

Pacific Journal of Mathematics

SOME MORE STATES MODELS FOR LINK INVARIANTS

ANDREW SOLOMON LIPSON

SOME MORE STATES MODELS FOR LINK INVARIANTS

ANDREW S. LIPSON

In this paper I present two new states models for a specialization of the Kauffman link invariant $F(a, z)$ and show that these models occur naturally together. The particular specialization we are concerned with has been investigated by Lickorish and Millett and identified with a probability generating function. Kauffman has also found a description (different from the one given here) of this evaluation of $F(a, z)$.

1. The Jones polynomial $V(t)$. We begin by briefly recalling Kauffman's states model for the Jones polynomial $V(t)$, upon which the results of this paper are modelled (see [1], [3] for more details). It is interesting to compare this with the models obtained below, especially in view of the fact that $V(t)$ may be obtained from the Kauffman link invariant $F(a, z)$ of [3] by making the right substitutions for a and z . In fact, $V(t) = F(t^{-3/4}, -(t^{-1/4} + t^{1/4}))$ (see [4]).

We start by defining the bracket polynomial $\langle \mathcal{D} \rangle$ of an unoriented link diagram \mathcal{D} . By a state ν of \mathcal{D} we mean (for the moment) an assignation to each crossing c in \mathcal{D} of one of the two alternatives shown in Figure 1, the value $\nu(c)$ of the state at that crossing being x or x^{-1} accordingly.

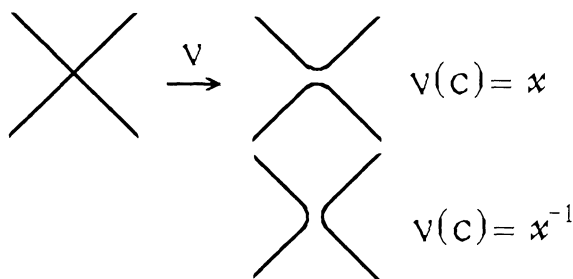


FIGURE 1

For a given state ν , we may 'undo' each crossing in \mathcal{D} in the indicated manner and then count the number of connected components of the resulting diagram $\nu(\mathcal{D})$. Write $|\nu|$ for this number. We then

$$\begin{aligned}
 \text{Diagram 1} &= x^2 \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + x^{-2} \text{Diagram 5} \\
 &= \text{Diagram 6} + (x^2 + x^{-2} - (x^2 + x^{-2})) \text{Diagram 7} = \text{Diagram 8}
 \end{aligned}$$

$$\begin{aligned}
 \text{Diagram 9} &= x \text{Diagram 10} + x^{-1} \text{Diagram 11} \\
 &= x \text{Diagram 12} + x^{-1} \text{Diagram 13} = \text{Diagram 14}
 \end{aligned}$$

FIGURE 2

define $\langle \mathcal{D} \rangle$ by

$$\langle \mathcal{D} \rangle = \sum_{\text{states } \nu} (-x^2 - x^{-2})^{|\nu|-1} \prod_{\text{crossings } c} \nu(c).$$

It is then easy to show that $\langle \mathcal{D} \rangle$ is invariant under the action of Reidemeister moves of types II and III (see [8]) on \mathcal{D} (see Figure 2), and the Reidemeister move I just multiplies $\langle \mathcal{D} \rangle$ by a power of $(-x)^3$.

Now giving \mathcal{D} an orientation, recall that the writhe w of \mathcal{D} is defined as the number of positive crossings minus the number of negative crossings (where these are defined as in Figure 3). It rapidly follows that $J(\mathcal{D}) = \langle \mathcal{D} \rangle (-x)^{-3w}$ is invariant under all three Reidemeister moves and is therefore a link invariant. Now an examination of the behaviour of $J(x)$ on skein triplets (i.e. triplets of link diagrams which are identical everywhere except in the neighbourhood of a particular crossing where they are as in Figure 3) leads to the discovery that it satisfies a linear relation and in fact that $J(x) = V(x^4)$.

