THE SPACE OF INFINITE-DIMENSIONAL COMPACTA AND OTHER TOPOLOGICAL COPIES OF \((l_j^2)^\omega\)

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THE SPACE OF INFINITE-DIMENSIONAL COMPACTA
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To Doug Curtis, on the occasion of his retirement

We show that there exists a homeomorphism from the hyperspace
of the Hilbert cube \(Q\) onto the countable product of Hilbert cubes
such that the \(\geq k\)-dimensional sets are mapped onto \(B^k \times Q \times Q \times \cdots\), where \(B\) is the pseudoboundary of \(Q\). In particular, the infinite-
dimensional compacta are mapped onto \(B^\omega\), which is homeomorphic
to the countably infinite product of \(l^2_f\). In addition, we prove for
\(k \in \{1, 2, \ldots, \infty\}\) that the space of uniformly \(\geq k\)-dimensional
sets in \(2^Q\) is also homeomorphic to \((l^2_f)^\omega\).

1. Introduction. If \(X\) is a compact metric space then \(2^X\) denotes
the hyperspace of \(X\) equipped with the Hausdorff metric. According
to Curtis and Schori [6] \(2^X\) is homeomorphic to the Hilbert cube \(Q\)
whenever \(X\) is a nontrivial Peano continuum.

Our primary interest is the subset of \(2^Q\) consisting of all infinite-
dimensional compacta. This space is an \(F_{\sigma\delta}\)-set in \(2^Q\) and one may
expect that it is homeomorphic to the countable product of the pre-
Hilbert space

\(l^2_f = \{x \in l^2 : x_i = 0\text{ for all but finitely many } i\}\).

We prove this conjecture. The space \((l^2_f)^\omega\) is in a sense maximal in the
class \(\mathcal{S}_{\sigma\delta}\) of absolute \(F_{\sigma\delta}\)-spaces and it has received a lot of attention
in recent years because of its topological equivalence to numerous
function spaces, see e.g. Dijkstra et al. [7].

For \(k \in \{0, 1, 2, \ldots, \infty\}\) we let \(\text{Dim}_{\geq k}(X)\) denote the subspace
consisting of all \(\geq k\)-dimensional elements of \(2^X\). We define
\(\text{Dim}_k(X)\) and \(\text{Dim}_{\leq k}(X)\) in the same way. Let \(\text{Dim}_{\geq k}(X)\) stand
for all uniformly \(\geq k\)-dimensional compacta in \(2^X\), i.e. spaces such
that every nonempty open subset is at least \(k\)-dimensional. The de-
default value here is \(X = Q\), i.e., \(\text{Dim}_{\geq k} = \text{Dim}_{\geq k}(Q)\) etc.

Let \(I\) stand for the interval \([0, 1]\). The Hilbert cube is denoted
by \(Q = \prod_{i=1}^{\infty} I\) with metric \(d(x, y) = \max\{2^{-i}|x_i - y_i| : i \in \mathbb{N}\}\).
The pseudointerior of \(Q\) is \(s = \prod_{i=1}^{\infty} (0, 1)\) and \(B = Q \setminus s\) is the
pseudoboundary.
Theorem 1.1. (a) There exists a homeomorphism \( \alpha \) from \( 2^Q \) onto \( Q^N = \prod_{i=1}^{\infty} Q \) such that for every \( k \in \{0, 1, 2, \ldots \} \),

\[
\alpha(\text{Dim}_{\geq k}) = \underbrace{B \times \cdots \times B}_k \times Q \times Q \times \cdots .
\]

This implies that \( \alpha(\text{Dim}_\infty) = B^N \).

(b) There exists a homeomorphism \( \beta \) from \( 2^Q \) onto \( Q^N \) such that for every \( k \in \{0, 1, 2, \ldots \} \),

\[
\beta(\text{Dim}_{\leq k}) = \underbrace{Q \times \cdots \times Q}_k \times s \times s \times \cdots .
\]

The pseudoboundary \( B \) is an absorber for the collection of \( \sigma \)-compacta \( \mathcal{J}_\sigma \). Furthermore, \( B^N \) is an absorber in \( Q^N \) for the collection \( \mathcal{J}_{\sigma \delta} \). For definitions see §2 and §3. The space \( B^N \) is homeomorphic to \( (l_j^2)^\omega \). If \( Y \) is an \( \mathcal{J}_{\sigma \delta} \)-absorber in \( Q \), i.e., the pair \( (Q, Y) \) is homeomorphic to \( (Q^N, B^N) \), then we have the following:

Theorem 1.2. There exists a homeomorphism \( \alpha \) from \( 2^Q \) onto \( Q^N \) such that for every \( k \in \{0, 1, 2, \ldots \} \),

\[
\alpha(\text{Dim}_{\geq k}) = \underbrace{Y \times \cdots \times Y}_k \times Q \times Q \times \cdots .
\]

This means that \( \overline{\text{Dim}}_{\geq k} \) is homeomorphic to \( B^N \) and \( (l_j^2)^\omega \) for \( k \in \{1, 2, \ldots, \infty\} \).

In the final section we illustrate the power of the technique that we developed to prove the main theorems by applying the method to function spaces \( C_p(X) \).

For an explanation of undefined terminology see van Mill [12].

2. Absorbing systems. Let \( \Gamma \) be an ordered set and let \( \mathcal{M}_\gamma \) be a collection of spaces for each \( \gamma \in \Gamma \). Each \( \mathcal{M}_\gamma \) is assumed to be topological and closed hereditary. Let \( \mathcal{M} \) stand for the whole system \( \mathcal{M}_\gamma \). Let \( X = (X_\gamma)_{\gamma \in \Gamma} \) be an order preserving indexed collection of subsets of a topological copy \( E \) of \( Q \), i.e., \( X_\gamma \subseteq X_{\gamma'} \) if and only if \( \gamma \leq \gamma' \). The system \( X \) is called \( \mathcal{M} \)-universal if for every order preserving system \( (A_\gamma)_{\gamma} \) in \( Q \) such that \( A_\gamma \in \mathcal{M}_\gamma \) for every \( \gamma \in \Gamma \), there is an embedding \( f: Q \to E \) with \( f^{-1}(X_\gamma) = A_\gamma \). The system \( X \) is called strongly \( \mathcal{M} \)-universal if for every order preserving system \( (A_\gamma)_{\gamma} \) in \( Q \) such that \( A_\gamma \in \mathcal{M}_\gamma \) for every \( \gamma \in \Gamma \), and for every map \( f: Q \to E \)
that restricts to a Z-embedding on some compact set \( K \), there exists a Z-embedding \( g: Q \to E \) that can be chosen arbitrarily close to \( f \) with the properties: \( g|K = f|K \) and \( g^{-1}(X_\gamma) \setminus K = A_\gamma \setminus K \) for every \( \gamma \). The system \( X \) is called reflexively universal if for every map \( f: E \to E \) that restricts to a Z-embedding on some compact set \( K \), there exists a Z-embedding \( g: E \to E \) that can be chosen arbitrarily close to \( f \) with the properties: \( g|K = f|K \) and \( g^{-1}(X_\gamma) \setminus K = X_\gamma \setminus K \) for every \( \gamma \). Observe that \( X \) is strongly \( \mathcal{M} \)-universal whenever \( X \) is \( \mathcal{M} \)-universal and reflexively universal. If \( X_\gamma \in \mathcal{M} \) then the converse is also true.

The system \( X \) is called \( \mathcal{M} \)-absorbing if

1. \( X_\gamma \in \mathcal{M} \) for every \( \gamma \in \Gamma \),
2. \( \bigcup\{X_\gamma : \gamma \in \Gamma\} \) is contained in a \( \sigma Z \)-set of \( E \), and
3. \( X \) is strongly \( \mathcal{M} \)-universal.

This notion appears to be a successful synthesis of the \( Q \)-matrices technique of van Mill [11] and the generalized absorbers of Bestvina and Mogilski [2]. The power of the method we introduce here comes mainly from the relative ease of application.

