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**A REMARK ON THE SYMMETRY OF SOLUTIONS TO
NONLINEAR ELLIPTIC EQUATIONS**

Ji Min

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This note gives a necessary and sufficient condition for solutions of second order elliptic equations to be radially symmetric.

1. Introduction.

1.1. In an elegant paper [GNN], Gidas-Ni-Nirenberg proved that the positive solutions of

$$(1) \quad \begin{cases} \Delta u = f(u) & \text{in } B, \\ u = 0 & \text{on } \partial B, \\ u \in C^2(\bar{B}), \end{cases}$$

must be radially symmetric. Here f is C^1 and B is the n -dimensional ball: $\{x \in R^n; |x| < 1\}$. Obviously a symmetric solution of (1) is not necessary to be positive. In this note, we give a necessary and sufficient condition for symmetric solutions of (1). The main result is the following

THEOREM 1. *Suppose $n \geq 2$. A solution u of (1) is radially symmetric if and only if its nodal set $\{x \in \bar{B}; u(x) = 0\}$ is radially symmetric.*

REMARK. It is interesting to note that Theorem 1 need not hold in case $n = 1$. For, $u = \sin x$ solves

$$u'' = -u \quad \text{in } [-\pi, \pi]$$

with the symmetric nodal set $\{0\} \cup \{-\pi, \pi\}$, but u is not radially symmetric since $\sin(-x) = -\sin x$.

It is clear that the result of [GNN] is a special case of Theorem 1 since the nodal set of a positive solution to (1) is the sphere ∂B .

In order to prove Theorem 1, we need the following two preliminary results.

THEOREM 2. *Let $u \in C^2(\bar{B})$ satisfy*

$$(2) \quad \Delta u = f(u) \quad \text{in } B.$$

If the nodal set of u consists of spheres with the center 0, then these spheres must be isolated unless $u \equiv 0$.

THEOREM 3. Let $n \geq 2$ and $u \in C^2(\bar{B})$ satisfy

$$(3) \quad \begin{cases} \Delta u = f(u) & \text{in } B, \\ u > 0 & \text{in } B \setminus \{0\}, \\ u = 0 & \text{on } \partial B. \end{cases}$$

Then $u > 0$ in B .

REMARK. In case $n = 1$, Theorem 3 need not hold. For example, let $u(x) = \sin(x - \frac{\pi}{2}) + 1$ for $x \in [-2\pi, 2\pi]$, we have

$$\begin{cases} u'' = 1 - u & \text{in } (-2\pi, 2\pi), \\ u > 0 & \text{in } (-2\pi, 2\pi) \setminus \{0\}, \\ u = 0 & \text{at } x = 0, -2\pi, 2\pi. \end{cases}$$

1.2. The proof of Theorem 3 is based on Lemma 12.1 in [GNN], we rewrite it in the form.

LEMMA A. Let $p = (p^1, p^2, \dots, p^n) \in \partial B$ with $p^1 > 0$. Assume for some $\varepsilon > 0$ that u is a C^2 function satisfying equation (2) in $\bar{\Omega}_\varepsilon$ where $\Omega_\varepsilon = B \cap \{x; |x - p| < \varepsilon\}$, $u > 0$ in $\bar{\Omega}_\varepsilon \setminus \partial B \cap \{x; |x - p| < \varepsilon\}$ and $u = 0$ on $\partial B \cap \{x; |x - p| < \varepsilon\}$. Then there exists $\delta > 0$ such that in $B \cap \{x; |x - p| < \delta\}$, $\frac{\partial u}{\partial x_1} < 0$.