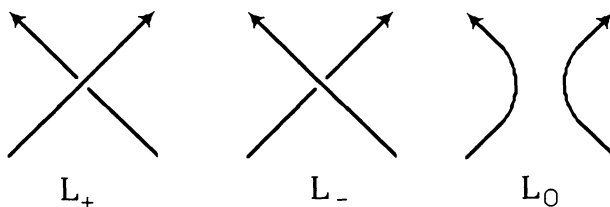


FIGURE 3

Note that this model combines both local (what is going on at each crossing) and global (how the crossings are joined up to give distinct components in $\nu(\mathcal{D})$) information about \mathcal{D} .

2. Two new states models. We proceed in a manner very similar to that used above, first defining a polynomial $[\mathcal{D}]$ for each link diagram \mathcal{D} in such a way that $[\mathcal{D}]$ is invariant under Reidemeister moves II and III and then multiplying by an appropriate factor to ensure invariance under Reidemeister move I as well. We first define a $[\cdot]$ -state of \mathcal{D} to be a labelling of each connected component of $(\mathcal{D}$ -crossing points) with either 1 or 2. A $[\cdot]$ -state is legal if at any given crossing, each label occurs an even number of times. Each crossing must then take one of the forms illustrated in Figure 4 (a and b are to be interpreted as distinct labels), and we take the value $\nu(c)$ of the state at that crossing to be as shown.

Now defining $[\mathcal{D}]$ by

$$[\mathcal{D}] = \sum_{\text{legal states } \nu} \prod_{\text{crossings } c} \nu(c)$$

we obtain different possibilities for states models depending on the particular values assigned to A, B, C and D . Of course, any values give a perfectly good function on link diagrams, but in order to obtain link invariants it is necessary that the result be unchanged by the three Reidemeister moves.

THEOREM 1. *Let $A = B = 0$ and $D = C^{-1}$. Then $R = [\cdot]C^2$ is a link invariant, where w is the writhe. Furthermore,*

$$\frac{1}{2}R(C) = (-1)^{c(L)-1} F(iC^{-1}, iC - iC^{-1})$$

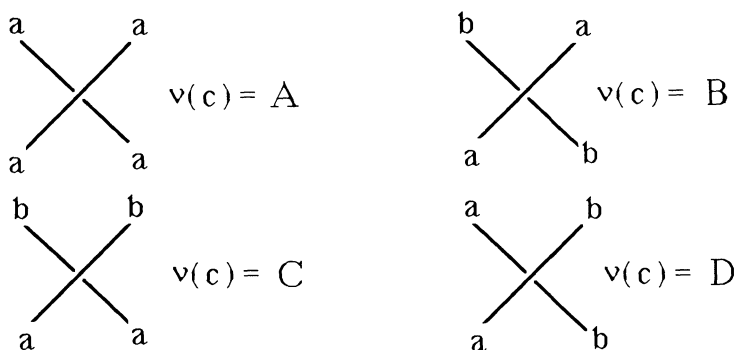


FIGURE 4

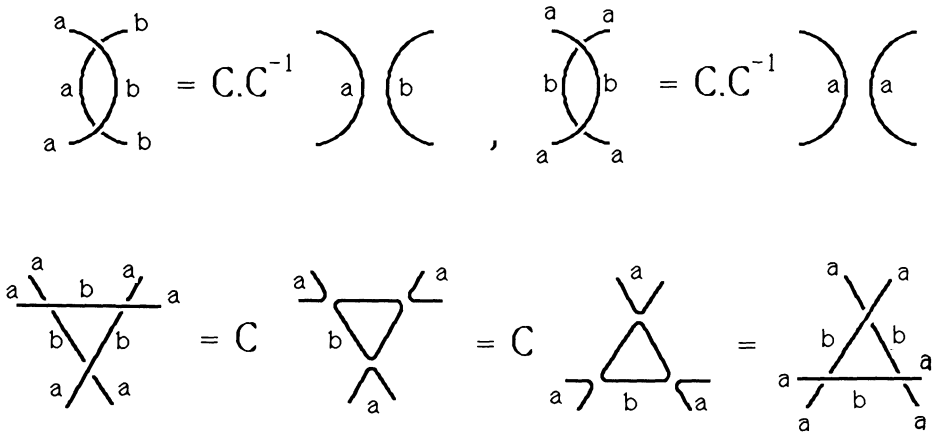


FIGURE 5

where $F(a, z)$ is the Kauffman link invariant and $c(L)$ is the number of components of the link L .

