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HOMEOTOPY GROUPS OF IRREDUCIBLE 3-MANIFOLDS WHICH MAY CONTAIN TWO-SIDED PROJECTIVE PLANES

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A description is obtained for the homeotopy group (the group of isotopy classes of homeomorphisms) of a compact irreducible sufficiently large 3-manifold (which may contain 2-sided projective planes). It is finitely presented, and modulo finite groups is either free, $GL(3, \mathbb{Z})$, or is built up in a certain way by extensions starting from 2-manifold homeotopy groups and finitely generated abelian groups.

0. Introduction. Manifolds containing 2-sided projective planes have been a stumbling block in the study of mappings of 3-manifolds. For orientable sufficiently large 3-manifolds with incompressible boundary (now called *Haken* 3-manifolds), Waldhausen's [W] seminal work provides (except for I-bundles) an isomorphism from the homeotopy group to the group of outer automorphisms of the fundamental group that preserve the peripheral structure. To establish this isomorphism, he shows that every proper homotopy equivalence is homotopic to a homeomorphism, and that (except for I-bundles) homotopic homeomorphisms are isotopic. This program can be extended to nonorientable manifolds which do not contain 2-sided projective planes [H2], [L], but when there are 2-sided projective planes present, the manifolds are no longer aspherical and the behavior of homotopy equivalences becomes more complicated. For example, for 3-manifolds that are nontrivial connected sums and contain a 2-sided projective plane, not every self-homotopy equivalence is homotopic to a homeomorphism [H3], [H4].

Swarup [S3] surmounted some of these difficulties. He studied the class of irreducible 3-manifolds containing 2-sided projective planes. In [S3], such a manifold is said to be *sufficiently large* if it has a hierarchy (a finite sequence of cuttings along 2-sided incompressible surfaces) which ends with a collection of 3-balls and $\mathbb{P}^2 \times I$'s. Swarup shows that if there is an isomorphism of fundamental groups between two sufficiently large irreducible boundary-irreducible 3-manifolds (that contain no fake $\mathbb{P}^2 \times I$), then there is *some* homeomorphism between the manifolds. In the process, he develops an obstruction to realizability of a given isomorphism by a homeomorphism.

An irreducible 3-manifold containing 2-sided projective planes contains a finite collection of disjoint 2-sided projective planes such that every 2-sided projective plane is isotopic to one of them. A minimal such collection is unique up to isotopy, and techniques of Laudenbach and Hatcher show that isotopies between homeomorphisms which preserve this collection are deformable to isotopies which preserve the collection (see §4.1). Therefore the study of the homeotopy group is effectively reduced to the study of irreducible 3-manifolds in which every projective plane is parallel to the boundary. For such a manifold N, the orientable double cover has boundary components which are 2-spheres; filling them in with 3-balls, one obtains a Haken 3-manifold. The covering transformation extends to an involution with isolated fixed points. The quotient of this extended involution is the space $\widehat{N}$ obtained by coning off each projective plane boundary component of N; it is a 3-orbifold, irreducible in the sense of **[B-S]**. For irreducible 3-orbifolds, Bonahon and Siebenmann have extended the characteristic submanifold theory of Jaco and Shalen and Johannson to the orbifold setting—not only the existence of a characteristic S^1-fibered suborbifold, but also its structural classification and the deformation of its homeomorphisms to fiber-preserving homeomorphisms so useful in the study of Seifert 3-manifolds succeed beautifully. With this in hand, it appears that a theory of homeotopy groups of Haken 3-orbifolds could be developed by tediously replicating the known (and already tedious) theory of homeotopy groups of Haken 3-manifolds. In our case, however, a much simpler approach can be used to investigate the homeotopy group $\mathscr{H}(N)$.

Bonahon **[B]**, in the Seifert-fibered case, and Boileau and Zimmermann **[B-Z]** studied equivariant deformations of involutions of Haken manifolds (this is generalized to orientation-preserving finite actions on Haken manifolds in **[Z2]**). Using the observation that the proof of one of their theorems can be adapted to orientation-reversing involutions, together with a major result of Tollefson, we can give a quick proof in §3.1 that $\mathscr{H}(N)$ is isomorphic, up to finite kernel and finite index in the range, to the centralizer in $\mathscr{H}(M)$ of the homeotopy class of the involution. The homeotopy group in the Haken case is sufficiently well-understood to allow a reasonably precise description of these centralizers, given in §3.2 and §3.3. Coupled with the projective plane splitting, this leads to our structure theorem for homeotopy groups:

THEOREM 4.2.3. *Let N be a sufficiently large irreducible 3-manifold with incompressible boundary. Then $\mathscr{H}(N)$ is isomorphic mod finite groups to a direct product of finitely many groups Z_i, each of which satisfies (at least) one of the following conditions.*

(1) Z_i is finite.

(2) Z_i contains a finitely generated free group of finite index.

(3) Z_i is isomorphic mod finite groups to $\mathrm{GL}(3, \mathbb{Z})$.

(4) There is an exact sequence $1 \to A \to Z_i \to Q \to 1$, where A is a finitely generated abelian group (possibly trivial) with torsion subgroup of order at most 2, and Q is isomorphic mod finite groups to a 2-manifold homeotopy group.

(5) There is an exact sequence $1 \to D \to Z_i \to R \to 1$, where D is a finitely generated abelian group (possibly trivial) and R has finite index in a direct product of finitely many groups which are extensions of the form described in (4).

In particular, this implies that $\mathscr{H}(N)$ is finitely-presented.

Using the Boileau-Zimmermann technique, it is possible to extend the Waldhausen-Heil isomorphism to the case of sufficiently large irreducible boundary-incompressible 3-manifolds in which every 2-sided projective plane is boundary-parallel **[K]**, but we emphasize that Theorem 4.2.3 gives a great deal of qualitative information about $\mathscr{H}(N)$.

After collecting some algebraic concepts and lemmas in §1, we develop some technical results about 2-manifolds (stated for convenience in the language of 2-orbifolds) in §2. The main technical work is carried out in §3, as described above. In §4, we first develop the characteristic collection $\mathscr{P}$ of 2-sided projective planes in N. In particular, we use methods of Hatcher to prove that the natural homomorphism $\mathscr{H}(N, \mathscr{P}) \to \mathscr{H}(N)$, induced by inclusion of spaces of mappings, is an isomorphism. This implies that $\mathscr{H}(N)$ is isomorphic mod finite groups to the direct product of the homeotopy groups of the manifolds obtained by splitting N along the projective planes in $\mathscr{P}$. Then, Theorem 4.2.3 can be deduced from the results in §3.

1. Some algebraic lemmas. To avoid interruptions in the geometric arguments in later sections, we will present various algebraic concepts and results in this section.

1.1. *Isomorphism mod finite groups.* Following the approach used by Serre for classes of abelian groups (see pp. 504–506 of **[S2]**) we say that G is isomorphic to H *mod finite groups*, and write $G \cong_f H$, if there is a group K for which there are homomorphisms $\phi_1 \colon K \to G$ and $\phi_2 \colon K \to H$ having finite kernels and images of finite index. The argument given in Lemma 9.6.9 of **[S2]** shows that this is an equivalence relation. We note the following easily verified facts:

1. $G \cong_f \{1\}$ if and only if G is finite.
2. $G \cong_f F$, where F is a finitely generated free group, if and only if G contains a finitely generated free group as a subgroup of finite index.
3. If $G_1 \cong_f H_1$ and $G_2 \cong_f H_2$, then $G_1 \times G_2 \cong_f H_1 \times H_2$.

Because a group is finitely generated (respectively, finitely presented) if and only if any finite index subgroup is finitely generated (respectively, finitely presented) (see §2.3 of **[M-K-S]**), we have also

4. If $G \cong_f H$, then G is finitely generated (respectively, finitely presented) if and only if H is finitely generated (finitely presented).

1.2. *Centralizers of involutions in exact sequences.* If α is an automorphism of a group H, define $\operatorname{fix}(\alpha)$ to be the subgroup of elements fixed by α. In our applications, the automorphism will be conjugation by an element of order 2, sometimes in a supergroup of H, in which case the fixed subgroup consists of the subgroup of elements of H that commute with the element.

LEMMA 1.2.1. *Let $1 \to A \to G \to Q \to 1$ be an exact sequence of groups with A abelian. Suppose τ is an automorphism of G of order 1 or 2 with $\tau(A) = A$, and that the subgroup $\mathscr{Z}_- = \{a \in A | \tau(a) = a^{-1}\}$ is finitely generated. Let $\overline{\tau}$ be the induced automorphism on Q. Then the image of the projection homomorphism $\operatorname{fix}(\tau) \to \operatorname{fix}(\overline{\tau})$ has finite index.*

Proof. Consider the endomorphism of A which sends a to $\tau(a)a^{-1}$. Its image is a subgroup of $\mathscr{Z}_-$; define A_0 to be the quotient of $\mathscr{Z}_-$ by this image. Since the image contains $2\mathscr{Z}_-$, and $\mathscr{Z}_-$ is assumed to be finitely generated, A_0 is finite. Conjugation induces an action of $\operatorname{fix}(\overline{\tau})$ on A_0. Define $\chi \colon \operatorname{fix}(\overline{\tau}) \to A_0$ as follows. For $\overline{g} \in \operatorname{fix}(\overline{\tau})$, choose any $g \in G$ that projects to $\overline{g}$. Since $\overline{g} \in \operatorname{fix}(\overline{\tau})$, we have $\tau(g) = ag$ for some $a \in A$. Also, $g = \tau^2(g) = \tau(a)ag$, so $a \in \mathscr{Z}_-$. Define $\chi(\overline{g}) = [a]$. Since $\tau(a_1 g) = \tau(a_1)ag = a\tau(a_1)a_1^{-1}a_1 g$, χ is well-defined. If $\tau(g_i) = a_i g_i$ for $i = 1, 2$, then $\tau(g_1 g_2) = a_1 a_2^{g_1} g_1 g_2$, hence χ is a crossed homomorphism. Since A_0 is finite, the kernel of χ has finite index. If $\overline{g}$ is in the kernel, then $\tau(g) = \tau(a)a^{-1}g$ for some $a \in A$, and $\tau(a^{-1}g) = \tau(a^{-1})\tau(a)a^{-1}g = a^{-1}g$, hence $\overline{g}$ is in the image of $\text{fix}(\tau) \to \text{fix}(\overline{\tau})$. This proves Lemma 1.2.1.

It is known **[D-S]** that the fixed subgroup of an involution of a finitely-generated free group is finitely generated. We will need the following slight extension of this fact:

LEMMA 1.2.2. *Let G be a finitely generated group which has a free subgroup of finite index, and let τ be an automorphism of G of order 1 or 2. Then* $\text{fix}(\tau)$ *contains a finitely generated free group of finite index.*

Proof. Let F be a finitely generated free group of finite index in G; replacing F by $F \cap \tau(F)$ we may assume that $\tau(F) = F$. By **[D-S]**, the fixed subgroup of the restriction of τ to F is finitely generated, hence $\text{fix}(\tau)$ has a finitely generated free subgroup of finite index.

1.3. *Centralizers of involutions in* $\text{GL}(3, \mathbb{Z})$. The 3-torus will be an exceptional case. For its homeotopy group, which is isomorphic to $\text{GL}(3, \mathbb{Z})$, we will need the following elementary observation.

LEMMA 1.3.1. *The centralizer of any element of order* 1 *or* 2 *in* $\text{GL}(3, \mathbb{Z})$ *is finitely presented. More precisely, the centralizer of $\pm I$ is* $\text{GL}(3, \mathbb{Z})$, *while the centralizer of any other involution is isomorphic mod finite groups to* $\text{GL}(2, \mathbb{Z})$.