As expected we have a uniqueness theorem for absorbing systems:

**Theorem 2.1.** If \( X \) and \( Y \) are both \( \mathcal{M} \)-absorbing systems in \( E \) respectively \( E' \) then \((E, X)\) and \((E', Y)\) are homeomorphic, i.e., there is a homeomorphism \( h: E \to E' \) such that \( h(X_\gamma) = Y_\gamma \) for all \( \gamma \in \Gamma \). If \( E = E' \) then the map \( h \) can be found arbitrarily close to the identity.

**Proof.** This is a standard back and forth argument. Obviously, we may assume that \( E = E' = Q \). Let \( \bigcup_\gamma X_\gamma \subset \bigcup_i A_i \) and let \( \bigcup_\gamma Y_\gamma \subset \bigcup_i B_i \), where \( \emptyset = A_0 \subset A_1 \subset A_2 \subset \cdots \) and \( \emptyset = B_0 \subset B_1 \subset B_2 \subset \cdots \) are sequences of Z-sets in \( Q \). By induction we shall construct sequences of homeomorphisms \( f_i: Q \to Q \) and \( g_i = f_i \circ \cdots \circ f_0 \) with the properties:

\[
A_i \cap X_\gamma = A_i \cap g_i^{-1}(Y_\gamma), \quad B_i \cap g_i(X_\gamma) = B_i \cap Y_\gamma, \quad f_i|\left(g_{i-1}(A_{i-1}) \cup B_{i-1}\right) = 1,
\]

where 1 denotes the identity map. Put \( f_0 = 1 \).

Assume that \( f_i \) has been constructed. Since \( X_\gamma \in \mathcal{M} \) and \( \mathcal{M} \) is topological and closed hereditary we have \( g_i(X_\gamma) \cap (g_i(A_{i+1}) \cup B_i) \in \mathcal{M} \). Put \( K = g_i(A_i) \cup B_i \) and observe that \( g_i(X_\gamma) \cap K = Y_\gamma \cap K \). Since \( Y \) is strongly universal we can find a Z-embedding \( \alpha: g_i(A_{i+1}) \cup B_i \to Q \) that fixes \( K \) and that has the property

\[
\alpha^{-1}(Y_\gamma) \cap g_i(A_{i+1}) = g_i(X_\gamma \cap A_{i+1}).
\]
Let \( \tilde{\alpha} \) be an extension of \( \alpha \) to a homeomorphism of \( Q \). Since \( \tilde{\alpha} \circ g_i(X) \) is just as \( X \) strongly universal we can find a \( Z \)-embedding \( \beta : \alpha \circ g_i(A_i) \cup B_{i+1} \to Q \) that fixes \( K' = \alpha \circ g_i(A_{i+1}) \cup B_i \) and that has the property

\[
\beta^{-1}(\tilde{\alpha} \circ g_i(X_y)) \cap B_{i+1} = Y_y \cap B_{i+1}.
\]

Let \( \tilde{\beta} \) be an extension of \( \beta \) to a homeomorphism of \( Q \). If we put \( f_{i+1} = \tilde{\beta}^{-1} \circ \tilde{\alpha} \) then one can easily verify the induction hypothesis for \( i+1 \). Since \( \tilde{\alpha} \) and \( \tilde{\beta} \) and hence \( f_{i+1} \) can be chosen arbitrarily close to the identity we may assume that \( h = \lim_{i \to \infty} g_i \) is a homeomorphism of \( Q \). The function \( h \) maps each \( X_y \) onto \( Y_y \).

3. Absorbing sequences in \( Q^N \). We shall now consider the special case that the system \( X \) is a decreasing sequence \( Q \supseteq X_1 \supseteq X_2 \supseteq \cdots \). Formally, this corresponds to choosing \( \Gamma = N \) with an inverted ordering. As a further simplification we assume that all the \( \mathcal{M}_\gamma \)'s are equal to a fixed \( \mathcal{M} \) and use the term \( \mathcal{M} \)-absorbing sequence. In addition, if \( \Gamma \) is a singleton then we call \( X \) an \( \mathcal{M} \)-absorber. Recall that the pseudoboundary \( B \) of \( Q \) is an \( \mathcal{T}_\sigma \)-absorber, where \( \mathcal{T}_\sigma \) is the collection of \( \sigma \)-compact spaces. Observe that if \( X \) is an \( \mathcal{M} \)-absorbing sequence and \( \mathcal{M} \) is closed under finite intersections then \( X_\infty = \bigcap_{i=1}^{\infty} X_i \) is an \( \mathcal{M}_\delta \)-absorber, where \( \mathcal{M}_\delta \) stands for the collection of countable intersections of elements of \( \mathcal{M} \).

Let \( X \) be a subset of \( Q \). We define three decreasing sequences of subsets of \( Q^N \):

\[
S_n(X) = \underbrace{X \times \cdots \times X}_n \times Q \times Q \times \cdots,
\]

\[
S'_n(X) = \{x \in Q^N : \text{at least } n \text{ of the } x_i \text{'s are in } X\},
\]

\[
S''_n(X) = \{x \in Q^N : x_i \in X \text{ for some } i \geq n\}.
\]

Note that \( S_n(X) \subset S'_n(X) \subset S''_n(X) \) and that \( S_\infty(X) = X^N \) and \( S'_\infty(X) = S''_\infty(X) \).

**Theorem 3.1.** If \( X \subset Q \) is strongly \( \mathcal{M} \)-universal then the sequences \( S(X), S'(X) \) and \( S''(X) \) are strongly \( \mathcal{M} \)-universal in \( Q^N \). If, in addition, \( \mathcal{M} \) is closed under finite intersections then \( X^N \) and \( S'_\infty(X) \) are strongly \( \mathcal{M}_\delta \)-universal.

**Proof.** Let \( \rho_n \) be a metric on \( Q \) such that

\[
\rho(x, y) = \max\{\rho_n(x_n, y_n) : n \in \mathbb{N}\}
\]
is a metric on $Q^N$. Consider a map $f: Q \to Q^N$ that restricts to a $Z$-embedding on some compactum $K$ and a sequence $Q \supset A_1 \supset A_2 \supset \cdots$ of elements of $\mathcal{M}$. We may assume that $f$ is a $Z$-embedding. Write $Q \setminus K$ as a union of compacta $(F_i)_{i=0}^\infty$ with $F_i \subset \text{int}(F_{i+1})$ and $F_0 = \emptyset$. Let $\varepsilon > 0$ and define the decreasing sequence $\varepsilon_i = \min\{2^{-i}\varepsilon, \frac{1}{2}\rho(f(K), f(F_i))\}$. Consider now the $n$-th component $f_n: Q \to Q$ of $f$. We shall construct a sequence $\alpha_0, \alpha_1, \ldots$ of functions from $Q$ into $Q$ with the following properties:

$$
\begin{align*}
\rho_n(\alpha_i, \alpha_{i-1}) &< \varepsilon_{i+1}, \\
\alpha_i|Q \setminus F_{i+1} &= f_n|Q \setminus F_{i+1}, \\
\alpha_i|F_i &\text{ is a } Z\text{-embedding,} \\
\alpha_i^{-1}(X) \cap F_i &= A_n \cap F_i.
\end{align*}
$$

Put $\alpha_0 = f_n$ and assume that $\alpha_i$ has been constructed. Using the strong $\mathcal{M}$-universality of $X$ we find a $Z$-embedding $\beta: F_{i+1} \to Q$, close to $\alpha_i|F_{i+1}$, with $\beta|F_i = \alpha_i|F_i$ and $\beta^{-1}(X) = A_n \cap F_{i+1}$. Extend $\beta$ to a map $\alpha_{i+1}: Q \to Q$ that restricts to $f$ on $Q \setminus F_{i+2}$.