Proof. It is once again fairly easy to show that $[\mathcal{D}]$ is invariant under the second and third Reidemeister moves (see Figure 5. There are two possible labellings to consider for Reidemeister move II, both shown, and four possibilities for Reidemeister move III, of which one is shown. The reader may easily check the others).

It is convenient to allow all possible labellings of arcs of (\mathcal{D} -crossing points) but to assign values of 0 to crossings of types not appearing in Figure 4 so that illegal states do not appear in the sum. Now, as before, Reidemeister move I just multiplies $[\mathcal{D}]$ by a power of C , so setting $R(\mathcal{D}) = [\mathcal{D}]C^w$ it follows that the polynomial R in C is a link invariant. It remains only to identify it with the claimed evaluation of F . Suppose given four unoriented link diagrams which are identical except in the vicinity of one particular crossing where they are as shown in Figure 6.

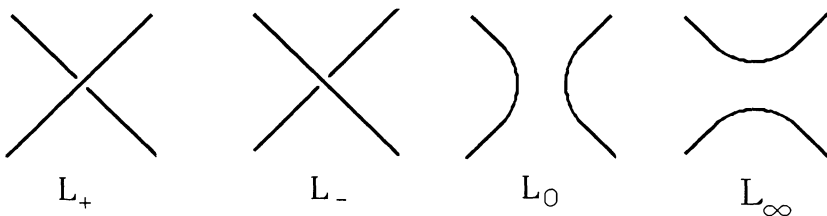


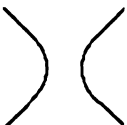
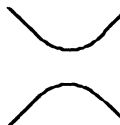




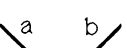



FIGURE 6

TABLE 1

				
	0	0	1	1
				
	C	C ⁻¹	0	1
				
	C ⁻¹	C	1	0
				

Now evaluating the bracket polynomial $[\cdot]$ on these four link diagrams we find its values are in the ratios given in Table 1 and it easily follows that

$$[\text{crossing}] - [\text{crossing}] = (C^{-1} - C)([\text{cup}] - [\text{cap}])$$

and we have already seen that

$$[\text{cup}] = -C^{-1}[\text{cap}] \quad , \quad [\text{cap}] = -C[\text{cup}]$$

This makes $R = [\cdot]C^w$ a case of the so-called ‘Dubrovnik polynomial’ $D(\alpha, \xi)$ defined by

$$\begin{aligned} \Lambda(\text{crossing}) - \Lambda(\text{crossing}) &= \xi(\Lambda(\text{cup}) - \Lambda(\text{cap})) \\ \Lambda(\text{cup}) &= \alpha \Lambda(\text{cap}) \quad , \quad \Lambda(\text{cap}) = \alpha^{-1} \Lambda(\text{cup}) \\ D(\alpha, \xi) &= \Lambda(\alpha, \xi) \cdot \alpha^{-w} \end{aligned}$$

In fact, $\frac{1}{2}R(C) = D(C^{-1}, C^{-1} - C)$. Now W. B. R. Lickorish pointed out some time ago that $D(\alpha, \xi)$ is really $F(a, z)$ in disguise via

$$D(\alpha, \xi) = (-1)^{c(L)-1} F(i\alpha, -i\xi)$$

where $c(L)$ is the number of components of the link L . Hence,

$$R(C) = w(-1)^{c(L)-1} F(iC^{-1}, iC - iC^{-1}).$$

This concludes the proof of Theorem 1.

The particular evaluation of F that we have encountered has already been investigated by Lickorish and Millett [6]. It is equal to

$$\frac{1}{2}(-1)^{c(L)-1} \sum_{X \subset L} a^{-4lk(X, L-X)}$$

the sum being over all sublinks of the link L . This is the probability generating function for the linking number of a randomly-chosen sublink with its complement!

We now look at another set of values for A, B, C and D :

THEOREM 2. *Let $A = \frac{1}{2}(x + x^{-1})$, $B = -\frac{1}{2}(x + x^{-1})$, $C = \frac{1}{2}(x - x^{-1})$ and $D = -\frac{1}{2}(x - x^{-1})$. Then $s = [\cdot]x^w$ is a link invariant, where w is the writhe. Furthermore, S as a polynomial in the variable x is identical to the polynomial R in the variable C of Theorem 1.*

Proof. It is again easy to check that Reidemeister move II has no effect upon the bracket polynomial $[\mathcal{D}]$ of a link diagram \mathcal{D} , see Figure 7.