Proof. As in Lemma 3.1 of **[T]**, elementary linear algebra shows that if A is a linear involution of a finitely generated free abelian group G, there is a splitting $G \cong H \oplus \mathbb{Z}$ for which $A(H) = H$; applying this twice in our case produces a basis for $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ for which the matrix of A is of the form

$$\begin{pmatrix} \pm 1 & x & y \\ 0 & \pm 1 & z \\ 0 & 0 & \pm 1 \end{pmatrix}.$$

Let

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

We need only consider one involution in each conjugacy class, and some conjugation by elementary matrices shows that every matrix written as above is conjugate to one of $\pm I$, $\pm A_0$, or $\pm A_1$ (first find conditions on x, y, and z in order for the matrix to be of order 2). The centralizer of $\pm I$ is all of $\mathrm{GL}(3, \mathbb{Z})$. Consider the centralizer of A_1. Multiplying on the left and on the right by

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

and equating the results shows that the centralizer consists precisely of the matrices in $\mathrm{GL}(3, \mathbb{Z})$ of the form

$$\begin{pmatrix} a & 2c & c \\ d & 2f+i & f \\ 0 & 0 & i \end{pmatrix}.$$

Define a homomorphism from the centralizer to $\mathbb{Z}/2$ by sending each matrix to the entry i in the third row of the third column. Define a homomorphism from the kernel of this homomorphism to $\mathrm{GL}(2, \mathbb{Z})$ by the assignment

$$\begin{pmatrix} a & 2c & c \\ d & 2f+1 & f \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} a & 2c \\ d & 2f+1 \end{pmatrix}.$$

This is clearly injective, with image equal to the subgroup consisting of all elements of the form $\left(\begin{smallmatrix} p & 2q \\ r & s \end{smallmatrix}\right)$. This subgroup has finite index, since it is the preimage of the subgroup of all elements of the form $\left(\begin{smallmatrix} 1 & 0 \\ r & 1 \end{smallmatrix}\right)$ in $\mathrm{GL}(2, \mathbb{Z}/2)$ under the homomorphism $\mathrm{GL}(2, \mathbb{Z}) \to \mathrm{GL}(2, \mathbb{Z}/2)$ induced by the reduction of the coefficients modulo 2. Thus the centralizer is isomorphic mod finite groups to $\mathrm{GL}(2, \mathbb{Z})$.

Since $-I$ is central in $\mathrm{GL}(3, \mathbb{Z})$, the centralizer of $-A_1$ is equal to the centralizer of A_1. The case of A_0 is similar: the centralizer is all matrices of the form

$$\begin{pmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & i \end{pmatrix},$$

hence is isomorphic to $\mathrm{GL}(2, \mathbb{Z}) \times \mathbb{Z}/2$. This completes the proof.

2. Centralizers of involutions in homeotopy groups in dimension 2.

2.1. *Equivariant isotopy in 2-fold coverings.* The first result in this section is needed to extend the Relative Baer Theorem to orientation-reversing involutions (Corollary 2.1.2), but is also needed to prove Proposition 2.2.1, which will be used heavily in §3.

Proposition 2.1.1. *Let F be a compact surface, not a projective plane or 2-sphere, and let $\partial_0 F$ be a subcollection of the components of ∂F. If F is a disc, assume that $\partial_0 F = \partial F$. Let P be a finite, possibly empty, collection of points in the interior of F. Let τ be an involution of F such that $\tau(\partial_0 F) = \partial_0 F$ and $\tau(P) = P$. Suppose that f is a homeomorphism of F, which commutes with τ, such that f is isotopic to the identity, relative to $\partial_0 F \cup P$. If P is empty, assume the following:*

(1) *F is not the Möbius band or Klein bottle.*

(2) *If F is a torus, $\mathrm{fix}(\tau) = \varnothing$, and τ is orientation-reversing, then the induced homeomorphism $\overline{f}$ of F/τ induces the identity outer automorphism on the fundamental group.*

(3) *If F is a torus or annulus and $\mathrm{fix}(\tau) \neq \varnothing$, then f leaves some component of $\mathrm{fix}(\tau)$ invariant, and if F is an annulus either $\partial_0 F = \varnothing$ or $\partial_0 F = \partial F$.*

Then f is τ-equivariantly isotopic to the identity, relative to $\partial_0 F \cup P$.

Proof. Suppose first that F is a disc and P is empty or consists of one point. The involutions of the disc are known to be linear (up to equivalence). In coordinates for which the involution is linear, the Alexander trick produces a τ-equivariant isotopy from f to the identity.

Next, suppose that F is a torus or annulus. If τ is as in (2) above, then F/τ is a Klein bottle, and by hypothesis $\overline{f}$ is isotopic to the identity. The lift of such an isotopy furnishes the required isotopy of f. If F is a torus, $\mathrm{fix}(\tau) = \varnothing$, and τ is orientation-preserving, then F/τ is a torus, and $\overline{f}$ induces the identity automorphism on its fundamental group (since f induces the identity on the fundamental group of F). Again, $\overline{f}$ is isotopic to the identity. If F is an annulus, $\mathrm{fix}(\tau) = \varnothing$, and τ is orientation-reversing, then F/τ is a Möbius band so $\overline{f}$ is isotopic to the identity relative to $\partial_0 F$. If F is an annulus, $\mathrm{fix}(\tau) = \varnothing$, and τ is orientation-preserving, then F/τ is an annulus and $\overline{f}$ does not interchange its boundary components: moreover, if $\partial_0 F = \partial F$, then $\overline{f}$ must be isotopic to the identity relative to $\partial(F/\tau)$ since f is isotopic to the identity relative to ∂F. Again, $\overline{f}$ is isotopic to the identity. So we may assume that $\mathrm{fix}(\tau)$ is nonempty. By (3), f leaves some component of $\mathrm{fix}(\tau)$ invariant. By Lemma 5.1 of [T], f is τ-equivariantly isotopic to the identity. If F is an annulus and $\partial_0 F = \partial F$, then since f is isotopic to the identity relative to ∂F, the τ-equivariant isotopy may be adjusted so that it is relative to ∂F.

By deleting the interiors of disjoint τ-invariant 2-cells, each containing an element of P, it suffices to prove the result for the case $P = \varnothing$, and from hypothesis (1) and the previous argument, we may now assume that $\chi(F) < 0$. We consider three cases.

Case I. Fix $(\tau) = \varnothing$.

In this case, $q : F \to F/\tau$ is a 2-sheeted covering. Since $\chi(F) < 0$, $\pi_1(F/\tau)$ is free of rank at least two, so the centralizer of $q_\#(\pi_1(F))$ in $\pi_1(F/\tau)$ is trivial. The result follows by the proof of Lemma 1.6 in **[B-H]**.

Case II. $\dim(\text{fix}(\tau)) = 0$.

Let $\text{fix}(\tau) = \{x_1, x_2, \dots, x_n\}$, and let $f(x_i) = y_i$. Since f commutes with τ, each y_i is contained in $\text{fix}(\tau)$. For some isotopy $F : 1_F \simeq f$, denote by α_i the trace at the point x_i. Using F, one constructs an isotopy $K : f\tau f^{-1} \simeq \tau$ whose trace at x_i is $\alpha_i \tau(\alpha_i^{-1})$. Since $f\tau f^{-1} = \tau$, this trace must be central in $\pi_1(F, x_i)$, hence is a contractible loop. Let $p \colon \widetilde{F} \to F$ be the universal covering. For each x_i, let $\tilde{x}_i$ be a point in the preimage of x_i, and let $\tilde{\tau}$ be the lift of τ to an involution of $\widetilde{F}$ that fixes $\tilde{x}_i$. If $\tilde{y}_i$ is the endpoint of the lift of α_i starting at $\tilde{x}_i$, then the contractibility of $\alpha_i \tau(\alpha_i^{-1})$ shows that $\tilde{\tau}(\tilde{y}_i) = \tilde{y}_i$. But $\text{fix}(\tilde{\tau})$ is connected and 0-dimensional, so $\tilde{x}_i = \tilde{y}_i$. Therefore α_i is a contractible loop at x_i, and it may be assumed that the isotopy F is relative to $\text{fix}(\tau) \cup \partial_0 F$. The result now follows by applying Case I to the complement of a τ-invariant regular neighborhood of $\text{fix}(\tau)$.

Case III. $\dim(\text{fix}(\tau)) = 1$.

Let J be a component of $\text{fix}(\tau)$ and note that $f(J) \subseteq \text{fix}(\tau)$. Suppose first that J is a simple closed curve. Since f is isotopic to the identity, it follows that if $f(J) \cap J = \varnothing$, then $J \cup f(J)$ bounds an annulus A. Furthermore, $A \cap \tau(A) = \partial A$, so F is a torus or Klein bottle, contradicting the assumption that $\chi(F) < 0$. The case when J is an arc and $J \cap f(J) = \varnothing$ is similar. So we may assume that f preserves each component of $\text{fix}(\tau)$. Since $\chi(F) < 0$, J is not isotopic to its inverse. Therefore we may change f by τ-equivariant isotopy so that its restriction to $\text{fix}(\tau)$ is the identity, and so that the isotopy from f to the identity is relative to $\text{fix}(\tau) \cup \partial_0 F$. For the components of $\text{fix}(\tau)$ which are arcs, and therefore meet some boundary components, we may assume that f is isotopic to the identity relative to those boundary components as well. The conclusion now follows by splitting F along each 1-dimensional component of $\text{fix}(\tau)$ to obtain a surface to which the previous cases apply. This completes the proof of Proposition 2.1.1.

COROLLARY 2.1.2 (*Relative Baer Theorem for involutions*). *Let F be a compact connected orientable surface with nonempty boundary and let $\widetilde{F}$ be its universal covering. Let τ be an involution on F and let $\mathscr{O}_{\widetilde{F}}$ be generated by all lifts of τ to $\widetilde{F}$, so that there is an exact sequence*

$$1 \to \pi_1(F) \to \mathscr{O}_{\widetilde{F}} \to \mathbb{Z}_2 \to 1 .$$

Let $\tilde{f}$ be a homeomorphism of $\widetilde{F}$, commuting with the elements of $\mathscr{O}_{\widetilde{F}}$, such that the restriction of $\tilde{f}$ to $\partial\widetilde{F}$ is the identity. Then there exists an $\mathscr{O}_{\widetilde{F}}$-equivariant isotopy from $\tilde{f}$ to the identity, which is constant on $\partial\widetilde{F}$.

Proof. Let f be the homeomorphism of F induced by $\tilde{f}$. The restriction of f to ∂F is the identity and f commutes with τ. Since its lift commutes with $\pi_1(F)$, f induces the identity outer automorphism on $\pi_1(F)$, so f is isotopic to the identity of F. Moreover, since $\tilde{f}$ is the identity on $\partial\widetilde{F}$, this isotopy may be chosen to be relative to ∂F. Let J be a component of the fixed-point set of τ. Then each component $\widetilde{J}$ of the preimage of J is the fixed-point set of some lift $\tilde{\tau}$ of τ, and since $\tilde{f}$ commutes with $\tilde{\tau}$, it follows that $\tilde{f}(\widetilde{J}) = \widetilde{J}$ and hence $f(J) = J$. The result now follows from Proposition 2.1.1.

2.2. *Homeotopy groups of* 2-*orbifolds.* We begin by briefly recalling some facts from the theory of orbifolds. References for orbifolds are **[T2]**, **[S1]**, **[D-M]**, and **[B-S]**.

An n-orbifold is a space locally isotopic to a quotient D^n/G where G is a finite subgroup of the orthogonal group $O(n)$ acting on $D = D^n$. An n-orbifold with boundary may also have boundary points, which are locally the image of a point of $D^{n-1} = \{(x_1, \ldots, x_{n-1}, 0)\} \subseteq D^n$ in a quotient of $D^n_+ = \{(x_1, \ldots, x_n) \in D^n | x_n \geq 0\}$ by a finite subgroup G which preserves D^n_+ (i.e. $G \subseteq O(n-1)$). Thus the boundary of an n-orbifold is an $(n-1)$-orbifold. The *local group* at $x \in D/G$ is defined to be the conjugacy class in $O(n)$ of the stabilizer of any point in the preimage of x under the quotient map $D \to D/G$.

The *exceptional set* is the set of points whose local group is not trivial. An *orbifold homeomorphism* is a homeomorphism of the underlying topological space which respects all the orbifold structure.

When $n = 2$, the finite subgroups of $O(2)$ are either cyclic or dihedral, and the underlying topological space of the 2-orbifold is always a 2-manifold. The exceptional set consists of the following three types of points:

(1) *cone points*, whose local group is cyclic acting by rotation about the origin in D^2. These are isolated points in the interior of the orbifold.