The $\alpha_i$'s obviously form a Cauchy sequence and we can define the continuous map $g_n = \lim_{i \to \infty} \alpha_i$. One may verify that $g_n$ has the following properties:

$$
\begin{align*}
\rho_n(g_n, f_n) &< \varepsilon, \\
\text{if } x \in F_{i+1} \setminus F_i \text{ then } \rho_n(g_n(x), f_n(x)) &< \rho(f(K), f(F_{i+1})), \\
g_n|K &= f_n|K, \\
g_n|F_i &\text{ is a } Z\text{-embedding for every } i, \\
g_n^{-1}(X) \setminus K &= A_n \setminus K.
\end{align*}
$$

Define $g = (g_n)_n: Q \to Q^N$. Note that $g$ is one-to-one and hence an embedding. The set $g(Q)$ is contained in the $\sigma Z$-set $f(K) \cup \bigcup_{i=0}^\infty g_1(F_i) \times Q \times Q \times \cdots$ and is therefore a $Z$-set. The maps $f$ and $g$ are $\varepsilon$-close and $f|K = g|K$. Let $x \in Q \setminus K$. If $x$ is an element of $A_n$ then $x \in \bigcap_{j=1}^n A_j$. Consequently, we have $g_j(x) \in X$ for $j = 1, 2, \ldots, n$. This means that $g(x) \in S_n(X) \subset S'_n(X) \subset S''_n(X)$. On the other hand, if $g(x)$ is an element of $S''_n(X)$ then $g_j(x) \in X$ for some $j \geq n$ and hence $x \in A_j \subset A_n$. This completes the proof.

Consider now the pseudoboundary $B$ of the Hilbert cube. This is an $\mathcal{F}_\sigma$-absorber in $Q$. The conditions (1) and (2) of the definition of absorbing system are trivially satisfied by $S(B)$, $S'(B)$ and $S''(B)$,
so we have:

**Corollary 3.2.** The sequences $S(B)$, $S'(B)$ and $S''(B)$ are $\mathcal{F}_\sigma$-absorbing and hence they are homeomorphic in $\mathbb{Q}^N$. Moreover, $B^N$ and $S'_{\infty}(B)$ are $\mathcal{F}_{\sigma}\delta$-absorbers.

Consider the $\sigma$-Z-set

$$\sigma = \{ x \in \mathbb{Q} : x_i = 0 \text{ for all but finitely many } i \}.$$ 

It is well known that $\sigma$ is homeomorphic to $l^2_\sigma$ and that it is a so-called fd-capset in $\mathbb{Q}$ or, in our terminology, an absorber for the strongly countable dimensional $\sigma$-compacta. It is easily verified by juggling coordinates that the system $S(\sigma)$ is homeomorphic to $S(B)$ in $\mathbb{Q}^N$ and hence $\mathcal{F}_\sigma$-absorbing. Observe that the following systems are all homeomorphic: $S(\sigma)$ in $\mathbb{Q}^N$, $S(\sigma \times I)$ in $(\mathbb{Q} \times I)^N$, $S(\sigma) \times I^N$ in $\mathbb{Q}^N \times I^N$, $S(\sigma) \times \mathbb{Q}^N$ in $\mathbb{Q}^N \times \mathbb{Q}^N$, $S(\sigma \times \mathbb{Q})$ in $(\mathbb{Q} \times \mathbb{Q})^N$ and finally $S(B)$ in $\mathbb{Q}^N$.

We can take this one step further:

**Corollary 3.3.** If $Y$ is an $\mathcal{F}_{\sigma}\delta$-absorber in $\mathbb{Q}$ then the sequences $S(Y)$, $S'(Y)$ and $S''(Y)$ are $\mathcal{F}_{\sigma}\delta$-absorbing and hence they are homeomorphic in $\mathbb{Q}^N$. Moreover, $Y^N$ and $S'_{\infty}(Y)$ are also $\mathcal{F}_{\sigma}\delta$-absorbers.

4. **The space of infinite-dimensional compacta.** In this section we prove Theorem 1.1. The following lemma is easily verified.

**Lemma 4.1.** If $X$ and $Y$ are compact spaces and if $F : X \to 2^Y$ is continuous then $G(A) = \bigcup \{ F(a) : a \in A \}$ defines a continuous map from $2^X$ into $2^Y$.

**Proposition 4.2.** The sequence $(\text{Dim} \geq k)_{k=1}^\infty$ is reflexively universal in $2^\mathbb{Q}$.

**Proof.** Let $F : 2^\mathbb{Q} \to 2^\mathbb{Q}$ be a map and let $K$ be a closed subset of $2^\mathbb{Q}$ such that $F|K$ is a Z-embedding. We may assume that $F$ is a Z-embedding. Let $e : 2^\mathbb{Q} \to I$ be a map with the properties: $e^{-1}(0) = F(K)$ and $e(A) \leq d(A, F(K))/4$ for each $A \in 2^\mathbb{Q}$. According to Curtis [5] the finite sets in $2^\mathbb{Q}$ contain an fd-capset and hence there exists a deformation $H_t$ of $2^\mathbb{Q}$ such that $H_0 = 1$ and $H_t(A)$ is finite for $t > 0$ and $A \in 2^\mathbb{Q}$. We may assume, moreover, that $d(H_t, 1) \leq 2t$ and that $H_t(A) \subset [0, 1 - t]^N$ for every $t$ and $A$. 
We shall use the vector addition and scalar multiplication operations that \( Q \) inherits from \( \mathbb{R}^N \). Define the homotopy \( \alpha_t: 2Q \to 2Q \) by

\[
\alpha_t(A) = \{0\} \cup \bigcup_{n=1}^{\infty} \left\{ \frac{1}{n} \right\} \times \frac{1}{n} \tilde{A},
\]

where \( \tilde{A} \) is the subset of \( \prod_{i=2}^{\infty} I \) that is obtained from \( A \) by a coordinate shift. Note that \( \alpha_t(A) \subset [0, t]^N \) and that \( \alpha_0(A) = \{0\} \). The map \( G: 2Q \to 2Q \) that approximates \( \iota \) is defined by

\[
G(A) = H_{\varepsilon(F(A))}(F(A)) + \alpha_{\varepsilon(F(A))}(A).
\]

The function \( G \) is continuous by Lemma 4.1 and the continuity of the homotopies \( H \) and \( \alpha \). Observe that \( d(G(A), F(A)) \leq 3\varepsilon(F(A)) \) for every \( A \in 2Q \). If \( A \in K \) then \( \varepsilon(F(A)) = 0 \) and hence \( G \) restricts to \( F \) on \( K \). Let \( A \) be an element of \( 2Q \setminus K \). Then \( t = \varepsilon(F(A)) > 0 \) and hence \( H_t(F(A)) \) is finite. So \( G(A) \) is a finite union of translates of \( \alpha_t(A) \) and consequently a union of a finite set and a countable collection of copies of \( A \). This means that \( G \) preserves dimension and

\[
G^{-1}(\dim_{\geq k}) \setminus K = \dim_{\geq k} \setminus K.
\]

We shall now show that \( G \) is one-to-one. The restriction of \( G \) to \( K \) is obviously one-to-one. If \( A \in 2Q \setminus K \) then \( d(G(A), F(A)) \leq 3\varepsilon(F(A)) < d(F(K), F(A)) \) and hence \( G(A) \) is not in \( G(K) = F(K) \). For the remaining case let \( A, B \in 2Q \setminus K \) such that \( G(A) = G(B) \). Let \( \pi: Q \to I \) be the projection onto the first coordinate and define the positive numbers \( r = \varepsilon(F(A)) \) and \( t = \varepsilon(F(B)) \). Select a point \( y = (a, x) \in G(A) = G(B) \) such that \( a = \min(\pi(G(A))) = \min(\pi(G(B))) \). Note that \( y \) is an element of both \( H_r(F(A)) \) and \( H_t(F(B)) \). Since the latter sets are finite we can define \( \lambda > 0 \) as one half of the distance of \( y \) towards the other points in \( H_r(F(A)) \cup H_t(F(B)) \).

