $s \geq 3$, then necessarily $k = 1$, i.e. the induced covering projection on the Seifert surfaces is an isomorphism. Therefore G leaves each fiber of \widetilde{M} setwise fixed, i.e. G is a transformation group of the Seifert fibration of M (see [S], §14). \square

A Seifert fibered space M has a unique geometric structure in the sense of Thurston and there are exactly six possible geometries for M determined by the following table (see [Sc1] Theorem 5.3, p. 477):

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$S^2 \times \mathbf{R}$	E^3	$H^2 \times \mathbf{R}$
$e \neq 0$	S^3	Nil	$\widetilde{\text{SL}}_2(\mathbf{R})$

Here χ is the orbifold Euler characteristic of the Seifert surface F , e is the Euler number of the Seifert bundle $M \rightarrow F$, and $S^2 \times \mathbf{R}$, S^3 , E^3 , Nil, $H^2 \times \mathbf{R}$, $\widetilde{\text{SL}}_2(\mathbf{R})$ are the six possible universal coverings (geometries) on which $\pi_1(M)$ acts by isometries. Note, both χ and e are rational numbers. If both M and F are orientable then the Seifert invariant of M is $(0, o; g|b; \alpha_1, \beta_1; \dots; \alpha_r, \beta_r)$, and

$$\chi = 2 - 2g - \sum_{i=1}^r \left(1 - \frac{1}{\alpha_i}\right), \quad e = - \left(b + \sum_{i=1}^r \frac{\beta_i}{\alpha_i}\right)$$

where g is the genus of F and the (α_i, β_i) , $i = 1 \cdots r$, are the invariants of the singular fibers. See [SC1] p. 427 and p. 437 respectively.

In §12 of [S] the following is proved: If M is a homology 3-sphere then $g = 0$ and the $\alpha_1, \dots, \alpha_r$ are relatively prime in pairs. Moreover, if $M \neq S^3$ then $r \geq 3$. Conversely, for any $r \geq 3$ pairwise coprime integers $\alpha_1, \dots, \alpha_r \geq 2$ there is a unique Seifert fibered homology 3-sphere with Seifert invariant $(0, o; o|b; \alpha_1, \beta_1; \dots; \alpha_r, \beta_r)$. We denote this homology 3-sphere by $\sum(\alpha_1, \dots, \alpha_r)$.

We have the following proposition ([Mi2], [N]).

PROPOSITION (4.2). *The geometry of a Seifert fibered homology 3-sphere is either modelled on S^3 or on $\widetilde{\mathrm{SL}}_2(\mathbf{R})$. The 3-sphere S^3 and the dodecahedral space $\Sigma(2, 3, 5)$ are modelled on S^3 . All other homology 3-spheres $\Sigma(\alpha_1, \dots, \alpha_r)$, $r \geq 3$, $(\alpha_1, \dots, \alpha_r) \neq (2, 3, 5)$ are modelled on $\widetilde{\mathrm{SL}}_2(\mathbf{R})$.*

Proof. Let M be a Seifert fibered space with Seifert invariant $(0, o; o|b; \alpha_1, \beta_1; \dots; \alpha_r, \beta_r)$. Then equation (3) in §12 of [S] states that

$$b\alpha_1 \cdots \alpha_r + \beta_1\alpha_2 \cdots \alpha_r + \alpha_1\beta_2\alpha_3 \cdots \alpha_r + \cdots + \alpha_1 \cdots \alpha_{r-1}\beta_r + \pm 1.$$

Hence $e = \pm 1/\alpha_1 \cdots \alpha_r \neq 0$. Consequently, the only possible geometries for M to be modelled on are S^3 , Nil, or $\widetilde{\mathrm{SL}}_2(\mathbf{R})$.