(2) *reflection points*, also called *silvered* points, whose local group is cyclic of order 2 acting by reflection across a line through the origin of D^2. A reflection point may be interior to the orbifold (although in the topological boundary of the underlying 2-manifold), or it may lie in the boundary, in which case it is a silvered endpoint of a 1-orbifold component of the boundary of the 2-orbifold.

(3) *corner reflectors*, whose local group is dihedral generated by the reflections through two distinct lines through the origin. These always lie in the interior of the orbifold (although in the topological boundary of the underlying 2-manifold), and are in the closure of two arcs of reflection points.

PROPOSITION 2.2.1. *Let B be a compact 2-orbifold, and let B_0 be a compact 1-orbifold contained in ∂B.*

(a) *$\mathcal{H}(B, B_0)$ is finitely-presented.*

(b) *if $\overline{\tau}$ is an involution of B preserving B_0, then the centralizer in $\mathcal{H}(B, B_0)$ of $\langle\overline{\tau}\rangle$ is isomorphic mod finite groups to $\mathcal{H}(B/\overline{\tau}, B_0/\overline{\tau})$.*

Proof. Let $F = |B|$ denote the compact 2-manifold underlying the orbifold B and let P be the set of cone points. Denote by $\partial_0 F$ the components of ∂F that contain either a corner reflector, an arc component of $|B_0|$, or a component of $|\partial B|$ which is a silvered interval. Let $\mathcal{H}(F \operatorname{rel} \partial_0 F \cup P)$ be the homeotopy group consisting of classes whose restriction to $\partial_0 F \cup P$ is the identity. Consider the subgroup $\mathcal{H}_1(F \operatorname{rel} \partial_0 F \cup P)$ consisting of those classes whose restriction to $\partial_1 F = \partial F - \partial_0 F$ is isotopic to the identity. If $\langle g\rangle \in \mathcal{H}_1(F \operatorname{rel} \partial_0 F \cup P)$, then g is the identity on $|B_0|$ and the exceptional set, and can therefore be viewed as an orbifold homeomorphism of B whose class represents an element in $\mathcal{H}(B, B_0)$. This defines a homomorphism $\mathcal{H}_1(F \operatorname{rel} \partial_0 F \cup P) \to \mathcal{H}(B, B_0)$.

LEMMA 2.2.2. *$\mathscr{H}_1(F \operatorname{rel} \partial_0 F \cup P) \to \mathscr{H}(B, B_0)$ is an injection onto a subgroup of finite index.*

Proof. Suppose $\langle g \rangle \in \mathscr{H}_1(F \operatorname{rel} \partial_0 F \cup P)$ is in the kernel. Then g is isotopic to the identity by an isotopy which preserves $|B_0|$, $|\partial B|$, and the exceptional set. Such an isotopy may be adjusted to be relative to $\partial_0 F \cup P$. Therefore $\langle g \rangle$ is trivial in $\mathscr{H}_1(F \operatorname{rel} \partial_0 F \cup P)$. This shows that the homomorphism is injective. The elements of $\mathscr{H}(B, B_0)$ act as a finite group of permutations on the components of the strata of the exceptional set. The elements which act trivially are exactly the image of $\mathscr{H}_1(F \operatorname{rel} \partial_0 F \cup P)$, which therefore has finite index. This completes the proof of Lemma 2.2.2.

Using Lemma 2.2.2, we will regard $\mathscr{H}_1(F \operatorname{rel} \partial_0 F \cup P)$ as a subgroup of $\mathscr{H}(B, B_0)$. Since $\mathscr{H}_1(F \operatorname{rel} \partial_0 F \cup P)$ is known to be finitely presented [H], it follows that $\mathscr{H}(B, B_0)$ is finitely presented, proving part (a) of Proposition 2.2.1.

In the proof of part (b), we will need the following lemma, which is a generalization of Theorem 6.1 of [T].

LEMMA 2.2.3. *Let F be a compact surface, which either has $\chi(F) < 0$, or is a disc, annulus, or Möbius band. Let $\partial_0 F$ be a subcollection of the components of ∂F, and if F is a disc, annulus, or Möbius band, then assume that $\partial_0 F = \partial F$. Let h be an involution of F such that $h(\partial_0 F) = \partial_0 F$. Suppose that g is an involution of F, isotopic to h relative to $\partial_0 F$. Then there exists a homeomorphism k, isotopic to the identity relative to $\partial_0 F$, such that $kgk^{-1} = h$.*

Proof. The proof is very similar to the argument in Theorem 6.1 of [T]. We will simply indicate the necessary changes. For one, all isotopies in the proof will be relative to $\partial_0 F$ instead of ∂F.

The orientability assumption for F is not needed to guarantee the existence of an h-hierarchy, as can be seen by an argument analogous to the proof of Theorem 3.6 of [T]. If F is nonorientable and J is an element of the h-hierarchy, then J may be one-sided.

The proof of Theorem 6.1 of [T] is broken into three lemmas, called Lemma 6.2, Lemma 6.3, and Lemma 6.4 there, and we now describe the modifications to the proofs of these.

Call the complex obtained by identifying an arc $x_0 \times I$ in the annulus $S^1 \times I$ to a point a pinched annulus. In the modified proof of Lemma 6.2, the product regions between homotopic loops may now be discs, annuli, and, for one-sided loops, pinched annuli. One again considers the cases of $\lambda g\lambda^{-1}(J) = J$ or $\lambda g\lambda^{-1}(J) \cap J = \varnothing$. If J is one-sided, however, then if $h(J) = J$ it necessarily follows that $\lambda g\lambda^{-1}(J) = J$, since $\lambda g\lambda^{-1}(J)$ will then be homotopic to J and cannot be disjoint from J. Thus in case (a), if $h(J) = J$ and $\lambda g\lambda^{-1}(J) \cap J = \varnothing$, then either J is a two-sided simple closed loop, or, since $\partial_0 F$ need not equal ∂F, J may also be an arc. In the latter case, there exists a disc D, containing $J \cup \lambda g\lambda^{-1}(J)$ in ∂D, so that $\partial D-(J\cup\lambda g\lambda^{-1}(J))$ consists of two arcs contained in $\partial_1 F = \partial F - \partial_0 F$. The proof for this additional case is similar. For case (b), where $h(J) \cap J = \varnothing$ and $\lambda g\lambda^{-1}(J) \cap J = \varnothing$, the following additional situations may arise: (i) D is a disc containing $h(J) \cup \lambda g\lambda^{-1}(J)$ in ∂D so that $D - (h(J) \cup \lambda g\lambda^{-1}(J))$ consists of two arcs contained in $\partial_1 F$, (ii) D is a pinched annulus with "boundary" $h(J) \cup \lambda g\lambda^{-1}(J)$, and (iii) D is a "triangle" with one vertex in $\partial_0 F$ and having edges $h(J)$, $\lambda g\lambda^{-1}(J)$, and an arc in $\partial_1 F$. The proofs for these additional three cases are analogous to the one given in case (b) of Lemma 6.2. Case (c) is unchanged, because as seen above, it cannot arise when J is a one-sided loop.

Lemma 6.3 of [T] is the same except for replacing ∂F by $\partial_0 F$ and requiring that the homotopy preserve the boundary.

For Lemma 6.4 of [T], consider first the case when J is an arc. Since the homotopy preserves the boundary, we take $x_0 \in \partial J$ and the proof is complete. Suppose J is a loop. Then F is not the Möbius band or annulus (because in those cases, the hierarchy is chosen to begin with an arc). Therefore the center of $\pi_1(F)$ is trivial and we can continue the proof as in [T]. In the case where $h(J) \cap J = \varnothing$ and J is one-sided, the neighborhood U of $h(J)$ is a Möbius band, but the homeomorphism called λ can be constructed in a similar fashion to the cases in Lemma 6.4. The remainder of the proof is unchanged from the proof of Theorem 6.1. This completes our discussion of Lemma 2.2.3.

We can now prove part (b) of Proposition 2.2.1. Let $\overline{F}$ be the compact 2-manifold underlying the orbifold $B/\overline{\tau}$ and let $\overline{P}$ be the cone points. Denote by $\partial_0 \overline{F}$ the components of $\partial \overline{F}$ that contain either a corner reflector, an arc component of $|B_0|/\overline{\tau}$, or a silvered interval component of $\partial(B/\overline{\tau})$. Define $\mathscr{H}_1(\overline{F} \operatorname{rel} \partial_0 \overline{F} \cup \overline{P})$ as above, and again using Lemma 2.2.2 regard it as a finite-index subgroup of $\mathscr{H}(B/\overline{\tau}, B_0/\overline{\tau})$. Denote by $\mathscr{H}_0(\overline{F} \operatorname{rel} \partial_0 \overline{F} \cup \overline{P})$ the finite-index subgroup of $\mathscr{H}_1(\overline{F} \operatorname{rel} \partial_0 \overline{F} \cup \overline{P})$ consisting of those elements that lift to

(B, B_0) and have some lift in $\mathscr{H}_1(F \operatorname{rel} \partial_0 F \cup P)$. Let C_1 denote the intersection $\mathscr{H}_1(F \operatorname{rel} \partial_0 F \cup P)$ with the centralizer in $\mathscr{H}(B, B_0)$ of $\langle\overline{\tau}\rangle$. Since C_1 has finite index in the centralizer of $\langle\overline{\tau}\rangle$ in $\mathscr{H}(B, B_0)$, part (b) of Proposition 2.2.1 will follow once we show that $\mathscr{H}_0(\overline{F} \operatorname{rel} \partial_0 \overline{F} \cup \overline{P})$ is isomorphic mod finite groups to C_1. By lifting we obtain a homomorphism $\lambda : \mathscr{H}_0(\overline{F} \operatorname{rel} \partial_0 \overline{F} \cup \overline{P}) \to C_1$ or $C_1/\langle\overline{\tau}\rangle$ according as $\langle\overline{\tau}\rangle$ is not or is contained in C_1.

Suppose that F is a projective plane, a disc, or a sphere, and that $\chi(F - P) \geq 0$. The involutions of those surfaces are well-known, up to equivalence, and upon consideration of the various cases one finds that $\mathscr{H}_1(F \operatorname{rel} \partial_0 F \cup P)$ and $\mathscr{H}_0(\overline{F} \operatorname{rel} \partial \overline{F} \cup \overline{P})$ are both finite. Now suppose that P is empty and that F is either a Klein bottle, a Möbius band, or an annulus with $\partial_0 F \neq \partial F$. Again, the involutions of F in these cases are known, up to equivalence, and $\mathscr{H}_1(F \operatorname{rel} \partial_0 F \cup P)$ and $\mathscr{H}_0(\overline{F} \operatorname{rel} \partial \overline{F} \cup \overline{P})$ are finite. Therefore the result holds in all these cases, and we exclude them in the remainder of the argument.

Let $\langle f \rangle \in C_1$. Then $f\overline{\tau}f^{-1}$ is isotopic to $\overline{\tau}$, relative to $\partial_0 F \cup P$. Suppose that F is a torus and $P = \varnothing$. By Theorem 6.1(ii) of [T] there exist involutions β_1, β_2, and β_3 associated with $\overline{\tau}$, each imbedded in actions of SO(2) commuting with $\overline{\tau}$, such that any involution homotopic to $\overline{\tau}$ is strongly equivalent to some $\overline{\tau}\beta_i$. This implies that the image of λ has finite index in C_1. In all remaining cases, by removing invariant discs about the points P, we may assume that $P = \varnothing$ and $\chi(F) < 0$. By Lemma 2.2.3, there exists a homeomorphism k, isotopic to the identity relative to $\partial_0 F$, such that $kf\overline{\tau}f^{-1}k^{-1} = \overline{\tau}$. This implies that we may isotope f, relative to $\partial_0 F$, to commute with $\overline{\tau}$, showing that λ is a surjection.