Let \( m \) and \( n \) be the first numbers that satisfy \( \frac{r}{m} \leq \lambda \) and \( \frac{t}{n} \leq \lambda \). We now have:

\[
(y + [0, \lambda]^N) \cap G(A) = \{y\} \cup \bigcup_{i=m}^{\infty} \{a + \frac{r}{m}\} \times (x + \frac{r}{m} \tilde{A})
\]

\[
= (y + [0, \lambda]^N) \cap G(B) = \{y\} \cup \bigcup_{i=n}^{\infty} \{a + \frac{t}{n}\} \times (x + \frac{t}{n} \tilde{B}).
\]

This implies:

\[
\{a + \frac{r}{m}\} \times (x + \frac{r}{m} \tilde{A}) = \{a + \frac{t}{n}\} \times (x + \frac{t}{n} \tilde{B}).
\]
This means that $\frac{r}{m} = \frac{i}{n}$ and $\frac{r}{m} \vec{A} = \frac{i}{n} \vec{B}$ and hence that $A = B$. So $G$ is one-to-one and therefore an embedding.

Observe that $\pi(G(A))$ is countable if $A \in 2^Q \setminus K$ so $G(A)$ is nowhere dense in $Q$. Since $D_t(A) = \{x \in Q : d(x, A) \leq t\}$ is a deformation of $Q$ through the complement of $G(2^Q \setminus K)$, we have that $G(2^Q \setminus K)$ is a $\sigma Z$-set. Consequently, $G(2^Q) \subset F(K) \cup G(2^Q \setminus K)$ is a $Z$-set and $G$ is a $Z$-embedding. This completes the proof.

Observing that $G$ preserves many other properties we find for instance:

**Corollary 4.3.** The sequence $\overline{(\operatorname{Dim}\geq k)}_{k=1}^\infty$ is reflexively universal in $2^Q$.

**Corollary 4.4.** The sequence consisting of the collections of compacta of cohomological dimension not less than $k$ is reflexively universal in $2^Q$.

**Corollary 4.5.** The transfinite sequence $\{A \in 2^Q : \operatorname{ind}(A) \geq \alpha\}_{\alpha<\omega_1}$ is reflexively universal in $2^Q$.

**Theorem 4.6.** The sequence $(\operatorname{Dim}\geq k)_{k=1}^\infty$ is $\mathcal{F}_\sigma$-absorbing in $2^Q$. Consequently, $\operatorname{Dim}_\infty$ is an $\mathcal{F}_{\sigma\delta}$-absorber.

**Proof.** Let $k, n \in \mathbb{N}$ and define

$\mathcal{G}_n = \{A \in 2^Q : \text{there is in } Q \text{ a finite open cover of } A \text{ with mesh } \leq 1/n \text{ and order } \leq k\}$. 

Obviously, $\mathcal{G}_n$ is an open subset of $2^Q$. Note that $\operatorname{Dim}\geq k = Q \setminus \bigcap_{n=1}^\infty \mathcal{G}_n$ is therefore an $F_\sigma$-set. According to Curtis [5] the finite sets in $2^Q$ contain an $\mathcal{F}$-capset and hence $\operatorname{Dim}_{\geq 1}$ is a $\sigma Z$-set.

In view of Proposition 4.2 it suffices to show that the system is $\mathcal{F}_\sigma$-universal. The space $\operatorname{Dim}_1(I)$ is an $\mathcal{F}_\sigma$-absorber in the Hilbert cube $2^I$. This can be found essentially in Kroonenberg [10] if we note that $H_t(A) = \{x \in I : d(x, A) \leq t\}$ is a deformation of $2^I$ through $\operatorname{Dim}_1(I)$, see also [1]. So the pair $(2^I, \operatorname{Dim}_1(I))$ is homeomorphic to $(Q, B)$. Corollary 3.2 now guarantees that $S'(\operatorname{Dim}_1(I))$ is an $\mathcal{F}_\sigma$-absorbing sequence in $(2^I)^\mathbb{N}$. Define the embedding $\alpha : (2^I)^\mathbb{N} \to 2^Q$ by $\alpha((P_i)_{i=1}^\infty) = \prod_{i=1}^\infty P_i$. Since $\prod_{i=1}^\infty P_i$ is $k$-dimensional if and only if precisely $k$ of the $P_i$'s are in $\operatorname{Dim}_1(I)$, we have

$$\alpha^{-1}(\operatorname{Dim}_{\geq k}) = S'_k(\operatorname{Dim}_1(I)).$$
The sequence $\text{Dim}_{\geq k}$ is then $\mathcal{F}_\sigma$-universal because $S'(\text{Dim}_1(I))$ is.

We find Theorem 1.1 by combining Theorem 2.1, Corollary 3.2 and Theorem 4.6. The fact that $(2^Q, (\text{Dim}_{\geq k})_{k=1}^\infty)$ is homeomorphic to $(Q^N, S(B))$ means that there exists a homeomorphism $\alpha: 2^Q \to Q^N$ such that

$$\alpha(\text{Dim}_{\geq k}) = \underbrace{B \times \cdots \times B}_{k \text{ times}} \times Q \times Q \times \cdots.$$ 

This implies that $\alpha(\text{Dim}_\infty) = B^N$, which space is homeomorphic to $(l_2^f)^\omega$. Observe that in view of the remark following Corollary 3.2 it is also possible to find an $\alpha'$ with

$$\alpha'(\text{Dim}_{\geq k}) = \underbrace{\sigma \times \cdots \times \sigma}_{k \text{ times}} \times Q \times Q \times \cdots.$$ 

Comparing $(2^Q, \text{Dim}_{\geq k})$ with $(Q^N, S''(B))$ we find part (b) of Theorem 1.1. There exists a homeomorphism $\beta$ from $2^Q$ onto $Q^N$ such that for every $k \in \{0, 1, 2, \ldots\}$,

$$\beta(\text{Dim}_{\leq k}) = \underbrace{Q \times \cdots \times Q}_{k \text{ times}} \times s \times s \times \cdots.$$ 

Note that

$$\beta(\text{Dim}_k) = \underbrace{Q \times \cdots \times Q}_{k-1 \text{ times}} \times B \times s \times s \times \cdots$$

and hence the pair $(\text{Dim}_{\leq k}, \text{Dim}_k), 0 < k < \infty$, is homeomorphic to $(Q \times s, B \times s)$, i.e., $\text{Dim}_k$ is a so-called $Z$-absorber in the topological Hilbert space $\text{Dim}_{\leq k}$.

Let $c\text{Dim}_{\geq k}$ stand for all elements of $2^Q$ with cohomological dimension at least $k$ with respect to for instance the group $\mathbb{Z}$.

**Question.** Is $c\text{Dim}_{\geq k}$ $\sigma$-compact?

Observe that it follows from the proof of Theorem 4.6 that the sequence $c\text{Dim}_{\geq k}$ is $\mathcal{F}_\sigma$-universal. If the answer to the question is yes then we have in view of Corollary 4.4 and the fact $c\text{Dim}_{\geq 1} = \text{Dim}_{\geq 1}$ that $c\text{Dim}_{\geq k}$ is $\mathcal{F}_\sigma$-absorbing and $c\text{Dim}_\infty$ is homeomorphic to $B^N$.

**5. Uniformly $\geq k$-dimensional compacta in $2^Q$.** This section is devoted to the proof of Theorem 1.2. Consider the following decreasing sequence of subsets of $(2^Q)^N$:

$$X_k = \{ P \in (2^Q)^N : P_i \in \text{Dim}_{\geq k} \text{ for infinitely many } i \}.$$
**Lemma 5.1.** The sequence \((X_k)_{k=1}^{\infty}\) is \(\mathcal{F}_{\sigma\delta}\)-universal.