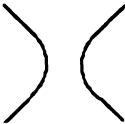
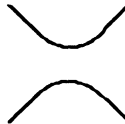








$$\begin{aligned} \begin{array}{c} a & & a \\ \diagdown & & / \\ & \bigcirc & \\ / & & \diagdown \\ a & & a \end{array} &= \begin{array}{c} a & & a \\ \diagdown & & / \\ & \bigcirc & \\ / & & \diagdown \\ a & & a \end{array} + \begin{array}{c} a & & a \\ \diagdown & & / \\ & \bigcirc & \\ / & & \diagdown \\ a & & a \end{array} = \left(\frac{1}{4}(x+x^{-1})^2 + \frac{1}{4}(x-x^{-1})^2\right) \begin{array}{c} a & & a \\ \diagdown & & / \\ & \bigcirc & \\ / & & \diagdown \\ a & & a \end{array} = \begin{array}{c} a & & a \\ \diagdown & & / \\ & \bigcirc & \\ / & & \diagdown \\ a & & a \end{array} \begin{array}{c} a & & a \\ \diagdown & & / \\ & \bigcirc & \\ / & & \diagdown \\ a & & a \end{array} \\ \begin{array}{c} b & & b \\ \diagdown & & / \\ & \bigcirc & \\ / & & \diagdown \\ a & & a \end{array} &= \begin{array}{c} a & & a \\ \diagdown & & / \\ & \bigcirc & \\ / & & \diagdown \\ a & & a \end{array} + \begin{array}{c} b & & b \\ \diagdown & & / \\ & \bigcirc & \\ / & & \diagdown \\ a & & a \end{array} = \frac{1}{4}(x+x^{-1})((x-x^{-1})-(x-x^{-1})) = 0 \\ \begin{array}{c} a & & b \\ \diagdown & & / \\ & \bigcirc & \\ / & & \diagdown \\ a & & b \end{array} &= \begin{array}{c} a & & b \\ \diagdown & & / \\ & \bigcirc & \\ / & & \diagdown \\ a & & b \end{array} + \begin{array}{c} a & & b \\ \diagdown & & / \\ & \bigcirc & \\ / & & \diagdown \\ a & & b \end{array} = \left(\frac{1}{4}(x-x^{-1})^2 + \frac{1}{4}(x+x^{-1})^2\right) \begin{array}{c} a & & b \\ \diagdown & & / \\ & \bigcirc & \\ / & & \diagdown \\ a & & b \end{array} = \begin{array}{c} a & & b \\ \diagdown & & / \\ & \bigcirc & \\ / & & \diagdown \\ a & & b \end{array} \begin{array}{c} a & & b \\ \diagdown & & / \\ & \bigcirc & \\ / & & \diagdown \\ a & & b \end{array} \\ \begin{array}{c} b & & a \\ \diagdown & & / \\ & \bigcirc & \\ / & & \diagdown \\ a & & b \end{array} &= \begin{array}{c} a & & b \\ \diagdown & & / \\ & \bigcirc & \\ / & & \diagdown \\ a & & b \end{array} + \begin{array}{c} b & & a \\ \diagdown & & / \\ & \bigcirc & \\ / & & \diagdown \\ a & & b \end{array} = -\frac{1}{4}(x+x^{-1})((x-x^{-1})-(x-x^{-1})) = 0 \end{aligned}$$

FIGURE 7

Reidemeister move III is equally easy but extremely tedious to check and adds no particular insight, so I leave it as an exercise for the computationally enthusiastic reader. As before, Reidemeister move I just multiplies $[\mathcal{D}]$ by a power of x , so setting $S(\mathcal{D}) = [\mathcal{D}]x^w$ it follows that the polynomial S in x is a link invariant. The rest of the proof is identical to that of Theorem 1 so I will do no more than provide in Table 2 the ratios of values of $\langle \mathcal{D} \rangle$ on the four link diagrams of Figure 6.