Suppose now that $\langle\overline{f}\rangle \in \mathscr{H}_0(\overline{F} \operatorname{rel} \partial\overline{F} \cup \overline{P})$ with $\lambda(\langle\overline{f}\rangle) = \langle f \rangle$ trivial. Then f commutes with $\overline{\tau}$ and is isotopic to the identity relative to $\partial_0 F \cup P$. By Proposition 2.1.1, f is τ-equivariantly isotopic to the identity, relative to $\partial_0 F \cup P$, unless $P = \varnothing$ and either property (2) or (3) of Proposition 2.1.1 is not satisfied. This implies that λ is injective except for these cases. If F is a torus and $\overline{\tau}$ is a fixed-point-free orientation-reversing involution, then $\overline{F}$ is a Klein bottle and $\mathscr{H}_0(\overline{F})$ is finite, implying that the kernel of λ is finite. Now assume that F is either a torus or annulus and $\operatorname{fix}(\tau) \neq \varnothing$. Suppose $\langle\overline{f}\rangle$ and $\langle\overline{g}\rangle$ are two elements in the kernel of λ, with images $\langle f \rangle$ and $\langle g \rangle$ respectively, such that f and g give the same permutation on the components of $\operatorname{fix}(\tau)$. Then by Proposition 2.1.1, fg^{-1} is τ-equivariantly isotopic to the identity, relative to $\partial_0 F$, which implies that $\langle\bar{f}\rangle = \langle\bar{g}\rangle$. Since there are only finitely many permutations of the components of $\mathrm{fix}(\tau)$, the kernel of λ is finite. This completes the proof of Proposition 2.2.1.

3. Centralizers of involutions in homeotopy groups in dimension 3. In §3.1, we will prove that for an orbifold $\mathscr{O}$ of the form M/τ, where M is Haken and τ is an involution, the orbifold homeotopy group $\mathscr{H}(\mathscr{O})$ is isomorphic mod finite groups to the centralizer of $\langle\tau\rangle$ in $\mathscr{H}(M)$. This will be used in our main application, and motivates the calculation of these centralizers. In §3.2, we treat the fibered cases—I-bundles and Seifert-fibered 3-manifolds—where τ is assumed to be fiber-preserving. The general Haken case is handled in §3.3. Apart from some special cases, one may assume that the involution preserves the characteristic submanifold Σ (in the sense of Johannson), and is fiber-preserving on Σ (it may permute the components nontrivially). Then, after passing to a subgroup of finite index in $\mathscr{H}(M)$, the centralizer of $\langle\tau\rangle$ can be compared to the centralizers of its restrictions to the components of Σ, and one obtains a qualitative description of the centralizer, stated as Theorem 3.3.2.

3.1. *Homeotopy groups and centralizers.*

THEOREM 3.1.1. *Let M be a Haken 3-manifold, and let τ be an involution of M such that M admits a τ-equivariant hierarchy. For the orbifold quotient $\mathscr{O} = M/\tau$, the homeotopy group $\mathscr{H}(\mathscr{O})$ is isomorphic mod finite groups to the centralizer of the homeotopy class $\langle\tau\rangle$ in $\mathscr{H}(M)$.*

Proof. Since $\pi_1^{\mathrm{orb}}(\mathscr{O})$ is finitely generated, it has only finitely many subgroups of index two. Let $\mathscr{H}_0(\mathscr{O})$ denote the subgroup of finite index in $\mathscr{H}(\mathscr{O})$ which preserves the image of $\pi_1(M)$; these are the homeotopy classes that lift to M. Let $\overline{\mathrm{Cent}}(\langle\tau\rangle)$ denote the quotient of $\mathrm{Cent}(\langle\tau\rangle)$ by the order 2 subgroup generated by $\langle\tau\rangle$. Lifting $\langle g\rangle$ to the coset of $\langle\tilde{g}\rangle$, where $\tilde{g}$ is either lift of g to M, defines a homomorphism Ψ from $\mathscr{H}_0(\mathscr{O})$ to $\overline{\mathrm{Cent}}(\langle\tau\rangle)$. The rest of the proof, showing that this lifting homomorphism is an isomorphism mod finite groups, will be broken into a sequence of lemmas.

LEMMA 3.1.2. *The image of Ψ has finite index in $\overline{\mathrm{Cent}}(\langle\tau\rangle)$.*

Proof. Recall that two involutions k_1 and k_2 of a space X are said to be *strongly equivalent* if there is a homeomorphism k of X such that $kk_1k^{-1} = k_2$ and k is isotopic to the identity. By Theorem 7.1 of [T] (which only requires the M have a τ-equivariant hierarchy—see the remark on p. 340 of [T]), the homeotopy class of an involution τ of M contains only finitely many strong equivalence classes of involutions (in fact, at most two unless M is the 3-torus, in which case there are at most eight). Observe that conjugation induces an action of $\mathrm{Cent}(\langle\tau\rangle)$ on this set of strong equivalence classes; let $\mathrm{Cent}_0(\langle\tau\rangle)$ denote the stabilizer of the strong equivalence class of τ. The image of $\mathrm{Cent}_0(\langle\tau\rangle)$ in $\overline{\mathrm{Cent}}(\langle\tau\rangle)$ is a subgroup of finite index, and each element in this subgroup contains a representative which commutes with τ and hence is in the image if Ψ. This proves Lemma 3.1.2.

For the injectivity, we begin by observing that the proof of Theorem 4.3 of **[B-Z]** can be extended easily to the case when τ is orientation-reversing.

LEMMA 3.1.3. *Let M be an orientable Haken 3-manifold and let τ be an involution such that M has a τ-equivariant hierarchy. Let $f\colon M \to M$ be a τ-equivariant homeomorphism which is isotopic to the identity. Then f is τ-equivariantly isotopic to the identity if and only if the induced homeomorphism $\overline{f}$ on the quotient M/τ induces the identity outer automorphism on $\pi_1^{\mathrm{orb}}(M/\tau)$.*

Proof. The argument of **[B-Z]** can be adapted straightforwardly to show that f is τ-equivariantly isotopic to the identity. We will summarize the additions needed. The reference **[M-S]** applies to the orientation-reversing case; this is needed at several places in case τ is orientation-reversing. In case τ is orientation-reversing, it is necessary to use Heil's **[H2]** extension of Waldhausen's results in 4.6(b) and 4.7(b)(ii). The Baer Theorem, stated as 4.11 in **[B-Z]**, is needed for orientation-reversing involutions of orientable surfaces; this appears in **[Z1]**. We omit more precise details of the adaptation.

LEMMA 3.1.4. *The kernel of Ψ is finite.*

Proof. If $\langle g\rangle$ is in the kernel, then there is a lift $\tilde{g}$ of g which is isotopic to the identity of M. By Lemma 3.1.3, g is isotopic to the identity unless it does not induce an inner automorphism on the orbifold fundamental group $\pi_1^{\mathrm{orb}}(\mathscr{O})$. Since g lifts to M, it induces an automorphism of the extension

$$1 \to \pi_1(M) \to \pi_1^{\mathrm{orb}}(\mathscr{O}) \to \mathbb{Z}/2 \to 1$$

and after conjugation, the induced automorphism is the identity on the subgroup $\pi_1(M)$.

SUBLEMMA. *Let A be a normal subgroup of the group B. Let $\mathrm{Aut}_0(B)$ denote the subgroup of $\mathrm{Aut}(B)$ consisting of automorphisms taking A to A, inducing the identity on A, and inducing the identity on the quotient B/A, and let $\mathrm{Out}_0(B)$ denote the image of $\mathrm{Aut}_0(B)$ in $\mathrm{Out}(B)$. Let $\mathcal{Z}$ denote the center of A. Then there is an isomorphism from $\mathrm{Out}_0(B)$ to a quotient of $H^1(B/A; \mathcal{Z})$.*

Proof. This lemma appears often in the literature in the case when A is abelian; for example, it is proved in **[M1]** under the assumptions that A is abelian and B is a semidirect product (in which case the quotient is $H^1(B/A; \mathcal{Z})$ itself), and as Theorem 8 in **[C-R]**, under the assumption that A is free abelian. Since we do not know a reference for the version needed here, we sketch the argument. Let C denote B/A. Notice that although the action of C on A is only defined up to inner automorphisms of A, the action on $\mathcal{Z}$ is well-defined. For each $\phi \in \mathrm{Aut}_0(B)$, define a crossed homomorphism α_ϕ by sending $b \in B$ to $b^{-1}\phi(b)$. This takes values in A since ϕ induces the identity on C, and for all $a \in A$ and $b \in B$ we have

$$bab^{-1} = \phi(bab^{-1}) = b\alpha_\phi(b)a\alpha_\phi(b)^{-1}b^{-1},$$

so $\alpha_\phi(b) \in \mathcal{Z}$. On A, α_ϕ vanishes, so it may be regarded as a crossed homomorphism from C to A. A crossed homomorphism α is the image of the element of $\mathrm{Aut}_0(B)$ defined by sending b to $b\alpha(Ab)$. The principal crossed homomorphisms correspond to the elements of $\mathrm{Inn}_0(B) = \mathrm{Inn}(B) \cap \mathrm{Aut}_0(B)$ that are conjugation by elements of $\mathcal{Z}$. Thus $\mathrm{Inn}_0(B)$ is carried to a subgroup of the group of crossed homomorphisms from C to $\mathcal{Z}$, which contains the principal crossed homomorphisms, yielding the assertion in the Sublemma.

Now in our case, $B/A \cong \mathbb{Z}/2$, so the cohomology $H^1(\mathbb{Z}/2; \mathcal{Z})$ with action given by the induced automorphism of τ is a quotient of $\mathcal{Z}_- = \{a \in \mathcal{Z} | \tau_\#(a) = a^{-1}\}$ by a subgroup which contains $2\mathcal{Z}_-$ (see for example p. 122 of **[M1]**). Since $\mathcal{Z}$ is (free) abelian of rank at most 3, this is finite. Lemma 3.1.4 follows.

Lemmas 3.1.2 and 3.1.4 immediately imply Theorem 3.1.1.

3.2. *The case of fibered* 3*-manifolds.* Fibered 3-manifolds (with 1-dimensional fiber) are those which are I-bundles or which are Seifert fibered. Let Σ be a fibered orientable 3-manifold. In $\partial\Sigma$ let F be a 2-dimensional submanifold which is a collection of tori and annuli, each of which is a union of fibers. Define $\mathscr{H}^f(\Sigma, F)$ to be the isotopy classes of fiber-preserving (i.e. taking each fiber to a possibly different fiber) homeomorphisms that preserve F. Define $\mathscr{G}(\Sigma, F)$ (respectively, $\mathscr{G}^f(\Sigma, F)$) to be the subgroup of $\mathscr{H}(\Sigma, F)$ (respectively, $\mathscr{H}^f(\Sigma, F)$) consisting of the isotopy classes whose restriction to F is isotopic to the identity map of F. Since F is assumed to consist of tori and annuli (but not squares, as is permissible in the general theory of boundary patterns in [J]), a fiber-preserving homeomorphism of F which is isotopic to the identity will be isotopic through fiber-preserving homeomorphism. Notice that $\mathscr{G}(\Sigma, F)$ is a normal subgroup of $\mathscr{H}(\Sigma, F)$.

Let B be a 2-orbifold, and B_0 a 1-suborbifold of ∂B. (It is not assumed that B_0 is a union of boundary components). Define $\mathscr{G}(B, B_0)$ to be the subgroup of $\mathscr{H}(B, B_0)$ consisting of the classes whose restriction to B_0 is isotopic to the identity. Since $|B_0|$ consists of arcs and circles, $\mathscr{G}(B, B_0)$ has finite index in $\mathscr{H}(B, B_0)$.

For any of the homeotopy groups discussed here, a "+" subscript, as in $\mathscr{H}^f_+(\Sigma, F)$, indicates the subgroup of orientation-preserving elements. When F is nonempty, $\mathscr{G}_+(\Sigma, F) = \mathscr{G}(\Sigma, F)$. In case B is a nonorientable 2-orbifold, then by convention $\mathscr{H}_+(B, B_0) = \mathscr{H}(B, B_0)$.