**Proof.** Let \(A_1 \supset A_2 \supset \cdots\) be a sequence of \(F_{\sigma\delta}\)-sets in \(Q\). Choose \(\sigma\)-compact sets \(A_k^n\) such that \(A_k^{n+1} \cup A_k^n \subset A_k^n\) and \(A_k = \bigcap_{n=1}^{\infty} A_k^n\). Since \((\text{Dim}_{\geq k})_{k=1}^{\infty}\) is \(\mathcal{F}_{\sigma\delta}\)-universal, Theorem 4.6, there exist embeddings \(f_n: Q \to 2^Q\) such that \(f_n^{-1}(\text{Dim}_{\geq k}) = A_k^n\). Put \(f = (f_n)_n: Q \to (2^Q)^N\). If \(x \in A_k\) then \(x \in A_k^n\) for all \(n\). So \(f_n(x) \in \text{Dim}_{\geq k}\) for all \(n\) and hence \(f(x) \in X_k\). If \(x \notin A_k\) then \(x \notin A_k^n\) for some \(j\), so \(x \notin A_k^n\) for all \(n \geq j\). Consequently, \(f_n(x) \notin \text{Dim}_{\geq k}\) for all \(n \geq j\) and \(f(x) \notin X_k\).

**Remark.** One may use the method of Theorem 3.1 to show that \((X_k)_k\) is in fact \(\mathcal{F}_{\sigma\delta}\)-absorbing in \((2Q)^N\).

**Proposition 5.2.** The sequence \((\overline{\text{Dim}_{\geq k}})_{k=1}^{\infty}\) is strongly \(\mathcal{F}_{\sigma\delta}\)-universal.

**Proof.** In view of Corollary 4.3 it suffices to show that the sequence is \(\mathcal{F}_{\sigma\delta}\)-universal. We shall prove that the system \(X_k\) can be embedded in \(\overline{\text{Dim}_{\geq k}}\).

Let \(G\) stand for the compact, multiplicative subspace \(\{0\} \cup \{2^{-m} : m = 1, 2, \ldots\}\) of \(I\). According to Curtis [5] there exists a deformation \(H_t: 2^Q \to 2^Q\) such that \(H_0 = 1\) and \(H_t(A)\) is finite if \(t > 0\). Let \(P = (P_m)_{m=1}^{\infty}\) be an element of \((2^Q)^N\). We define the continuous function \(F: G \times (2^Q)^N \to 2^Q\) by

\[
F_0(P) = \{0\} \quad \text{and} \quad F_{2^{-m}}(P) = 2^{-m}P_m \cup \{0\}.
\]

We shall define inductively a sequence of compacta \((A_n)_{n=1}^{\infty}\) such that

\[
A_n \subset (G \times Q)^{n-1} \times G,
\]

i.e., the \(n\) odd coordinates are in \(G\) and the \(n-1\) even ones in \(Q\). Put \(A_1(P) = G\) and

\[
A_{n+1}(P) = \bigcup \{(x, a) \times H_{ab}(F_a(P)) \times \{ab\} : (x, a) \in A_n(P) \text{ and } b \in G\}.
\]

Here \((x, a) \in A_n\) means that \(x \in (G \times Q)^{n-1}\) and \(a \in G\). Note that since \(ab < a\) the odd components of the points in \(A_n\) form a decreasing sequence. Applying Lemma 4.1 we find that every \(A_n\) is a compactum that depends continuously on \(P\). We identify each \(A_n\) with its copy \(A_n \times \{(0, 0, \ldots)\}\) in \((G \times Q)^N \subset (I \times Q)^N\). The Hilbert
cube \( Q' = (I \times Q)^N \) is equipped with the metric \( \rho = \max_{i \in \mathbb{N}} \rho_i \), where \( \rho_{2j-1} \) is a standard metric on \( I \) that is bounded by \( 2^{-2j+1} \) and \( \rho_{2j} \) is a standard metric on \( Q \) that is bounded by \( 2^{-2j} \). Observe that \( \pi_n(A_{n+1}) = A_n \), where \( \pi_n \) is the projection from \( Q' \) onto \( (I \times Q)^{n-1} \times I \). This implies that \( \rho(\pi_n, 1) \leq 2^{-2n} \) and \( \rho(A_n, A_{n+1}) \leq 2^{-2n} \) so that \( (A_n(P))_{n=1}^{\infty} \) is a Cauchy sequence of maps. So \( \alpha(P) = \lim_{n \to \infty} A_n(P) \) defines a continuous map from \( (2^Q)^N \) into \( 2^Q \). In addition, we find that \( \alpha(P) = \bigcap_{n=1}^{\infty} \pi_n^{-1}(A_n) \). Since 0 is an element of every \( F_i(P) \) we have \( A_n \subseteq A_{n+1} \). This implies that \( \alpha(P) \) is the closure of \( Y = \bigcup_{n=1}^{\infty} A_n \) in \( Q' \).

We show by induction that
\[
A'_n = \{(x, a) \in A_n : a \neq 0\}
\]
is countable. This is obviously true for \( A'_1 \). Let \( (x, a, p, ab) \) be an element of \( A'_{n+1} \). So \( ab \neq 0 \), \( (x, a) \in A_n \) and \( p \in H_{ab}(F_a(P)) \). This implies \( a \neq 0 \) and \( (x, a) \in A'_n \) and hence we have:
\[
A'_{n+1} = \bigcup \{((x, a) \times H_{ab}(F_a(P)) \times \{ab\} : (x, a) \in A'_n \text{ and } b \in G \setminus \{0\}\}.
\]
This is a countable union of finite sets because \( H_{ab}(F_a(P)) \) is finite if \( ab \neq 0 \). Consequently, the set \( A'_{n+1} \) is countable.

Assume that \( P \notin X_k \). We shall prove that 0 has a neighbourhood in \( \alpha(P) \) with dimension less than \( k \). Since \( P \notin X_k \) there exists an \( m \) such that \( \dim(P_i) < k \) for all \( i \geq m \). So if we put \( c = 2^{-m} \) then \( \dim(F_a(P)) < k \) for \( a \leq c \). Let \( C \) consist of all points in \( Q' \) whose first component is less than or equal to \( c \). We shall prove inductively that \( \dim(A_n \cap C) < k \). Obviously, we have \( \dim(A_1 \cap C) = 0 \). Assume that \( \dim(A_n \cap C) < k \) and consider
\[
A_{n+1} \cap C = \bigcup \{((x, a) \times H_{ab}(F_a(P)) \times \{ab\} : (x, a) \in A_n \cap C \text{ and } b \in G\}.
\]
If \( a = 0 \) then \( ab = 0 \) and \( H_{ab}(F_a(P)) = \{0\} \). Consequently, we have:
\[
A_{n+1} \cap C = (A_n \cap C) \cup \bigcup \{((x, a) \times H_{ab}(F_a(P)) \times \{ab\} : (x, a) \in A'_n \cap C \text{ and } b \in G\}.
\]
Note that the \( H_{ab}(F_a(P)) \) in this expression is either finite or homeomorphic to \( F_a(P) \). Since the odd components of points form a decreasing sequence in \( G \) we have that \( a \leq c \) whenever \( (x, a) \) is a
point in $A_n \cap C$. So every $F_a(P)$ is less than $k$-dimensional. Since $A'_n$ is countable, the set $A_{n+1} \cap C$ is a countable union of $< k$-dimensional compacta and therefore $\dim(A_{n+1} \cap C) < k$. Note that $\alpha(P) \cap C = \bigcap_{n=1}^{\infty} \pi^{-1}_n(A_n \cap C)$. Since $\pi^{-1}_n(A_n \cap C)$ is the product of a $< k$-dimensional compactum and a Hilbert cube of diameter $\leq 2^{-2n}$, there is for every $n$ an open cover of $\pi^{-1}_n(A_n \cap C)$ (and hence of $\alpha(P) \cap C$) with mesh $\leq 2^{-2n}$ and order $\leq k$. Consequently, we have $\dim(\alpha(P) \cap C) < k$ and

$$\alpha(P) \notin \overline{\dim}_{\geq k}(Q').$$

Consider now the case $P \in X_k$. This means that $\dim(F_a(P)) \geq k$ for infinitely many $a \in G$. Let $(x, 0) \in A_n$. We show by induction that $A_{n+1}$ is at least $k$-dimensional at this point, i.e., every neighbourhood of the point in $A_{n+1}$ has dimension no less than $k$. First, consider $0 \in A_1$. We have:

$$A_2 = \bigcup_{a, b \in G} \{a\} \times H_{ab}(F_a(P)) \times \{ab\}.$$

Selecting $b = 0$ we find

$$\lim_{a \to 0} \{a\} \times H_0(F_a(P)) \times \{0\} = \lim_{a \to 0} \{a\} \times F_a(P) \times \{0\} = \{0\}$$

and hence $A_2$ is $\geq k$-dimensional at $0$.