In order to explain where these states models come from, I will briefly return to the definition of the new bracket polynomial $[\mathcal{D}]$ in §2.

TABLE 2

					
		$\frac{1}{2}(X+X^{-1})$	$\frac{1}{2}(X+X^{-1})$	1	1
		$-\frac{1}{2}(X+X^{-1})$	$-\frac{1}{2}(X+X^{-1})$	0	0
		$\frac{1}{2}(X-X^{-1})$	$\frac{1}{2}(X-X^{-1})$	0	1
		$-\frac{1}{2}(X-X^{-1})$	$-\frac{1}{2}(X-X^{-1})$	1	0

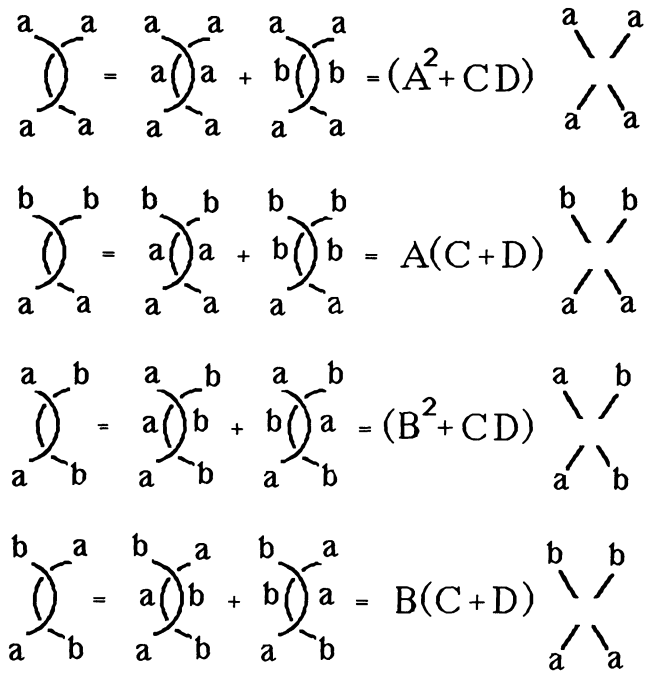


FIGURE 8

It is clear from Figure 8 that to obtain invariance under Reidemeister move II we are led to the relations:

$$\begin{aligned}
 A^2 + CD &= 1, \\
 A(C + D) &= 0, \\
 B^2 + CD &= 1, \\
 B(C + D) &= 0.
 \end{aligned}$$

Now there are two cases: either we may assume $A = 0$ and $B = 0$, in which case $CD = 1$, or alternatively we have $C = -D$ and then $B^2 = A^2 = -C^2$. These two solutions lead to the states models of Theorems 1 and 2.

I conclude this section by noting the provocative resemblance between the substitutions in $F(a, z)$ which provide the Jones polynomial, which is reducible to

$$F(a^3, -(a + a^{-1}))$$

and Lickorish/Millett's probability generating function, which is reducible to

$$F(a, -(a + a^{-1})).$$

3. Special evaluations of the Jones polynomial. There are also special similar states models for particular evaluations of the Jones polynomial $V(t)$. Let n be a positive integer greater than 1, and proceed as in §3 to define a bracket polynomial $[\cdot]$ on link diagrams. However, instead of labelling each component of (\mathcal{D} -crossing points) in a diagram \mathcal{D} with 1 or 2, we allow ourselves n distinct labels. Again, a legal state is one in which, at each crossing point, each label occurs an even number of times. The conditions put on A , B , C and D by requiring invariance under Reidemeister moves II and III are:

$$\begin{aligned} B &= 0, \\ A &= C + C^{-1}, \\ D &= -C, \\ A^2 &= 2 - n \end{aligned}$$

(there is another solution but it is trivial). Precisely the same arguments as used in §2 to identify R show that the link invariant provided by this model is equivalent to the Jones polynomial $V(C^4)$. Noting that

$$2 - n = A^2 = C^2 + C^{-2} + 2$$

we have

THEOREM 3. *Using n distinct labels on the components of (\mathcal{D} -crossing points) and setting $A = C + C^{-1}$, $B = 0$, $D = -C$ and $A^2 = 2 - n$, the bracket polynomial $[\mathcal{D}]$ is invariant under Reidemeister moves II and III and after renormalising with respect to the writhe provides a link invariant. Furthermore, this invariant is reducible to the Jones polynomial $V(t)$ when $t^{1/2} + t^{-1/2} = -n$.*

It is interesting to note that, for small n , these values of t seem to include some which are already known to be interesting (see, e.g. [5], [7]). Recent work by V. F. R. Jones [2] and V. G. Turaev [9] may show why these values are significant.