Given a fibered 3-manifold (Σ, F), let B be the quotient orbifold for Σ, and let B_0 be the image of F in ∂B. When Σ is an I-bundle, B is a 2-manifold, which is orientable if and only if Σ is a product I-bundle. When Σ is Seifert-fibered, B is an orbifold, possibly nonorientable, whose only singularities are cone points corresponding to the exceptional orbits. A fiber-preserving homeomorphism f of Σ, preserving F, induces a homeomorphism $\overline{f}$ of B, preserving B_0. We call $\overline{f}$ the *projection* of f. Sending $\langle f \rangle$ to $\langle \overline{f} \rangle$ induces a homomorphism $\mathscr{H}^f(\Sigma, F) \to \mathscr{H}(B, B_0)$, and we also refer to this (or its restrictions to subgroups of $\mathscr{H}^f(\Sigma, F)$) as projection.

LEMMA 3.2.1. *Suppose that Σ is an I-bundle over B. Then projection is an isomorphism from $\mathscr{G}^f_+(\Sigma, F)$ to $\mathscr{G}_+(B, B_0)$ if F is nonempty, and from $\mathscr{G}^f_+(\Sigma)$ to $\mathscr{G}(B)$ if F is empty.*

Proof. Suppose first that Σ is a product I-bundle $B \times I$. If F is nonempty, then the restriction of each element of $\mathscr{G}^f(\Sigma, F)$ to F is isotopic to the identity, so all elements preserve the orientation of the fibers. In particular, they cannot interchange $B \times \{0\}$ and $B \times \{1\}$, so the projection $\mathcal{G}_+^f(\Sigma, F) \to \mathcal{G}_+(B, B_0)$ is an isomorphism with inverse defined by sending $\langle g \rangle$ to $\langle g \times 1_I \rangle$. If F is empty, then the projection $\mathcal{G}_+^f(\Sigma) \to \mathcal{G}(B)$ is an isomorphism with inverse defined by sending $\langle g \rangle$ to $\langle g \times 1_I \rangle$, if g is orientation-preserving, and to $\langle g \times r \rangle$, where r is reflection in the I-fibers, when g is orientation-reversing.

Now suppose that Σ is a twisted I-bundle over the nonorientable surface B. For $\langle g \rangle \in \mathcal{G}_+(B, B_0) = \mathcal{G}(B, B_0)$, there is a unique orientation-preserving lift $\tilde{g}$ of g to a homeomorphism of the orientable double cover $\widetilde{B}$. The restriction of $\tilde{g}$ to the preimage $\widetilde{B}_0$ is isotopic to the identity. Since Σ is the mapping cylinder of the projection from $\widetilde{B}$ to B, and $\tilde{g}$ commutes with the covering transformation, the homeomorphism $\tilde{g} \times 1_I$ of $\widetilde{B} \times I$ induces a homeomorphism of Σ whose restriction to F is isotopic to the identity. This defines a homomorphism $\mathcal{G}_+(B, B_0) \to \mathcal{G}_+^f(\Sigma, F)$ which is an inverse to the projection.

PROPOSITION 3.2.2. *Suppose that Σ is an I-bundle over B and F is a (possibly empty) 2-manifold in $\partial\Sigma$ which is a union of fibers. Suppose τ is a fiber-preserving involution of (Σ, F), and let $\bar{\tau}$ denote the involution induced on B. Then the subgroup consisting of elements of $\mathcal{G}_+^f(\Sigma, F)$ that commute with $\langle \tau \rangle$ is isomorphic mod finite groups to $\mathcal{H}(B/\bar{\tau}, B_0/\bar{\tau})$.*

Proof. Suppose first that F is nonempty. By Lemma 3.2.1, the projection provides an isomorphism from $\mathcal{G}_+^f(\Sigma, F)$ to $\mathcal{G}_+(B, B_0)$. Since the projection of $h\tau h^{-1}$ is $\overline{h}\bar{\tau}\overline{h^{-1}}$, the subgroup of $\mathcal{G}_+^f(\Sigma, F)$ consisting of the elements that commute with $\langle \tau \rangle$ corresponds under projection to the subgroup of $\mathcal{G}_+(B, B_0)$ consisting of the elements that commute with $\langle \bar{\tau} \rangle$. Since $\mathcal{G}_+(B, B_0)$ has finite index in $\mathcal{H}(B, B_0)$, Proposition 2.2.1 shows that the latter subgroup is isomorphic mod finite groups to $\mathcal{H}(B/\bar{\tau}, B_0/\bar{\tau})$.

We also need a technical result.

LEMMA 3.2.3. *Suppose that Σ is an I-bundle over B and F is the preimage of ∂B in $\partial\Sigma$. Then $\mathcal{H}^f(\Sigma, F) \to \mathcal{H}(\Sigma, F)$ is an isomorphism.*

Proof. This is well-known, although we cannot find this precise statement in the literature. The proof is similar to the analogous result for Seifert-fibered manifolds (injectivity is sketched on p. 85 of [W], and surjectivity is as in Theorem VI.19 of **[J1]**) but is easier. A more general result in the context of boundary patterns may be found in **[J]** (see especially Proposition 5.13 there).

The Seifert-fibered case is more complicated. It will be convenient later on if we allow the homeotopy class to be the identity element. Of course, its centralizer is the entire homeotopy group.

THEOREM 3.2.4. *Let Σ be a Seifert-fibered 3-manifold and let F be a (possibly empty) 2-manifold in $\partial\Sigma$ which is a collection of tori and annuli, each of which is a union of fibers. Let τ be the identity homeomorphism or a fiber-preserving involution of Σ such that $\tau(F) = F$. Let C be the subgroup of elements in $\mathscr{G}(\Sigma, F)$ that commute with the homeotopy class of τ in $\mathscr{H}(\Sigma, F)$. Then (at least) one of the following is true:*

(1) *C is finite.*

(2) *C contains a finitely generated free group of finite index.*

(3) $C \cong \mathrm{GL}(3, \mathbb{Z})$.

(4) *There is an exact sequence $1 \to A \to C \to Q \to 1$, where A is finitely generated abelian with torsion subgroup of order at most 2, and Q is isomorphic mod finite groups to a 2-manifold homeotopy group.*

The proof of this theorem will be given as a sequence of lemmas. First, we consider the "exceptional cases": those Seifert-fibered 3-manifolds which admit homeomorphisms not isotopic to fiber-preserving homeomorphisms. The first lemma mentions the *Hantsche-Wendt manifold*, which is the closed flat 3-manifold given by the Seifert invariants $\{-1; (n_2, 1); (2, 1), (2, 1)\}$ (see **[O**, pp. 133, 138], **[C-V**, pp. 478–481], **[W1]**, **[H-W]**) .

LEMMA 3.2.5. *If Σ is any of the following 3-manifolds, then the centralizer C of the identity element or any involution in $\mathscr{H}(\Sigma)$ is as described below. Thus C is finite or contains a finitely generated free group of finite index unless Σ is the 3-torus and the homeotopy class induces $\pm I$ on $\pi_1(\Sigma)$, in which case C is isomorphic to* $\mathrm{GL}(3, \mathbb{Z})$.

(1) *If $\Sigma = S^1 \times S^1 \times I$, then C is finite or virtually finitely generated free.*

(2) *If Σ is the orientable twisted I-bundle over the Klein bottle, then C is finite.*

(3) *Suppose Σ is an S^1-bundle over the torus. If Σ is the 3-torus, then $C = \mathrm{GL}(3, \mathbb{Z})$ if the homeotopy class induces $\pm I$ on $\pi_1(\Sigma)$,*

otherwise C contains a finitely generated free group of finite index. If Σ is not the 3-torus, then C contains a finitely generated free group of finite index.

(4) *Suppose Σ is an orientable S^1-bundle over the Klein bottle. If the Euler class is zero, then C contains a finitely generated free group of finite index. If the Euler class is nonzero, then C is finite.*

(5) *If Σ is a Seifert-fibered Haken 3-manifold which fibers over S^2 with three exceptional orbits, then C is finite.*

(6) *If Σ is the Hantsche-Wendt manifold, then C is finite.*

Proof. For (1) and (2), Σ is an I-bundle and Proposition 3.2.2 applies. In (1), the quotient surface is a torus, with homeotopy group $\mathrm{GL}(2, \mathbb{Z})$. There are three conjugacy classes of involutions in $\mathrm{GL}(2, \mathbb{Z}) : -I$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$. The first is central, so its centralizer $\mathrm{GL}(2, \mathbb{Z})$ contains a free group of finite index (see for example pp. 100–101 of **[M-K-S]**). The other two involutions have finite centralizers. Statement (2) is immediate, since the homeotopy group of the Klein bottle is finite. For (3), suppose first that the Euler class of the bundle is 0. Then Σ is a 3-torus, with homeotopy group isomorphic to $\mathrm{GL}(3, \mathbb{Z})$, and Lemma 1.3.1 implies that the centralizers are as specified. If the Euler class is nonzero, then by Proposition 3.4.3 of **[M]**, $\mathscr{H}(\Sigma)$ is virtually finitely generated free, and Lemma 1.2.2 applies. For (4), suppose first that the Euler class of the bundle is 0. Then by Proposition 3.4.4 of **[M]**, $\mathscr{H}(\Sigma)$ is virtually finitely generated free, and Lemma 1.2.2 applies. If the Euler class is nonzero, then by Proposition 3.4.4 of **[M]**, $\mathscr{H}(M)$ is finite. For (5) and (6), the homeotopy groups are finite by Proposition 3.4.5 of **[M]** and by **[C-V]**, respectively. This completes the proof of Lemma 3.2.5.

LEMMA 3.2.6. *Let Σ be an irreducible orientable Seifert fibered 3-manifold with infinite fundamental group. Let F be a (possibly empty) 2-manifold in $\partial\Sigma$ which is a union of fibers. If Σ contains an incompressible 2-manifold which is a union of fibers, then the natural homomorphisms $\mathscr{G}_+^f(\Sigma, F) \to \mathscr{G}_+(\Sigma, F)$ and $\mathscr{H}_+^f(\Sigma, F) \to \mathscr{H}_+(\Sigma, F)$ are injective. If either*

(a) *F is nonempty, or*

(b) *Σ does not fiber over the 2-sphere with three exceptional orbits, Σ is not an S^1-bundle over the annulus or Möbius band, Σ is not an S^1-bundle over the torus or Klein bottle which admits a cross section, and Σ is not the Hantsche-Wendt manifold,*

then the natural homomorphism is surjective.

Proof. The proof of injectivity is sketched in [**W**, p. 85]. Surjectivity in case (a) is proved by the argument in [**J1**, Lemma VI.19], and in case (b) it is proved in [**W1**] (see also [**O**, Theorem 8.7]).

The following result is essentially Lemma 25.2 and Proposition 25.3 of [**J**].

LEMMA 3.2.7. *Let Σ be an orientable Seifert-fibered 3-manifold, and F a submanifold of $\partial\Sigma$. Let (B, B_0) be the quotient orbifold of (Σ, F). Then the projection homomorphism $p: \mathscr{G}_+^f(\Sigma, F) \to \mathscr{H}(B, B_0)$ has finitely-generated abelian kernel (possibly trivial) with torsion subgroup of order at most 2. If F is nonempty, then the image is $\mathscr{G}_+(B, B_0)$, and in any case the index of the image is finite.*

Instead of giving a detailed proof, it seems more useful to explain what is going on geometrically in Lemma 3.2.7. The image of $\mathscr{G}^f(\Sigma, F)$ contains all Dehn twists about circles, since these extend to Dehn twists about vertical tori in Σ, and these Dehn twists generate $\mathscr{G}_+(B, B_0)$. When F is nonempty, the fiber direction must be preserved, so the projected homeomorphism must be orientation-preserving, and the image is precisely $\mathscr{G}_+(B, B_0)$. The kernel of p consists of the elements that can be represented by homeomorphism that preserve each fiber—the "vertical" homeomorphisms. Suppose first that the underlying manifold $|B|$ is orientable. Let B_1 be a submanifold of ∂B consisting of all boundary circles that are not entirely contained in B_0. Choose a standard set of generators for $H_1(|B|, |B_1|)$, consisting of a dual pair of simple closed curves for each handle of F, together with a collection of $d-1$ arcs in $B - B_0$ each running between two boundary components neither of which is entirely contained in B_0 (where d is the total number of such boundary components). The preimages in Σ of this set of generators are a collection of vertical tori and annuli, and similarly to pp. 191–193 of [**J**], it can be proved that the Dehn twists about them generate the kernel of p and the isotopy relations among them correspond to the homological relations among the generators. Thus the kernel of p is isomorphic to $H_1(|B|, |B_1|)$, so is finitely generated abelian. The exceptional fibers have no effect here; a vertical Dehn twist about a torus bounding a neighborhood of an exceptional fiber is isotopic, taking each fiber to itself, to the identity. If $|B|$ is nonorientable, then regard $|B|$ as having crosscaps, and instead of a pair of dual curves in the handles, choose the one-sided circles in these crosscaps as homology generators. The preimage in Σ of each one-sided circle is a one-sided Klein bottle K in M. Let $N(K)$ be a fibered neighborhood of K in M. In Lemma 25.1 of [J], Johannson constructs a vertical homeomorphism supported in $N(K)$; on K it is a Dehn twist about the unique (up to isotopy) two-sided nonseparating simple closed curve in K (which is a fiber of the Seifert fibering), and since the lift of this to the orientable double cover of K is isotopic to the identity, this Dehn twist extends to a vertical homeomorphism of $N(K)$ which is the identity on the boundary torus, so extends to M. Johannson's analysis of this homeomorphism (Lemma 25.1 of [J]) shows that its square is isotopic to a vertical Dehn twist about the torus $\partial N(K)$, and thus the correspondence between isotopy of vertical homeotopy classes and homological equivalence of the elements of $H_1(|B|, |B_1|)$ extends to the case when $|B|$ is nonorientable. This completes our explanation of Lemma 3.2.7. A more detailed explanation in a somewhat different context may be found in [M]; a detailed proof can be obtained by modifying pp. 188–195 of [J].