2. Proofs.

2.1. *Proof of Theorem 2.* We may assume that the nodal set of u is $\bigcup_{\lambda \in \Lambda} S(\lambda)$ where $\Lambda \subset [0, 1]$ and $S(\lambda) = \{x \in R^n; |x| = \lambda\}$. It needs to be proved that the set Λ contains only isolated points unless $u \equiv 0$. Suppose that there is a sequence $\{\lambda_i\} \subset \Lambda$ with $\lambda_i \rightarrow \bar{\lambda}$. Using the polar coordinates $x = r\xi$ where $\xi \in S^{n-1}$ and $r^2 = x_1^2 + x_2^2 + \dots + x_n^2$, we obtain that $u = \frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial r^2} = 0$ for $r = \bar{\lambda}$, which implies that

$$u(0) = \frac{\partial u}{\partial x_i}(0) = \frac{\partial^2 u}{\partial x_l^2}(0) = 0 \quad (l = 1, 2, \dots, n)$$

when $\bar{\lambda} = 0$, and that $u = D_\xi u = D_\xi^2 u = 0$ on $S(\bar{\lambda})$ when $\bar{\lambda} > 0$. Thus, in both cases, $u = \Delta u = 0$ on $S(\bar{\lambda})$, and, from (2) we conclude that $f(0) = 0$. Set

$$c(x) = \int_0^1 f'(tu(x)) dt.$$

In case $\bar{\lambda} > 0$, we have

$$\begin{cases} \Delta u - c(x)u = 0 & \text{in } \{x; |x| < \bar{\lambda}\}, \\ u = \frac{\partial u}{\partial r} = 0 & \text{on } S(\bar{\lambda}), \end{cases}$$

and obtain $u = 0$ in B by uniqueness of solutions to Cauchy's problem of linear elliptic equations. Now it remains to consider the case $\bar{\lambda} = 0$. Set

$$w(x) = \cos Nx_1 \cdot \cos Nx_2 \cdot \dots \cdot \cos Nx_n,$$

where N is taken to be large enough so that

$$(4) \quad c(x) + N^2 \geq 0.$$

Put $u = w \cdot v$ for $|x| < \frac{\pi}{2N}$. It is easy to see that

$$\begin{cases} \Delta w = -N^2 w \\ w > 0 \end{cases} \quad \text{in } \left\{x; |x| < \frac{\pi}{2N}\right\}$$

and $S(\lambda_i) \subset \{x; |x| < \frac{\pi}{2N}\}$ for i large enough since $\lambda_i \rightarrow 0$ as $i \rightarrow \infty$. On account of (2), it follows

$$\begin{cases} \Delta v + \frac{\nabla W}{W} \nabla v - (c(x) + N^2)v = 0 & \text{in } \{x; |x| < \lambda_i\}, \\ v = 0 & \text{on } S(\lambda_i). \end{cases}$$

Because of (4), a well-known maximum principle for second order linear elliptic equations can be applied, and that $v = 0$ is obtained, so $u = 0$ for $|x| < \lambda_i$, and in turn $u = 0$ in B . The proof is completed.

2.2. *Proof of Theorem 3.* Suppose for contradiction that $u(0) = 0$. Automatically $\nabla u(0) = 0$. For $0 \leq \lambda < 1$, denote $\Sigma_\lambda = \{x \in B; x_1 > \lambda\}$; $T_\lambda = \{x \in B; x_1 = \lambda\}$, and for $x \in \Sigma_\lambda$, denote by x^λ the reflexion of x with respect to T_λ , denote by Σ'_λ the reflexion of Σ_λ with respect to T_λ . Set

$$\Lambda = \left\{ \lambda \in (0, 1); u(x^\lambda) > u(x) \text{ in } \Sigma_\lambda, \frac{\partial u}{\partial x_1} < 0 \text{ on } T_\lambda \right\},$$

which is not empty by Lemma A and a similar argument to [GNN]. First of all we prove $\inf \Lambda \in \Lambda$. Indeed, there holds

$$\begin{cases} u(x^\alpha) \geq u(x) & \text{in } \Sigma_\alpha, \\ \frac{\partial u}{\partial x_1} \geq 0 & \text{on } T_\alpha \end{cases}$$

where $\alpha = \inf \Lambda$. Letting $w(x) = u(x^\alpha)$ for $x \in \Sigma_\alpha$ and

$$c(x) = \int_0^1 f'(u + t(w - u)) dt,$$

we have

$$\begin{cases} \Delta(w - u) - c(x)(w - u) = 0, \\ (w - u) \geq 0 \quad \text{in } \Sigma_\alpha, \\ (w - u) = 0 \quad \text{on } T_\alpha. \end{cases}$$