Since $g = 0$, $\chi = 2 - \sum_{i=1}^r (1 - 1/\alpha_i)$. If M is not S^3 then $r \geq 3$ and it is easy to show the following

(1) $\chi > 0$ only for the unordered triples $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$

(2) $\chi = 0$ only for the unordered triples $(2, 3, 6)$, $(2, 4, 4)$, $(3, 3, 3)$ and for the 4-tuple $(2, 2, 2, 2)$. The only homology 3-sphere in this list comes from $(2, 3, 5)$ since all of the other unordered r -tuples $(\alpha_1, \dots, \alpha_r)$ are not relatively prime in pairs. \square

The homology 3-spheres $\Sigma(\alpha_1, \dots, \alpha_r)$, $r \geq 3$, have representations as follows (see [N]).

Let $a_{ij} \in \mathbb{C}$, $i = 1, \dots, r-2$, $j = 1, \dots, r$, be such that every $(r-2) \times (r-2)$ submatrix of the $(r-2) \times r$ matrix $A = (a_{ij})$ is non-singular. Then

$$\begin{aligned} V_A(\alpha_1, \dots, \alpha_r) \\ = \{(z_1, \dots, z_r) \in \mathbb{C}^r : a_{i1}z_1^{\alpha_1} + \cdots + a_{ir}z_r^{\alpha_r} = 0, i = 1, \dots, r-2\} \end{aligned}$$

is a complex algebraic surface which is non-singular except at 0. Let $S^{2r-1} = \{(z_1, \dots, z_r) \in \mathbb{C}^r : |z_1|^2 + \cdots + |z_r|^2 = 1\}$ be the unit sphere in \mathbb{C}^r .

Then

$$\Sigma(\alpha_1, \dots, \alpha_r) = V_A(\alpha_1, \dots, \alpha_r) \cap S^{2r-1}.$$

In particular, the diffeomorphism type of $\Sigma(\alpha_1, \dots, \alpha_r)$ is independent of the matrix A .

If $r = 3$, we may choose $A = (1, 1, 1)$ and obtain

$$\Sigma(\alpha_1, \alpha_2, \alpha_3) = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^{\alpha_1} + z_2^{\alpha_2} + z_3^{\alpha_3} = 0\} \cap S^5.$$

Let $G = \mathbb{Z}_n$ be a cyclic group and suppose that n is relatively prime to each of $\alpha_1, \dots, \alpha_r$. Then \mathbb{Z}_n acts on $\sum(\alpha_1, \dots, \alpha_r)$ without fixed points as a transformation group of the Seifert fibration as follows:

$$\begin{aligned} \mathbb{Z}_n \times \sum(\alpha_1, \dots, \alpha_r) &\longrightarrow \sum(\alpha_1, \dots, \alpha_r) \\ (t^i, (z_1, \dots, z_r)) &\longrightarrow (\xi^{\alpha/\alpha_1 i} z_1, \dots, \xi^{\alpha/\alpha_r i} z_r) \end{aligned}$$

where t is a generator of \mathbb{Z}_n , $\alpha = \alpha_1 \cdots \alpha_r$ and ξ is a primitive n th root of unity. We call any conjugate of this action a standard action of \mathbb{Z}_n on the homology 3-sphere $\sum(\alpha_1, \dots, \alpha_r)$.

PROPOSITION (4.3). *Let G be a group acting fixed point freely on the homology 3-sphere $\sum(\alpha_1, \dots, \alpha_r)$. Then $G = \mathbb{Z}_n$ and the action is standard.*

Proof. $\sum(\alpha_1, \dots, \alpha_r)/G$ has a Seifert fibered structure (see [Sc2] p. 35). By Theorem (4.1), $G = \mathbb{Z}_n$ is a transformation group of the Seifert fibration of $\sum(\alpha_1, \dots, \alpha_r)$ induced by the regular covering $p: \sum(\alpha_1, \dots, \alpha_r) \rightarrow \sum(\alpha_1, \dots, \alpha_r)/G$. But the Seifert fibration of $\sum(\alpha_1, \dots, \alpha_r)$ is unique (see [S] Satz 12). \square