We can now complete the proof of Theorem 3.2.4. Let τ be a fiber-preserving involution of Σ. The class $\langle\tau\rangle$ is an element of $\mathscr{H}^f(\Sigma, F)$. Conjugation by $\langle\tau\rangle$ preserves the normal subgroup $\mathscr{G}^f(\Sigma, F)$, and moreover it preserves the subgroup of vertical homeotopy classes. Let μ denote conjugation by $\langle\tau\rangle$. By Lemma 3.2.7, there is an exact sequence

$$1 \to A \to \mathscr{G}^f(\Sigma, F) \to \mathscr{H}_1(B, B_0) \to 1,$$

where A is finitely generated abelian with torsion subgroup of order at most 2, and $\mathscr{H}_1(B, B_0)$ has finite index in $\mathscr{H}(B, B_0)$. The latter is isomorphic mod finite groups to a 2-manifold homeotopy group, by Lemma 2.2.2. By Lemma 1.2.1, there is a subsequence

$$1 \to A_1 \to \mathrm{fix}(\mu) \to \mathrm{fix}(\overline{\mu})$$

where $\overline{\mu}$ is the induced involution on $\mathscr{H}_1(B, B_0)$ and the image of $\mathrm{fix}(\mu)$ has finite index. By Proposition 2.2.1 and Lemma 2.2.2, $\mathrm{fix}(\overline{\mu})$ is isomorphic mod finite groups to a 2-manifold homeotopy group. Theorem 3.2.4 follows.

3.3. *The case of Haken* 3-*manifolds.* We now consider the general case of M a Haken 3-manifold and τ an involution of M. Let Σ denote a characteristic submanifold for M in the sense of [J].

LEMMA 3.3.1. Σ *can be chosen so that* $\tau(\Sigma) = \Sigma$.

Proof. This follows from Theorem 14 of **[B-S]**, after resolving the minor technical differences between their characteristic toric splitting and Johannson's characteristic submanifold.

We can now give the main result of this section.

THEOREM 3.3.2. *Let M be a Haken 3-manifold and let τ be the identity homeomorphism or an involution of M. Let C be the centralizer of $\langle\tau\rangle$ in $\mathscr{H}(M)$. Then (at least) one of the following is true:*

(1) *C is finite.*

(2) *C contains a finitely generated free group of finite index.*

(3) *M is the 3-torus and $C \cong \mathrm{GL}(3, \mathbb{Z})$.*

(4) *There is an exact sequence $1 \to A \to Z \to Q \to 1$, where Z is a subgroup of finite index in C, A is a finitely generated abelian group (possibly trivial) with torsion subgroup of order at most 2, and Q is isomorphic mod finite groups to a 2-manifold homeotopy group.*

(5) *There is an exact sequence $1 \to D \to Z \to R \to 1$, where Z is a subgroup of finite index in C, D is a finitely generated abelian group (possibly trivial) and R has finite index in a direct product of finitely many groups Z_i which are extensions having the form described in* (4).

Proof. Choose a characteristic submanifold Σ so that $\tau(\Sigma) = \Sigma$. By **[T1]**, for Seifert fibered spaces, and by **[M-S]**, for I-bundles, we may choose the fibered structure on the components of Σ so that the restriction of τ to each τ-invariant component is fiber-preserving. For pairs of components that are interchanged, we can simply transfer the fibered structure on one of the components to the other using τ, to make τ fiber-preserving. From now on, we will assume that the fibered structure on Σ is τ-invariant.

If M fibers over the circle with torus fiber and attaching homomorphism whose trace, as a matrix in $\mathrm{GL}(2, \mathbb{Z})$, has absolute value at least 3 (i.e. if M admits a *Sol* structure) then by Proposition 4.1.2 of **[M]**, $\mathscr{H}(M)$ is finite, and Theorem 3.3.2 holds. Assume from now on that M is not such a manifold; then, by Proposition 4.1.1 of **[M]**, the natural homomorphism $\mathscr{H}(M, \Sigma) \to \mathscr{H}(M)$ is an isomorphism.

LEMMA 3.3.3. *Assume that M is not a torus bundle over S^1 which admits a Sol structure. Then the image of the homomorphism $\mathscr{H}(M, \Sigma) \to \mathscr{H}(\overline{M - \Sigma}, \mathrm{Fr}(\Sigma))$ induced by restriction is finite.*

Proof. By the argument in **[J**, Corollary 27.6**]**, the subgroup of $\mathscr{H}(M, \Sigma)$ generated by Dehn twists about admissible essential tori and annuli in Σ has finite index. Thus, there is a subgroup of finite index in $\mathscr{H}(M, \Sigma)$ which can be represented by homeomorphisms which are the identity on $\overline{M - \Sigma}$. The lemma follows.

Notice in particular that when Σ is empty, the restriction in Lemma 3.3.3 is the identity, so $\mathscr{H}(M)$ is finite and Theorem 3.3.2 is proved for this case. When M is fibered (i.e. when $M = \Sigma$), Theorem 3.3.2 has been proved in §3.2. So for the remainder of §3.3, we will *assume that the characteristic submanifold* Σ *is not empty and is not equal to* M. In particular, the frontier F_i of each component Σ_i of Σ is nonempty.

Define $\mathscr{K}(M, \Sigma)$ to be the kernel of the homomorphism in Lemma 3.3.3. Recall the groups $\mathscr{G}^f(\Sigma_i, F_i)$ defined at the beginning of §3.2.

LEMMA 3.3.4. *There is a surjective homomorphism*

$$\phi : \mathscr{K}(M, \Sigma) \to \prod \mathscr{G}^f(\Sigma_i, F_i)$$

whose kernel is the finitely generated abelian subgroup $\mathscr{D}(M, \Sigma)$ *generated by Dehn twists about the components of the frontier of* Σ.

Proof. Since the frontier of each Σ_i is nonempty, the argument of [**J1**, Lemma VI.19] shows that each element of $\mathscr{K}(M, \Sigma)$ is representable by a homeomorphism whose restriction to each (Σ_i, F_i) is fiber-preserving. Using Lemmas 3.2.3 and 3.2.6, this fiber-preserving homeomorphism is unique up to fiber-preserving isotopy (preserving F). Therefore the restriction homomorphism ϕ is well-defined. Since the elements of $\mathscr{G}^f(\Sigma_i, F_i)$ have representatives which are the identity on the frontier of Σ_i, ϕ is surjective. Any element of the kernel of ϕ is isotopic to a homeomorphism which is the identity outside a neighborhood of the components of the frontier of Σ, giving the description of $\mathscr{D}(M, \Sigma)$.

We can now complete the proof of Theorem 3.3.2. Let $S = \overline{M - \Sigma}$. Consider the disjoint union $(\coprod \Sigma_i) \coprod S$. There is a restriction homomorphism $\rho : \mathscr{H}(M, \Sigma) \to \mathscr{H}((\coprod \Sigma_i) \coprod S)$. We have a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathscr{D}(M, \Sigma) & \longrightarrow & \mathscr{K}(M, \Sigma) & \xrightarrow{\phi} & \prod \mathscr{G}^f(\Sigma_i, F_i) & \longrightarrow & 1 \\
 & & \downarrow = & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathscr{D}(M, \Sigma) & \longrightarrow & \mathscr{H}(M, \Sigma) & \xrightarrow{\rho} & \operatorname{im}(\rho) & \longrightarrow & 1
\end{array}$$

In this diagram, the homomorphism from $\mathscr{K}(M, \Sigma)$ to $\mathscr{H}(M, \Sigma)$ is inclusion, and that from $\prod \mathscr{G}^f(\Sigma_i, F_i)$ to $\operatorname{im}(\rho)$ is induced by extending representatives by the identity on S. These are injective, and since the image of $\mathscr{H}(M, \Sigma)$ has finite index, so does the image of $\prod \mathscr{G}^f(\Sigma_i, F_i)$. Notice that conjugation by $\langle\tau\rangle$ induces an automorphism of this entire diagram. We denote the automorphism induced in $\prod \mathscr{G}^f(\Sigma_i, F_i)$ by μ.

Rename the components of Σ as $U_1, U_2, \ldots, U_r, V_1, V_2, \ldots, V_r, W_1, W_2, \ldots, W_s$, where $\tau(U_i) = V_i$ and $\tau(W_j) = W_j$. Put $\rho_i = \langle\tau|_{U_i}\rangle$ and $\sigma_j = \langle\tau|_{W_j}\rangle$. Suppose that

$$(f_1, \ldots, f_r, g_1, \ldots, g_r, h_1, \ldots, h_s) \in \left(\prod \mathscr{G}^f(U_i, \operatorname{Fr}(U_i))\right) \times \left(\prod \mathscr{G}^f(V_i, \operatorname{Fr}(V_i))\right) \times \left(\prod \mathscr{G}^f(W_j, \operatorname{Fr}(W_j))\right).$$

Observe that this element is fixed by μ if and only if $g_i = \rho_i f_i \rho_i^{-1}$ and $h_j = \sigma_j h_j \sigma_j^{-1}$. Thus the subgroup of fixed elements in $\prod \mathscr{G}^f(\Sigma_i, F_i)$ is isomorphic to

$$\left(\prod_{i=1}^{r} \mathscr{G}^f(U_i, \operatorname{Fr}(U_i))\right) \times \left(\prod_{j=1}^{s} \operatorname{Cent}_{\mathscr{H}(W_j, \operatorname{Fr}(W_j))}(\langle\sigma_j\rangle) \cap \mathscr{G}^f(W_j, \operatorname{Fr}(W_j))\right).$$

If U_i is an I-bundle, then Proposition 3.2.2 shows that $\mathscr{G}^f(U_i, \operatorname{Fr}(U_i))$ is isomorphic mod finite groups to a 2-manifold homeotopy group, while if U_i is Seifert fibered, Lemma 3.2.7 shows it is of the form specified in part (4) of Theorem 3.3.2. If W_j is an I-bundle, then Proposition 2.2.1 and Lemma 2.2.2 show that $\operatorname{Cent}_{\mathscr{H}(W_j, \operatorname{Fr}(W_j))}(\langle\sigma_j\rangle) \cap \mathscr{G}^f(W_j, \operatorname{Fr}(W_j))$ is isomorphic mod finite groups to a 2-manifold homeotopy group, while if W_j is Seifert-fibered then Theorem 3.2.4 shows it has one of the forms listed in Theorem 3.3.2. Applying Lemma 1.2.1 to the top row of the diagram, we obtain an exact sequence

$$1 \to D \to \mathscr{H}(M, \Sigma) \cap \operatorname{Cent}_{\mathscr{H}(M, \Sigma)}(\langle\tau\rangle) \to Q \to 1$$

where D is abelian and Q has finite index in the subgroup of elements of $\prod \mathscr{G}^f(\Sigma_i, F_i)$ fixed by μ. This completes the proof of Theorem 3.3.2.