Then for $K > 0$,

$$\Delta(w - u) - (K + c(x)) \cdot (w - u) = -K(w - u) \leq 0 \quad \text{in } \Sigma_\alpha.$$

Taking K large enough, we may apply the Hopf maximum principle to $(w - u)$ and obtain that either

$$(5) \quad (w - u) = 0 \quad \text{in } \Sigma_\alpha$$

or

$$(6) \quad \begin{cases} w(x) > u(x) \quad \text{in } \Sigma_\alpha, \\ \frac{\partial}{\partial \bar{n}}(w - u)(p) < 0, \end{cases}$$

where $p \in \partial \Sigma_\alpha$ such that $(w - u)(p) = 0$ and $\bar{n} = \bar{n}(p)$ is the outward normal vector of $\partial \Sigma_\alpha$ at p . Then (5) cannot hold since $n \geq 2$ and $u = 0$ on ∂B ; $u > 0$ in $B \setminus \{0\}$. Now (6) holds, then $u(x^\alpha) > u(x)$ in Σ_α , and on T_α ,

$$2 \frac{\partial u}{\partial x_1} = \frac{\partial}{\partial(-x_1)}(w - u) < 0$$

since $(w - u) = 0$, which means $\alpha \in \Lambda$. Next it is easy to see that $\alpha \geq \frac{1}{2}$. If $\alpha = \frac{1}{2}$, let $p_0 = (1, 0, \dots, 0) \in \partial B$, then $p_0^\alpha = 0$, and

$$(w - u)(p_0) = u(p_0^\alpha) - u(p_0) = 0.$$

By (6) we have

$$\frac{\partial}{\partial x_1}(w - u)(p_0) < 0, \quad \text{i.e.} \quad -\frac{\partial u}{\partial x_1}(0) - \frac{\partial u}{\partial x_1}(p_0) < 0.$$

Then we get

$$\frac{\partial u}{\partial x_1}(0) > -\frac{\partial u}{\partial x_1}(p_0) \geq 0,$$

a contradiction since $\nabla u(0) = 0$. Thus $\alpha > \frac{1}{2}$. In this case we claim that there exists $\alpha_0 < \alpha$ such that $\alpha_0 \in \Lambda$, which will contradict the assumption $\alpha = \inf \Lambda$ and our proof would then be completed. To

this end, we assume again for contradiction that there exists a sequence $\{\alpha_i\}$ with $\alpha_i \rightarrow \alpha$ but $\alpha_i \notin \Lambda$ which means that either

$$(7) \quad u(a_i^{\alpha_i}) \leq u(a_i) \quad \text{for some } a_i \in \Sigma_{\alpha_i}$$

or

$$(8) \quad \frac{\partial u}{\partial x_1}(b_i) \geq 0 \quad \text{for some } b_i \in T_{\alpha_i}.$$

The latter cannot always remain true for any subsequence of $\{i\}$ since, otherwise, it implies that $\frac{\partial u}{\partial x_1} \geq 0$ at some point on T_α when $\{b_i\}$ do not approach ∂B , contradicting $\alpha \in \Lambda$, and that there exists a point in any neighborhood of b such that $\frac{\partial u}{\partial x_1} \geq 0$ when $b_i \rightarrow b \in \partial B$, contradicting Lemma A since $b = (b^1, \dots, b^n)$ with $b^1 = \alpha > 0$. Now let $a_i \rightarrow \bar{a} \in \bar{\Sigma}_\alpha$. From (7) $u(\bar{a}^\alpha) \leq u(\bar{a})$, and $\bar{a} \in \partial \Sigma_\alpha$ by $\alpha \in \Lambda$. But because $\alpha > \frac{1}{2}$, for $x \in \partial \Sigma_\alpha \setminus \bar{T}_\alpha \subset \partial B$, where \bar{T}_α is the closure of T_α , obviously $u(x^\alpha) > 0 = u(x)$. Thus we further have $\bar{a} \in \bar{T}_\alpha$. Let L_i be the segment joining $a_i^{\alpha_i}$ and a_i , having $(1, 0, \dots, 0)$ as the tangent vector. From (7) it is seen that there exists $y_i \in L_i$ such that $\frac{\partial u}{\partial x_1}(y_i) \geq 0$. Since $\bar{a} \in \bar{T}_\alpha$, y_i must also tend to \bar{a} . And automatically $\frac{\partial u}{\partial x_1}(\bar{a}) \geq 0$, which leads to a contradiction when $\bar{a} \in T_\alpha$. Then $\bar{a} \in \partial \bar{T}_\alpha \subset \partial B$. But we have seen that $\frac{\partial u}{\partial x_1}(y_i) \geq 0$ and $y_i \rightarrow \bar{a}$, which contradicts Lemma A. Thus we complete the proof.