Assume that the induction hypothesis is valid for points $(x, 0)$ in $A_n$. If $(y, 0) \in A_{n+1}$ then $y = (x, a, p)$, where $(x, a) \in A_n$ and $p \in H_0(F_a(P)) = F_a(P)$. If $a = 0$ then $F_a(P) = \{0\}$ and $p = 0$. This means that $(y, 0) = (x, 0, 0, 0) \in A_n$ and by induction $A_{n+1}$ and therefore $A_{n+2}$ are $\geq k$-dimensional at the point. If $a \neq 0$ then for $b, c \in G$ we have:

$$\{(x, a)\} \times H_{ab}(F_a(P)) \times \{ab\} \times H_{abc}(F_{ab})(P) \times \{abc\} \subset A_{n+2},$$

where we denote $F_{ab}(P)$ simply by $F_a$. Since $\lim_{b \to 0} H_{ab}(F_a) = H_0(F_a) = F_a$ in $2^Q$ we can find points $p_b \in H_{ab}(F_a)$ such that $\lim_{b \to 0} p_b = p$. Selecting $c = 0$ we find

$$\lim_{b \to 0} \{(x, a, p_b, ab)\} \times F_{ab} \times \{0\} = \{(x, a, p, 0, 0, 0)\}.$$

Since $F_{ab}$ is $\geq k$-dimensional for infinitely many $b$'s we have that $A_{n+2}$ is $\geq k$-dimensional at $(y, 0, 0, \ldots) = (x, a, p, 0, 0, \ldots)$. This completes the induction.

If $x$ is an element of $A_n$ then $(x, 0, 0)$ is in $A_{n+1}$ and hence $A_{n+2}$ is $\geq k$-dimensional at $x$. Consequently, the set $Y = \bigcup_{n=1}^{\infty} A_n$
is \( k \)-dimensional at each of its points. So its closure \( \alpha(P) \) is an element of \( \text{Dim}_{\geq k}(Q') \) and we have:

\[
\alpha^{-1}(\text{Dim}_{\geq k}(Q')) = X_k.
\]

This does not quite complete the proof of Proposition 5.2 since \( \alpha \) is not one-to-one. This can easily be fixed, however. Define the map \( \beta \) from \((2^Q)^N \) into the hyperspace of \( Q'' = I \times Q' \times \prod_{i=1}^{\infty} Q \) by

\[
\beta(P) = (\{0\} \times \alpha(P) \times \{(0, 0, \ldots)\}) \cup (\{1\} \times Q' \times \prod_{i=1}^{\infty} P_i).
\]

The map \( \beta \) is obviously one-to-one and hence an embedding. Note that \( \beta(P) \) is a topological sum of a copy of \( \alpha(P) \) and a uniformly infinite-dimensional space, so we retain the property

\[
\beta^{-1}(\text{Dim}_{\geq k}(Q'')) = X_k.
\]

We may conclude that \( (\text{Dim}_{\geq k}(Q''))^\infty_{k=1} \) is \( \mathcal{F}_{\sigma\delta} \)-universal just as \( (X_k)_{k=1}^\infty \).

**Theorem 5.3.** The sequence \( (\text{Dim}_{\geq k})(Q'')^\infty_{k=1} \) is \( \mathcal{F}_{\sigma\delta} \)-absorbing and \( \text{Dim}_{\infty} \) is an \( \mathcal{F}_{\sigma\delta} \)-absorber in \( 2^Q \).

**Proof.** Note that \( \text{Dim}_{\geq 1} \) is contained in the \( \sigma Z \)-set \( \text{Dim}_{\geq 1} \). It remains to be shown that every \( \text{Dim}_{\geq k} \) is in \( \mathcal{F}_{\sigma\delta} \). Let \( \{O_i : i \in \mathbb{N}\} \) be a countable open basis for the topology of \( Q \) and let \( k \in \mathbb{N} \). Write every \( O_i \) as a countable union of compacta \( F_i^1 \subset F_i^2 \subset \cdots \). Define the collections

\[
\mathcal{G}_i^j = \{A \in 2^Q : \text{there is in } Q \text{ an finite open cover } \mathcal{U} \text{ of } A \cap F_i^j \text{ with mesh } \leq 1/j \text{ and order } \leq k\}.
\]

If \( A \in \mathcal{G}_i^j \) and \( \mathcal{U} \) is such a cover then put \( \epsilon = \rho(A, F_i^j \setminus \bigcup \mathcal{U}) \). Observe that if \( \rho(A, B) < \epsilon \) then \( B \cap F_i^j \) is also covered by \( \mathcal{U} \) and hence \( \mathcal{G}_i^j \) is open in \( 2^Q \). So \( \mathcal{G}_i = \bigcap_{j=1}^{\infty} \mathcal{G}_i^j \) is a \( G_\delta \)-set. Since a countable union of \( k \)-dimensional compacta is again \( k \)-dimensional one easily verifies that an element \( A \) of \( 2^Q \) is in \( \mathcal{G}_i \) if and only if \( \dim(A \cap O_i) < k \). The collection \( \mathcal{F}_i^j = \mathcal{G}_i \setminus \{A \in 2^Q : A \cap O_i = \emptyset\} \) is obviously also \( G_\delta \). Observe that \( \bigcup_{i=1}^{\infty} \mathcal{F}_i^j \) is precisely the complement of \( \text{Dim}_{\geq k} \) in \( 2^Q \). This shows that \( \text{Dim}_{\geq k} \) is in \( \mathcal{F}_{\sigma\delta} \).

We find Theorem 1.2 by combining Theorem 2.1, Corollary 3.3 and Theorem 5.3. If \( Y \) is an \( \mathcal{F}_{\sigma\delta} \)-absorber in \( Q \) then there exists
a homeomorphism $\alpha$ from $2^Q$ onto $Q^N$ such that for every $k \in \{0, 1, 2, \ldots\}$,

$$\alpha(\overline{\dim}_{\geq k}) = Y \times \cdots \times Y \times Q \times Q \times \cdots.$$ 

Note that $\overline{\dim}_{\geq k}$, $0 < k \leq \infty$, is an $\mathcal{F}_\sigma\delta$-absorber and hence homeomorphic to $B^\mathbb{N}$ and $(l_2^2)^\omega$.

6. Function spaces in the topology of pointwise convergence. In this section the Hilbert cube $Q$ is represented by $\hat{\mathbb{R}}^N$, where $\hat{\mathbb{R}}$ stands for the compactification $[-\infty, \infty]$. Consequently, $\mathbb{R}^N$ is the pseudointerior of $Q$. If $X$ is countable metric space then $C_p(X)$ denotes the space of continuous, realvalued functions on $X$ endowed with the topology of pointwise convergence. Define the following subspaces of $\mathbb{R}^N$:

$$c_0 = \left\{ x \in \mathbb{R}^N : \lim_{i \to \infty} x_i = 0 \right\}$$

and for $n \in \mathbb{N}$

$$\Sigma_n = \left\{ x \in \mathbb{R}^N : |x_i| \leq 2^{-n} \text{ for all but finitely many } i \right\}.$$ 

Observe that $\Sigma = (\Sigma_n)_n$ is a decreasing sequence of $\sigma Z$-sets in $Q$ with the property that its intersection is $c_0$. The aim of this section is to show that $c_0$ and $C_p(X)$ are $\mathcal{F}_\sigma\delta$-absorbers in the Hilbert cubes $\hat{\mathbb{R}}^N$ respectively $\hat{\mathbb{R}}^X$. This is an improvement over the result of Dobrowolski, Gul\'ko and Mogilski [8] and, independently, Cauty [3] that $c_0$ and $C_p(X)$ are homeomorphic to $(l_2^2)^\omega$.