4. Homeotopy groups of 3-manifolds.

4.1. *The characteristic* $\mathbb{P}^2$ *splitting.* A loop in $SO(3, \mathbb{R}) \subseteq \mathrm{Diff}(S^2)$, based at the identity and regarded as a diffeomorphism of $S^2 \times I$, induces a diffeomorphism of $\mathbb{P}^2 \times I$ which represents an element of $\mathscr{H}(\mathbb{P}^2 \times I \operatorname{rel} \mathbb{P}^2 \times \partial I)$. Such an element is called a *rotation* about $\mathbb{P}^2 \times \{0\}$. It is known that $\pi_1(SO(3, \mathbb{R}), 1_{\mathbb{R}}) \cong \mathbb{Z}/2$, generated by the loop that rotates once about a fixed axis (in fact, $SO(3, \mathbb{R})$ is diffeomorphic to $\mathbb{P}^3$), so the square of a rotation is trivial, and any two rotations defined using a noncontractible loop in $SO(3, \mathbb{R})$ are isotopic. In Lemma 4.1.1 below we will calculate that $\mathscr{H}(\mathbb{P}^2 \times I \operatorname{rel} \mathbb{P}^2 \times \partial I) \cong \mathbb{Z}/2$ generated by a nontrivial rotation. If P is a 2-sided projective plane in a 3-manifold N, contained in or disjoint from ∂N, then a rotation about P defined on a collar of P extends using the identity to a diffeomorphism of N, again called a rotation about P.

LEMMA 4.1.1. *The relative homeotopy group* $\mathscr{H}(\mathbb{P}^2 \times I \operatorname{rel} \mathbb{P}^2 \times \partial I)$ *is of order* 2, *generated by a rotation about* $\mathbb{P}^2 \times \{0\}$.

Proof. First we show that $\mathscr{H}(\mathbb{P}^2 \times I \operatorname{rel} \mathbb{P}^2 \times \partial I)$ is generated by a rotation. Let h be a homeomorphism of $\mathbb{P}^2 \times I$ which is the identity on $\mathbb{P}^2 \times \partial I$. Let C be a 1-sided simple closed curve in $\mathbb{P}^2$, and let A be the annulus $C \times I$. Fix a rotation r that leaves C invariant in each $\mathbb{P}^2 \times \{s\}$. Let p be a point on C, and let a be the arc $p \times I$. We may deform $h(\operatorname{rel} \partial I)$ so that in a neighborhood of $\mathbb{P}^2 \times \partial I$, $h(A) \cap A$ is contained in a. Now deform h, fixing a neighborhood of $\mathbb{P}^2 \times \partial I$, so that $h(C \times (0, 1))$ is transverse to A. The intersections will be a single arc and a collection of simple closed curves. By irreducibility, these simple closed curves may be eliminated by isotopy. Now $h(A) \cap A$ winds some number of times around A, compared to a; changing h by some r^k we may assume that $h(a) = a$ and in fact that h restricts to the identity on a. Deform h to preserve a tubular neighborhood $W = D^2 \times I$ of a. It twists the D^2-factor of this neighborhood some number of times. Let r_1 be a nontrivial rotation which preserves a; since $\pi_1(SO(3, \mathbb{R})) \cong \mathbb{Z}/2$, r_1 is isotopic to r. Changing h by r_1^l, for some l, we may assume that h is the identity on this neighborhood. The complement of $(\mathbb{P}^2 \times \partial I) \cup W$ is a solid Klein bottle K. Deform h to be the identity on a meridional 2-disc of K, then use the Alexander trick to complete the isotopy from h (changed by $\langle r^{k+l} \rangle$) to the identity. Since r^2 is isotopic to the identity relative to $\mathbb{P}^2 \times \partial I$, we can obtain an isotopy from h to either r or the identity. By **[H4]**, r is nontrivial, proving Lemma 4.1.1.

Let N be a compact irreducible 3-manifold. Consider a subspace $\mathscr{P} = \bigcup_{i=1}^{n} P_i$ where $\{P_1, \dots, P_n\}$ is a collection of disjoint imbedded nonparallel non-boundary-parallel 2-sided projective planes in N. By the Haken-Kneser Finiteness Theorem (see Theorem III.24 of **[J1]** for a general version) there is an upper bound for the size of such a collection.

PROPOSITION 4.1.2. *Let N be a compact irreducible 3-manifold which contains no fake $\mathbb{P}^2 \times I$. Let $\mathscr{P} = \bigcup_{i=1}^{n} P_i$ where $\{P_1, \dots, P_n\}$ is a maximal collection of disjoint nonparallel non-boundary-parallel 2-sided projective planes imbedded in the interior of N. Then the inclusion* $\mathrm{Diff}(N, \mathscr{P}) \to \mathrm{Diff}(N)$ *induces an isomorphism $\mathscr{H}(N, \mathscr{P}) \cong \mathscr{H}(N)$.*

Proof. Using Theorem 1 of **[N]**, the collection $\mathscr{P}$ is unique up to ambient isotopy. Consequently, the image of $\mathscr{P}$ under any diffeomorphism of N is isotopic to $\mathscr{P}$, and therefore $\mathrm{Diff}(N, \mathscr{P}) \to \mathrm{Diff}(N)$ is surjective on path components.

To prove injectivity, we will use the following consequence of Hatcher's method from **[H1]**.

LEMMA 4.1.3. *Let $\mathbb{P}^2 \times I$ be imbedded in the interior of an irreducible 3-manifold N, and let i_0 denote the inclusion of $\mathbb{P}^2$ to $\mathbb{P}^2 \times \{0\}$. Let P be a copy of $\mathbb{P}^2$, and let* $\mathrm{Imb}_0(P, N)$ *denote the space of smooth imbeddings of P into N which are isotopic to i_0. Let* $\mathrm{Imb}'_0(P, N)$ *denote the subspace of* $\mathrm{Imb}_0(P, N)$ *consisting of the imbeddings f with the property that $f(P)$ is disjoint from some $\mathbb{P}^2 \times \{s\}$ for some s (depending on f). Then the inclusion map* $\mathrm{Imb}'_0(P, N) \to \mathrm{Imb}_0(P, N)$ *is a homotopy equivalence.*

Proof. The argument is exactly as in **[H1]**, but is easier since for each of the isotopies there is only one 3-ball to push across, so one can ignore the steps involving the basepoints called p_t in **[H1]**.

We must be a bit careful in applying Hatcher's lemma, though, since to our knowledge, results analogous to statements (a) and (b) on the last page of **[H1]** have not been proved for $\mathbb{P}^2$.

SUBLEMMA 4.1.4. *Let $\mathbb{P}^2 \times I$ be imbedded in the interior of a 3-manifold N, and let i_s denote the inclusion of $\mathbb{P}^2$ to $\mathbb{P}^2 \times \{s\}$. Let P be a copy of $\mathbb{P}^2$, and let* $\mathrm{Imb}_1(P \times I, N \times I)$ *denote the space of smooth imbeddings j of $P \times I$ into $N \times I$ which are level preserving (i.e. the restriction j_t of j to $P \times \{t\}$ has image in $N \times \{t\}$), and* *such that* $j_0 = i_0$ *and* $j_t(P)$ *is disjoint from* $\mathbb{P}^2 \times \{1\}$ *for all* t. *Then* $\mathrm{Imb}_1(P \times I, N \times I)$ *is path connected. Moreover, if* $j_1(P)$ *lies in* $(\mathbb{P}^2 \times I) \times \{1\}$ *and the trace of the isotopy is homotopic into* $\mathbb{P}^2 \times I$, *then there is a path from* j *to the constant imbedding* H *defined by* $H_t = i_0$, *such that each imbedding on the path maps* $P \times \{1\}$ *into an arbitrarily small neighborhood of* $(\mathbb{P}^2 \times I) \times \{1\}$.

Proof. Let J be the constant imbedding $J_t = i_1$. Given $j \in \mathrm{Imb}_1(P \times I, N \times I)$, regard $j \cup J$ as a one-parameter family of imbeddings of the subspace $\mathbb{P}^2 \times \{0\} \cup \mathbb{P}^2 \times \{1\}$ into N. By parameterized isotopy extension, there is an isotopy K_t of N, with K_0 equal to the identity of N, which extends $j \cup J$. Let L_s be the restriction of K_t to $\mathbb{P}^2 \times \{s\}$, so that $L_0 = j$ and $L_1 = J$. Let h_s be a one-parameter family of diffeomorphisms of N with h_0 equal to the identity and $h_s(x, s) = (x, 0)$ for $(x, s) \in \mathbb{P}^2 \times \{s\} \subseteq \mathbb{P}^2 \times I$, and such that h_s is the identity outside an arbitrarily small given neighborhood of $\mathbb{P}^2 \times I$. Then $h_s \circ L_s$ is a path in $\mathrm{Imb}_1(P \times I, N \times I)$ from j to H. The trace condition in the last sentence of the Sublemma implies that $L((\mathbb{P}^2 \times I) \times \{1\}) \subseteq (\mathbb{P}^2 \times I) \times \{1\}$, so the last sentence follows.

We can now complete the proof of Proposition 4.1.2. Suppose h is a homeomorphism of N, preserving $\mathscr{P}$, which is isotopic to the identity. Since the projective planes in $\mathscr{P}$ are pairwise nonisotopic, this implies that h takes each projective plane in $\mathscr{P}$ to itself. Changing h by isotopy in a neighborhood of the boundary of N, we may assume that the isotopy h_t from $h = h_0$ to the identity fixes the boundary. By induction, it suffices to consider a single projective plane P_1 with $h(P_1) = P_1$, and to show that h is isotopic to the identity by an isotopy which fixes the boundary and takes P_1 to P_1 at each level. We may assume that h fixes a basepoint in P_1. If h reverses the local orientation at that basepoint, we may change it by an isotopy that moves the basepoint around an orientation-reversing loop in P_1; thus we may assume that h preserves the local orientation.

We will show that the isotopy may be chosen so that its trace at a point in P_1 is homotopic into P_1. Since h preserves the local orientation at the basepoint, the trace of the isotopy lifts to the orientable double cover. The orientable double covering of N is a connected sum of aspherical 2-manifolds and $S^2 \times S^1$'s, so it has torsion-free fundamental group. Therefore if the trace is nontrivial, it has infinite order. Since $h(P_1) = P_1$, the trace of the isotopy lies in the centralizer of $\pi_1(P_1)$ in $\pi_1(N)$. Applying Corollary 4.2 of [S4] shows that N

must be a homotopy $\mathbb{P}^2 \times S^1$. Since we are assuming that N contains no fake $\mathbb{P}^2 \times I$, this implies that $N = \mathbb{P}^2 \times S^1$. In this case there is an isotopy from the identity to the identity, which moves $\mathbb{P}^2$ around the S^1 factor. Changing our isotopy by some multiple of this one, we may assume the trace of the isotopy at a basepoint in P_1 is homotopic into P_1.

Let $\mathbb{P}^2 \times I$ be a collar neighborhood of $P_1 = \mathbb{P}^2 \times \{\frac{1}{2}\}$. By Lemma 4.1.3, we may assume that each $h_t(P_1)$ is disjoint from some $\mathbb{P}^2 \times \{s_t\}$. By compactness, there are finitely many intervals $I_1, I_2, \ldots, I_n$ which cover I, and corresponding s-values $s_1, s_2, \ldots, s_n$ such that $h_1(P_1)$ is disjoint from $\mathbb{P}^2 \times \{s_i\}$ for all $t \in I_i$. Proceeding inductively using Sublemma 4.1.4, we may deform the isotopy to one which takes $P_1 = \mathbb{P}^2 \times \{\frac{1}{2}\}$ to itself at all times, although this isotopy will no longer end at the identity. By the last sentence of Sublemma 4.1.4, we may choose the end level of the isotopy to be the identity outside $\mathbb{P}^2 \times I$. Applying Lemma 4.1.1. shows that this end level is isotopic preserving P_1 to either the identity or a rotation about P_1. Suppose the latter. Then that rotation is isotopic to the identity of N, fixing the boundary. Its orientation-preserving lift to the orientable double cover of N is a rotation about a 2-sphere S, which must also be isotopic to the identity, fixing the boundary. By **[H3]**, **[H4]**, the fact that it is homotopic to the identity implies that S bounds a connected sum of $S^2 \times S^1$'s and (certain kinds of) 3-manifolds with finite fundamental group. But this implies that one of the complementary components of P_1 has no boundary, which is impossible since P_1 cannot be null-cobordant. This completes the proof of Proposition 4.1.2.