Thus we have the following

COROLLARY (4.4). *Let M be a Seifert fibered 3-manifold and let $p: \widetilde{M} \rightarrow M$ be a regular covering by a homology 3-sphere with cyclic group of covering transformations. Then either $\widetilde{M} = S^3$ or $\widetilde{M} = \sum(\alpha_1, \dots, \alpha_r)$ and the action is standard.*

5. Proof of Theorem 2. Let W be an irreducible orientable compact 3-manifold with ∂W a torus and with $H_1(W) = \mathbb{Z}$, W not a solid torus. (e.g. let X be any irreducible homology 3-sphere with $\pi_1(X) \neq 1$ and $S^1 \subset X$ a 1-sphere not nullhomotopic in X , or $X = S^3$ and $S^1 \subset S^3$ a knot. Then $W = \overline{X - N(S^1)}$, where $N(S^1)$ is a regular neighborhood of S^1 in X , is such an irreducible orientable compact 3-manifold). Note that ∂W is incompressible in W . By Proposition (2.1) there is a connected proper 2-sided surface $F \subset W$ such that ∂F is a 1-sphere in ∂W , and ∂F intersects S^1 transversally in exactly one point. Let $\partial W = S^1 \times \partial F$ be such that $[S^1]$ is a generator of $H_1(W)$.

By a result of [Ha], there is a 1-sphere $S_0^1 \subset M_0$ which is null homotopic in M_0 and such that $C = \overline{M_0 - N(S_0^1)}$ is a fiber bundle over

a 1-sphere with fiber a proper surface F_0 , where $N(S_0^1) = S_0^1 \times D_0^2$ is a regular neighborhood of S_0^1 in M_0 . Note that C is irreducible and the torus ∂C is incompressible in C . Let $g: (W, \partial W) \rightarrow (N(S_0^1), \partial N(S_0^1))$ be a map such that $g|: \partial W \rightarrow \partial N(S_0^1)$ is an isomorphism, $g(F) = D_0^2$ and $g_*[S^1] = [S_0^1]$ in $H_1(\partial N(S_0^1))$. Define

$$M = W \cup C/x = g(x), \quad x \in \partial W$$

and the map $f: M \rightarrow M_0$ by

$$f(x) = \begin{cases} x, & x \in C, \\ g(x), & x \in W. \end{cases}$$

The closed 3-manifold M is orientable, irreducible and the torus $\partial W = \partial C$ is incompressible in M . The map $f: M \rightarrow M_0$ has degree 1.

Let $p: \widetilde{M} \rightarrow M$ be the regular covering induced from the regular covering $p_0: \widetilde{M}_0 \rightarrow M_0$ by the degree 1 map $f: M \rightarrow M_0$. The same Mayer-Vietoris sequence argument as in the proof of Theorem (5.1) of [LS1] applies to prove that \widetilde{M} is a homology 3-sphere.

Lastly, M and M_0 cannot be homotopy equivalent since $\pi_1(M)$ and $\pi_1(M_0)$ cannot be isomorphic. To see this, note that $f_*: \pi_1(M) \rightarrow \pi_1(M_0)$ is an epimorphism with $\ker(f_*) \neq 1$. Thus if $\pi_1(M) \cong \pi_1(M_0)$, then $\pi_1(M) \cong \pi_1(M)/\ker(f_*)$ and $\pi_1(M)$ is not Hopfian. But M is sufficiently large and therefore $\pi_1(M)$ is residually finite and hence Hopfian (see [He2]), a contradiction. \square

REMARK. Since \widetilde{M} is orientable and irreducible it follows from a result of [Du] that \widetilde{M} is also irreducible.

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Received August 15, 1989 and in revised form February 28, 1991. Research partially supported by NSERC grants A3503 and A7218.

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Pacific Journal of Mathematics

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