**Proposition 6.1.** The system $\Sigma$ is $\mathcal{F}_\sigma$-universal in $Q$.

**Proof.** We shall use the following fact: if $A$ is an $\mathcal{F}_\sigma$-absorber in $Q$ and $A'$ is a $\sigma Z$-set then for every $\sigma$-compactum $C$ in $Q$ there is an embedding $f: Q \to Q$ such that $f^{-1}(A) = C$ and $f(Q \setminus C) \cap A' = \emptyset$. This can be seen as follows. The proof of Theorem 2.1 shows that if $A_1 \supset A_2$ is an $\mathcal{F}_\sigma$-absorbing system in $Q$ then there is a homeomorphism $h: Q \to Q$ such that $h(A) = A_2$ and $h(A') \subset A_1$. Such a system exists by Corollary 3.2 and it has the required property.

Let $A_1 \supset A_2 \supset \cdots$ be a sequence of $\sigma$-compacta in $Q$. Let $\alpha$ be a bijection from $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$ and define $N_i = \{ \alpha(i, j) : j \in \mathbb{N} \}$. For every $i \in \mathbb{N}$ define the Hilbert cube $Q_i = [-2^{-i+1}, 2^{-i+1}]^N$. It is easily verified with the capset characterization theorem in Curtis [4] that

$$C_i = \{ x \in Q_i : |x_{\alpha(i, j)}| \leq 2^{k-j} \text{ for some } k \}$$
is an $\mathcal{F}_\sigma$-absorber in $Q_i$. Observe that for every $x \in C_i$ we have $\lim_{j \to \infty} x_\alpha(i, j) = 0$. Define in $Q_i$ the $\sigma Z$-set

$$D_i = \{x \in Q_i : |x_\alpha(i, j)| \leq 2^{-i} \text{ for all but finitely many } j \}.$$ 

Let $f_i : Q \to Q_i$ be an embedding such that $f_i^{-1}(C_i) = A_i$ and $f_i(Q \setminus A_i)$ does not meet $D_i$. Consider the embedding $f = (f_i)_{i \in \mathbb{N}} : Q \to \prod_{i=1}^\infty Q_i \subseteq Q$. Let $x \in A_n$. If $i > n$ then we have $f_\alpha(x) \in Q_i$ and hence all components of $f_\alpha(x)$ are in $[-2^{-n}, 2^{-n}]$. If $i \leq n$ then we have $x \in A_i$ and hence $f_\alpha(x) \in C_i$. Note that only finitely many components of $f_\alpha(x)$ are outside $[-2^{-n}, 2^{-n}]$ and hence only finitely many components of $f_\alpha(x)$ are outside this interval. This means that $f_\alpha(x)$ is an element of $\Sigma_n$. If $x \notin A_n$ then we have $f_\alpha(x) \notin D_n$. This means that infinitely many components of $f_\alpha(x)$ have absolute value greater than $2^{-n}$ and hence $f_\alpha(x) \notin \Sigma_n$. So we may conclude that $f^{-1}(\Sigma_n) = A_n$.

A subset $A$ is locally homotopy negligible in $X$ if for every map $f : M \to X$ from an absolute neighbourhood retract $M$ and for every open cover $\mathcal{U}$ of $X$ there exists a homotopy $h : M \times [0, 1] \to X$ such that $\{h({\{x}\} \times [0, 1])\}_{x \in M}$ refines $\mathcal{U}$, $h(x, 0) = f(x)$ and $h(M \times (0, 1)) \subseteq X \setminus A$. According to Theorem 2.4 in Toruńczyk [13] $A$ is locally homotopy negligible if the above condition is satisfied for $M = Q$.

For a space $X$ and $* \in X$ we define the weak cartesian product

$$W(X, *) = \{x \in X^N : x_i = * \text{ for all but finitely many } i \}.$$ 

Let $\Gamma$ be an ordered set. The following lemma is an adaptation to our needs of Proposition 3.2 in Dobrowolski, Gul'ko and Mogilski [8].

**Lemma 6.2.** Let $X = (X_\gamma)_{\gamma \in \Gamma}$ be an order preserving system in $Q$ such that $Q \setminus \bigcap_{\gamma \in \Gamma} X_\gamma$ is locally homotopy negligible in $Q$ and let $* \in \bigcap_{\gamma \in \Gamma} X_\gamma$. Assume that there exists a homeomorphism $\Phi : Q \to Q^N$ satisfying

$$W(X_\gamma, *) \subset \Phi(X_\gamma) \subset X^N_\gamma$$

for all $\gamma \in \Gamma$. Then $X$ is reflexively universal.

**Proof.** Let $f : Q \to Q$ be a map that restricts to a Z-embedding on some compact set $K$ and let $\epsilon : Q \to (0, 1)$ be a continuous function. We can assume that $f(Q \setminus K) \subseteq \bigcap_{\gamma \in \Gamma} X_\gamma \setminus f(K)$. We choose a metric $d$ on $Q^N$ so that $d(x, x') \leq 2^{-k-2}$ if $x$ and $x'$ agree on the first $k$
coordinates. Let \( e' \): \( \beta^N \to (0, 1) \) be a Lipschitz function such that if maps \( f_1, f_2 : Q \to Q^N \) are \( e' \)-close, then \( \Phi^{-1} \circ f_1 \) and \( \Phi^{-1} \circ f_2 \) are \( \epsilon \)-close. Define \( \delta : Q^N \to [0, 1) \) by \( \delta(x) = \min\{\epsilon(x), d(x, \Phi \circ f(K))\} \). Let \( \phi_i \) be the \( i \)-th component of the map \( \Phi \circ f \). By local homotopy negligibility of \( Q \setminus \bigcap_{\gamma \in \Gamma} X_\gamma \) there exists a homotopy \( h : [0, 1] \times Q \to Q \) with \( h(0, x) = x \), \( h((0, 1] \times Q) \subset \bigcap_{\gamma \in \Gamma} X_\gamma \) and \( h(1, x) = * \). Define a homotopy \( H_k : [0, 1] \times Q \to Q \) by

\[
H_k(t, x) = \begin{cases} h(2t - 2t^k, x), & \text{if } \frac{1}{2} \leq t \leq 1, \\ h_k(2t, x), & \text{if } 0 \leq t \leq \frac{1}{2}, \end{cases}
\]

where \( h_k : [0, 1] \times Q \to Q \) is a homotopy such that \( h_k((0, 1] \times Q) \subset \bigcap_{\gamma \in \Gamma} X_\gamma \), \( h_k(0, x) = \phi_k(x) \) and \( h_k(1, x) = * \). For \( x \in \{ y \in Q : 2^{-k-1} \leq \delta(\Phi \circ f(y)) \leq 2^{-k} \} \), \( k = 1, 2, \ldots \), define

\[
f'(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_k(x), H_{k+1}(-k - \log_2 \delta(\Phi \circ f(x)), x), x, x, h(-k - \log_2 \delta(\Phi \circ f(x)), x), *, *, \ldots)
\]

and extend \( f' \) on \( K \) by \( f'|K = \Phi \circ f|K \). By the construction \( f' : Q \to Q^N \) is a continuous, one-to-one map which is \( \epsilon' \)-close to \( \Phi \circ f \). Moreover, \( (f')^{-1}(X_\gamma^N) \setminus K = X_\gamma \setminus K \) and \( f'(X_\gamma \setminus K) \subset W(X_\gamma, *) \). Hence, the map \( g = \Phi^{-1} \circ f' \) is a \( Z \)-embedding which is \( \epsilon \)-close to \( f \) and satisfies \( g^{-1}(X_\gamma) \setminus K = X_\gamma \setminus K \).