COROLLARY 4.1.5. *Let N be a compact, irreducible 3-manifold, and let $\mathscr{P} = \bigcup_{i=1}^n P_i$ be a maximal collection of disjoint nonparallel non-boundary-parallel 2-sided projective planes in the interior of N. Let $N_1, N_2, \ldots, N_k$ be the components that result from cutting N along $\mathscr{P}$, and let $\coprod_{j=1}^k N_j$ be their disjoint union. Then there is an exact sequence*

$$1 \to \mathscr{R} \to \mathscr{H}(N) \to \mathscr{H}\left(\coprod_{j=1}^{k} N_j\right)$$

in which $\mathscr{R}$ is the finite subgroup generated by rotations about the P_i, and the image of $\mathscr{H}(N)$ in $\mathscr{H}(\coprod_{j=1}^k N_j)$ has finite index.

Proof. By Proposition 4.1.2, the homomorphism $\mathscr{H}(N, \mathscr{P}) \to \mathscr{H}(N)$ induced by inclusion is an isomorphism. The restriction map

$\mathscr{H}(N, \mathscr{P}) \to \prod_{j=1}^{k} \mathscr{H}(N_j)$ has kernel consisting of homeotopy classes containing representatives which are the identity outside a small neighborhood of $\mathscr{P}$. Using Lemma 4.1.1, these homeomorphisms are isotopic to products of rotations about the P_i. On the other hand, all such rotations are in the kernel. The subgroup of $\mathscr{H}(\coprod_{j=1}^{k} N_j)$ consisting of the homeotopy classes that permute the N_j and their projective plane boundary components trivially has finite index, and since any homeomorphism of $\mathbb{P}^2$ is isotopic to the identity, this subgroup is in the image of $\mathscr{H}(N, \mathscr{P})$. This completes the proof.

4.2. *Sufficiently large irreducible* 3-*manifolds.* Let M be an irreducible 3-manifold which contains no fake $\mathbb{P}^2 \times I$. Following **[S3]**, define a *hierarchy* for M to be a sequence of pairs (M_j, F_j), $1 \leq j \leq n$, such that

(1) $M_1 = M$.

(2) Each F_j is a 2-sided incompressible surface in M_j.

(3) M_{j+1} is the closure of the complement of a regular neighborhood of F_j in M_j.

(4) Each component of M_n is either a 3-cell or is homeomorphic to $\mathbb{P}^2 \times I$. If M has a hierarchy, then M is said to be *sufficiently large*.

By **[W]**, **[H2]**, and **[S3]**, M is sufficiently large if it satisfies any of the following conditions:

(a) M contains no 2-sided projective plane and M contains a 2-sided incompressible surface $F_1 \neq S^2$.

(b) M contains no 2-sided projective plane and M has nonempty boundary.

(c) ∂M is incompressible, every projective plane in M is parallel into ∂M, and M contains a 2-sided incompressible surface $F_1 \neq S^2, \mathbb{P}^2$.

(d) ∂M is incompressible, every projective plane in M is parallel into ∂M, and $H_1(M; \mathbb{Z})$ is infinite.

(e) ∂M is incompressible, every projective plane in M is parallel into ∂M, and ∂M contains at least 4 projective planes.

We also have

COROLLARY 4.2.1. *Let N be a compact, irreducible* 3-*manifold, and let $\mathscr{P} = \bigcup_{i=1}^{n} P_i$ be a maximal collection of disjoint nonparallel nonboundary-parallel* 2-*sided projective planes in the interior of N. Let $N_1, N_2, \ldots, N_k$ be the components that result from cutting N along $\mathscr{P}$. Then N is sufficiently large if and only if each N_i is sufficiently large.*

Proof. If each N_i has a hierarchy, then so does N. On the other hand, suppose N does. Since each P_i is 2-sided and incompressible and N is irreducible, any incompressible surface in N is isotopic into the complement of $\mathscr{P}$. So one may assume that F_1 lies in some N_i. Induction on the hierarchy length now shows that all the N_i are sufficiently large.

For a compact 3-manifold V, define $\widehat{V}$ to be the 3-orbifold obtained by coning off each projective plane boundary component.

PROPOSITION 4.2.2. *The homomorphism* $\mathscr{H}(V) \to \mathscr{H}(\widehat{V})$ *induced by coning is an isomorphism.*

Proof. This is easy using the fact that any projective plane P in the interior of $\mathbb{P}^2 \times I$ is isotopic to $\mathbb{P}^2 \times \{\frac{1}{2}\}$ (see Lemma 1.1. of [**N**]).

We can now state our main result.

THEOREM 4.2.3. *Let* N *be a sufficiently large irreducible* 3-*manifold with incompressible boundary. Then* $\mathscr{H}(N)$ *is isomorphic mod finite groups to a direct product of finitely many groups* Z_i, *each of which satisfies* (*at least*) *one of the following conditions.*

(1) Z_i *is finite.*

(2) Z_i *contains a finitely generated free group of finite index.*

(3) Z_i *is isomorphic mod finite groups to* $\mathrm{GL}(3, \mathbb{Z})$.

(4) *There is an exact sequence* $1 \to A \to Z_i \to Q \to 1$, *where* A *is a finitely generated abelian group* (*possibly trivial*) *with torsion subgroup of order at most* 2, *and* Q *is isomorphic mod finite groups to a* 2-*manifold homeotopy group.*

(5) *There is an exact sequence* $1 \to D \to Z_i \to R \to 1$, *where* D *is a finitely generated abelian group* (*possibly trivial*) *and* R *has finite index in a direct product of finitely many groups which are extensions having the form described in* (4).

Proof. If N is orientable, then the theorem follows by taking τ to be the identity homeomorphism in Theorem 3.3.2. So we may assume that N is nonorientable. If N contains any 2-sided projective planes, then by Corollary 4.1.5, we may assume that they are parallel into the boundary. By Proposition 4.2.2, $\mathscr{H}(N)$ is isomorphic to the orbifold homeotopy group $\mathscr{H}(\widehat{N})$. There is a 2-fold covering of $\widehat{N}$ by an orientable manifold M. Since N is sufficiently large, M is Haken.

By Theorem 3.1.1, $\mathscr{H}(\widehat{N})$ is isomorphic mod finite groups to the centralizer of the covering involution in $\mathscr{H}(M)$. Now apply Theorem 3.3.2.

COROLLARY 4.2.4. *Let N be a sufficiently large irreducible 3-manifold with incompressible boundary. Then $\mathscr{H}(N)$ is finitely presented.*

REFERENCES

[B-H] J. Birman and H. Hilden, *On isotopies of homeomorphisms of Riemann surfaces*, Ann. of Math., **97** (1973), 424–439.

[B] F. Bonahon, Involutions et fibrés de Seifert dans les variétés de dimension 3, Thèse de 3e cycle, Orsay, 1979.

[B-S] F. Bonahon and L. Siebenmann, *The characteristic toric splitting of irreducible compact 3-orbifolds*, Math. Ann., **278** (1987), 441–479.

[B-Z] M. Boileau and B. Zimmermann, *The π-orbifold group of a link*, Math. Z., **200** (1989), 187–208.

[C-R] P. Conner and F. Raymond, *Deforming homotopy equivalences to homeomorphisms in aspherical manifolds*, Bull. Amer. Math. Soc., **83** (1977), 36–85.

[C-V] L. S. Charlap and A. T. Vasquez, *Compact flat Riemannian manifolds* III; *the group of affinities*, Amer. J. Math., **95** (1973), 471–494.

[D-M] M. Davis and J. Morgan, *Finite group actions on homotopy 3-spheres*, in *The Smith Conjecture*, ed. J. Morgan and H. Bass, Academic Press, Orlando (1984), 181–226.

[D-S] J. Dyer and P. Scott, *Periodic automorphisms of free groups*, Comm. Algebra, **3** (3) (1975), 195–201.

[H-W] W. Hantsche and W. Wendt, *Drei dimensionale Euklidische Raumformen*, Math. Ann., **110** (1934), 593–611.

[H] J. Harer, *The cohomology of the moduli space of curves*, in *Theory of Moduli*, ed. E. Sernesi, Springer-Verlag Lecture Notes in Math., vol. 1337, 1988, pp. 138–221.

[H1] A. Hatcher, *On the diffeomorphism group of* $S^1 \times S^2$, Proc. Amer. Math. Soc., **83** (1981), 427–430.

[H2] W. Heil, *On* $\mathbb{P}^2$*-irreducible 3-manifolds*, Bull. Amer. Math. Soc., **75** (1969), 772–775.

[H3] H. Hendriks, *Applications de la théorie d'obstruction en dimension 3*, Bull. Soc. Math. France Mémoire, **53** (1977), 81–196.

[H4] ——, *Obstruction theory in 3-dimensional topology: classification theorems*, Bull. Amer. Math. Soc., **83** (1977), 737–738.

[J1] W. Jaco, *Lectures on 3-Manifold Topology*, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics No. 43 (1980).

[J] K. Johannson, *Homotopy Equivalences of 3-Manifolds with Boundaries*, Springer-Verlag Lecture Notes in Math., vol. 761, 1979.

[K] J. Kalliongis, *Realizing automorphisms of the fundamental group of irreducible 3-manifolds containing two-sided projective planes*, preprint.

[L] F. Laudenbach, *Topologie de la dimension trois. Homotopie et isotopie*, Astérisque, **12** (1974), 1–152.

[M1] S. Mac Lane, *Homology*, Springer-Verlag, Berlin, 1967.

[M-K-S] W. Magnus, A. Karrass, and D. Solitar, *Combinatorial Group Theory*, Dover, New York, 1976.

[M] D. McCullough, *Virtually geometrically finite mapping class groups of* 3-*manifolds*, J. Differential Geometry, **33** (1991), 1–65.

[M-S] W. Meeks and P. Scott, *Finite group actions on* 3-*manifolds*, Invent. Math., **86** (1986), 287–346.

[N] S. Negami, *Irreducible* 3-*manifolds with non-trivial* π_2, Yokohama Math. J., **29** (1981), 133–144.

[O] P. Orlik, *Seifert Manifolds*, Springer-Verlag Lecture Notes in Math., vol. 291, 1972.

[S1] P. Scott, *The geometries of* 3-*manifolds*, Bull. London Math. Soc., **15** (1983), 401–487.

[S2] E. Spanier, *Algebraic Topology*, McGraw-Hill, 1966.

[S3] G. A. Swarup, *Homeomorphisms of compact* 3-*manifolds*, Topology, **16** (1977), 119–130.

[S4] ———, *Projective planes in irreducible* 3-*manifolds*, Math. Z., **132** (1973), 305–317.

[T2] W. Thurston, *The geometry and topology of* 3-*manifolds*, mimeographed notes, Princeton University.

[T] J. Tollefson, *Involutions of sufficiently large* 3-*manifolds*, Topology, **20** (1981), 323–352.

[T1] ———, *Involutions of Seifert fiber spaces*, Pacific J. Math., **71** (1978), 519–529.

[W1] F. Waldhausen, *Eine Klasse von* 3-*dimensionale Mannigfaltigkeiten*, I, II, Invent. Math., **3** (1967), 308–333; **4** (1967), 87–117.

[W] ———, *On irreducible* 3-*manifolds which are sufficiently large*, Ann. of Math., **87** (1968), 56–88.

[Z1] H. Zieschang, *On the homeotopy groups of surfaces*, Math. Ann., **206** (1973), 1–21.

[Z2] B. Zimmermann, *Isotopies of Haken* 3-*orbifolds*, Quart. J. Math., (2) **40** (1989), 371–376.

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