REFERENCES

- [1] V. F. R. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc., **12** (1985), 103–111.
- [2] ———, *Seminars at Cambridge*, March 1987.
- [3] L. H. Kauffman, *States models for knot polynomials*, (to appear).
- [4] W. B. R. Lickorish, *A relationship between link polynomials*, Math. Proc. Camb. Phil. Soc., **100** (1986), 109–112.

- [5] W. B. R. Lickorish and K. C. Millett, *Some evaluations of link polynomials*, Comment. Math. Helvetici, **61** (1986), 349–359.
- [6] ———, *An evaluation of the F -polynomial of a link*, (to appear).
- [7] A. S. Lipson, *An evaluation of a link polynomial*, Math. Proc. Camb. Phil. Soc., **100** (1986), 361–364.
- [8] K. Reidemeister, *Knotentheorie*, *Ergebn. Math. Grenzgeb. Bd. 1*; Springer-Verlag, Berlin, 1932.
- [9] V. G. Turaev, *The Yang-Baxter equation and invariants of links*, LOMI preprint E-3-87, Leningrad 1987.

Received September 17, 1987.

DEPARTMENT OF PURE MATHEMATICS AND MATHEMATICAL STATISTICS
16 MILL LANE
CAMBRIDGE, CG2 15B, ENGLAND

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

V. S. VARADARAJAN
(Managing Editor)
University of California
Los Angeles, CA 90024-1555-05

HERBERT CLEMENS
University of Utah
Salt Lake City, UT 84112

F. MICHAEL CHRIST
University of California
Los Angeles, CA 90024-1555

THOMAS ENRIGHT
University of California, San Diego
La Jolla, CA 92093

NICHOLAS ERCOLANI
University of Arizona
Tucson, AZ 85721

R. FINN
Stanford University
Stanford, CA 94305

VAUGHAN F. R. JONES
University of California
Berkeley, CA 94720

STEVEN KERCKHOFF
Stanford University
Stanford, CA 94305

C. C. MOORE
University of California
Berkeley, CA 94720

MARTIN SCHARLEMANN
University of California
Santa Barbara, CA 93106

HAROLD STARK
University of California, San Diego
La Jolla, CA 92093

ASSOCIATE EDITORS

R. ARENS

E. F. BECKENBACH
(1906–1982)

B. H. NEUMANN

F. WOLF
(1904–1989)

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA
UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the 1991 *Mathematics Subject Classification* scheme which can be found in the December index volumes of *Mathematical Reviews*. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024-1555-05.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* (ISSN 0030-8730) is published monthly except for July and August. Regular subscription rate: \$190.00 a year (10 issues). Special rate: \$95.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) is published monthly except for July and August. Second-class postage paid at Carmel Valley, California 93924, and additional mailing offices. Postmaster: send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Copyright © 1992 by Pacific Journal of Mathematics

Pacific Journal of Mathematics

Vol. 152, No. 2

February, 1992

Edoardo Ballico , On the restrictions of the tangent bundle of the Grassmannians	201
Edward Burger , Homogeneous Diophantine approximation in S -integers	211
Jan Dijkstra, Jan van Mill and Jerzy Mogilski , The space of infinite-dimensional compacta and other topological copies of $(I_f^2)^\omega$	255
Mike Hoffman , Multiple harmonic series	275
Wu Hsiung Huang , Superharmonicity of curvatures for surfaces of constant mean curvature	291
George Kempf , Pulling back bundles	319
Kjeld Laursen , Operators with finite ascent	323
Andrew Solomon Lipson , Some more states models for link invariants	337
Xiang Yang Liu , Bloch functions of several complex variables	347
Madabusi Santanam Raghunathan , A note on generators for arithmetic subgroups of algebraic groups	365
Marko Tadić , Notes on representations of non-Archimedean $SL(n)$	375