Let \( \Phi : \hat{R}^N \to (\hat{R}^N)^N \) be any map that simply rearranges coordinates. It is easily seen that with this map the system \( \Sigma \) satisfies the conditions of Lemma 6.2. So we have:

**Theorem 6.3.** The system \( \Sigma \) is \( \mathcal{F}_\sigma \)-absorbing and \( c_0 \) is an \( \mathcal{F}_\sigma \delta \)-absorber in \( Q \).

The space \( R_f^N \) is defined as \( W(R, 0) \). This space is homeomorphic to \( l_2^f \) and furthermore the pair \( (\hat{R}^N, R_f^N) \) is homeomorphic to \( (I^N, \sigma) \). This means, according to §3 that there exists a homeomorphism \( \alpha : Q \to Q^N \) such that for every \( k \in N \),

\[
\alpha(\Sigma_k) = \underbrace{R_f^N \times \cdots \times R_f^N}_{k \text{ times}} \times Q \times Q \times \cdots.
\]

Consequently, \( c_0 \) is mapped by \( \alpha \) onto \( (R_f^N)^N \). In [9, Question 6.11] the following problem is posed. Does there exist a homeomorphism from \( R^N \) onto \( (R^N)^N \) that maps \( c_0 \) onto \( (R_f^N)^N \)? Such a homeomorphism cannot exist because \( c_0 \) is contained in the \( \sigma \)-compactum
consisting of bounded sequences where as \((\mathbb{R}_{f}^{N})^{N}\) contains a copy of \(\mathbb{R}^{N}\) that is closed in \((\mathbb{R}^{N})^{N}\).

**Lemma 6.4.** If \(A\) is strongly \(\mathcal{M}\)-universal in \(Q\) and \(X\) is locally homotopy negligible in a compact absolute retract \(M\) then \(A \times (M \setminus X)\) is strongly \(\mathcal{M}\)-universal in \(Q \times M\).

**Proof.** This is similar to the proof of Theorem 3.1. Let \(f = (f_{1}, f_{2})\) be a Z-embedding of \(Q\) in \(Q \times M\). Let \(K\) and \(C\) be subsets of \(Q\) such that \(K\) is closed and \(C\) is an element of \(\mathcal{M}\). Select a map \(\varepsilon: Q \to I\) such that \(\varepsilon^{-1}(0) = K\) and \(\varepsilon(x) \leq \rho(f(x), f(K))\) for each \(x \in Q\). Just as in the proof of Theorem 3.1 we can find a map \(g_{1}: Q \to Q\) such that \(f_{1}\) and \(g_{1}\) are \(\varepsilon\)-close, \(g_{1}^{-1}(A) \setminus K = C \setminus K\), \(g_{1}|Q \setminus K\) is a one-to-one map whose range is a \(\sigma Z\)-set. Since \(X\) is locally homotopy negligible we can find a map \(g_{2}: Q \to M\) such that \(f_{2}\) and \(g_{2}\) are \(\varepsilon\)-close and \(g_{2}(Q \setminus K) \subset M \setminus X\). The map \(g = (g_{1}, g_{2})\) is a Z-embedding of \(Q\) into \(Q \times M\) with \(g|K = f|K\) and \(g^{-1}(A \times (M \setminus X)) \setminus K = C \setminus K\).

**Theorem 6.5.** If \(X\) is a countable, nondiscrete metric space then \(C_{p}(X)\) is an \(\mathcal{F}_{\sigma\delta}\)-absorber in \(\hat{\mathbb{R}}^{X}\).

This means that there exists a homeomorphism \(\beta: \hat{\mathbb{R}}^{X} \to \mathbb{Q}^{N}\) such that \(\beta(C_{p}(X)) = (\mathbb{R}_{f}^{N})^{N}\).

**Proof.** It is well known (and easily verified) that \(C_{p}(X)\) is an element of \(\mathcal{F}_{\sigma\delta}\). Let \(A\) be a convergent sequence in \(X\). Observe that \(\bigcup_{n=1}^{\infty} \{f \in \hat{\mathbb{R}}^{X}: |f(a)| \leq n\) for every \(a \in A\)\) is a \(\sigma Z\)-set that contains \(C_{p}(X)\). It remains to be shown that \(C_{p}(X)\) is strongly \(\mathcal{F}_{\sigma\delta}\)-universal.

We first prove this for the convergent sequence \(\tilde{N} = \mathbb{N} \cup \{\infty\}\). In \(\hat{\mathbb{R}}\) and \(\hat{\mathbb{N}}\) we use the following arithmetic: \(1/0 = \infty\) and \(\infty + a = \infty\) if \(a\) is finite. Define the following continuous function from \(\hat{\mathbb{R}}\) into \(\hat{\mathbb{R}}^{\tilde{N}}\):

\[
\Psi(r)(n) = \text{sign}(r) \min\{|r|, n\}.
\]

Note that \(\Psi(r)(n)\) is finite if \(n \neq \infty\) and \(\lim_{n \to \infty} \Psi(r)(n) = \Psi(r)(\infty) = r\). This means that \(\Psi(\mathbb{R})\) is a subset of \(C_{p}(\tilde{N})\). If \(f \in \hat{\mathbb{R}}^{\tilde{N}}\) then \(\hat{f}\) is the extension of \(f\) over \(\tilde{N}\) that assigns 0 to \(\infty\). It is easily seen that \(\Phi(f, r) = \hat{f} + \Psi(r)\) is a well-defined map from \(\hat{\mathbb{R}}^{\tilde{N}} \times \hat{\mathbb{R}}\) onto \(\hat{\mathbb{R}}^{\tilde{N}}\). Observing that \(\Phi^{-1}(h) = (h - \Psi(h(\infty))|\mathbb{N}, h(\infty))\) we find that \(\Phi\) is a homeomorphism. Note that \(\Phi(c_{0} \times \mathbb{R}) = C_{p}(\tilde{N})\). According
to Lemma 6.4 \( c_0 \times \mathbb{R} \) is strongly \( \mathcal{F}_{\sigma_\delta} \)-universal in \( Q \times \mathbb{R} \) and hence \( C_p(\mathbb{N}) \) is strongly \( \mathcal{F}_{\sigma_\delta} \)-universal in \( \mathbb{R}\mathbb{N} \).

We use a similar argument to reduce the problem for \( C_p(X) \) to \( C_p(\mathbb{N}) \). Let \( d \) be a metric on \( X \) and let \( A \) be a convergent sequence in \( X \). We may assume that \( C_p(A) \) is strongly \( \mathcal{F}_{\sigma_\delta} \)-universal in \( \mathbb{R}^A \). Choose a retraction \( r \) from \( X \) onto \( A \). The formula

\[
\Psi(g)(x) = \text{sign}(g(r(x))) \min \{|g(r(x))|, 1/d(x, r(x))\}
\]

defines a continuous selection that extends every \( g \in \mathbb{R}^A \) to an element of \( \mathbb{R}^X \). The map \( \Psi \) has the following properties: \( \Psi(g)|A = g \), \( \Psi(g)|X \setminus A \) has its values in \( \mathbb{R} \) and \( \Psi(C_p(A)) \subset C_p(X) \). If \( f \in \mathbb{R}^X \setminus A \) then \( \hat{f} \) is the extension of \( f \) over \( X \) with zeros. As above it is easily seen that \( \Phi(f, g) = \hat{f} + \Psi(g) \) is a well-defined map from \( \mathbb{R}^X \setminus A \times \mathbb{R}^A \) onto \( \mathbb{R}^X \) and a homeomorphism. Let \( C_p(X, A) \) stand for \( \{f|X \setminus A : f \in C_p(X) \text{ and } f|A = 0\} \) and note that \( \Phi(C_p(X, A) \times C_p(A)) = C_p(X) \). It is easily seen that the complement of \( C_p(X, A) \) in \( \mathbb{R}^X \setminus A \) is locally homotopy negligible and hence Lemma 6.4 implies that \( C_p(X) \) is strongly \( \mathcal{F}_{\sigma_\delta} \)-universal in \( \mathbb{R}^X \). This completes the proof of Theorem 6.5.

\section*{References}


Received April 15, 1990 and in revised form May 3, 1991.

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