2.3. *Proof of Theorem 1.* Denote $B(\lambda) = \{x \in R^n; |x| < \lambda\}$. The necessity is obvious. For sufficiency, by Theorem 2, the nodal set of u must be $\bigcup_{i=1}^k S(\lambda_i)$ where $0 \leq \lambda_1 < \lambda_2 < \dots < \lambda_k = 1$. We further prove $\lambda_1 > 0$.

Indeed suppose $\lambda_1 = 0$, i.e. $u(0) = 0$. We see that there are no nodal points of u in $B(\lambda_2) \setminus \{0\}$, which, together with the fact that $B(\lambda_2) \setminus \{0\}$ is path-connected (since $n \geq 2$), implies that u is positive (or negative) in $B(\lambda_2) \setminus \{0\}$. Then from Theorem 3 we have $u(0) > 0$ (or $u(0) < 0$) also. It contradicts $u(0) = 0$, which shows $\lambda_1 > 0$.

Now in $B(\lambda_1)$, u is positive (or negative). It allows us to apply the result of [GNN] to conclude that u is radially symmetric in $B(\lambda_1)$. It is clear that

$$(9) \quad \frac{\partial u}{\partial r} = \text{const.} \quad \text{on } S(\lambda_1).$$

Let $T: R^n \rightarrow R^n$ be any rotation transform. Since equation (2) is invariant under the transform T , $v = u(Tx)$ also solves (2). On

$S(\lambda_1)$, obviously $v = u$, and $\frac{\partial v}{\partial r} = \frac{\partial u}{\partial r}$ by (9). Then $(v - u)$ is a solution to the Cauchy problem

$$\Delta w = \left(\int_0^1 f'(tv + (1-t)u) dt \right) \cdot w \quad \text{in } B,$$
$$w = \frac{\partial w}{\partial r} = 0 \quad \text{on } S(\lambda_1)$$

and constantly equals 0 by the uniqueness of the Cauchy problem, i.e. $u(x) = u(Tx)$ in B for any rotation transforms T , which means u is radially symmetric in B . We finish the proof of our main theorem.

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Pacific Journal of Mathematics

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Patrick Robert Ahern and Carmen Cascante , Exceptional sets for Poisson integrals of potentials on the unit sphere in \mathbf{C}^n , $p \leq 1$	1
David Peter Blecher , The standard dual of an operator space	15
Patrick Gilmer , Real algebraic curves and link cobordism	31
Simon M. Goberstein , On orthodox semigroups determined by their bundles of correspondences	71
John Kalliongis and Darryl John McCullough , Homeotopy groups of irreducible 3-manifolds which may contain two-sided projective planes	85
Yuji Konishi, Masaru Nagisa and Yasuo Watatani , Some remarks on actions of compact matrix quantum groups on C^* -algebras	119
Guojun Liao and Luen-Fai Tam , On the heat equation for harmonic maps from noncompact manifolds	129
John Marafino , Boundary behavior of a conformal mapping	147
Ji Min , A remark on the symmetry of solutions to nonlinear elliptic equations	157
Paul Nevai and Walter Van Assche , Compact perturbations of orthogonal polynomials	163
Kyril Tintarev , Level set maxima and quasilinear elliptic problems	185