LEVEL SET MAXIMA AND QUASILINEAR ELLIPTIC PROBLEMS

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The present paper studies existence of solutions to the problem
\[ \rho A'(x) = B'(x) \]
where \( A \) and \( B \) are Fréchet differentiable functionals on a Banach space. For every given value of \( A(x) = t \) we prove existence of a solution \( x \) and present an expression for the eigenvalue \( \rho = \rho(t) \). The result is applied to quasilinear elliptic equations.

1. Introduction. A typical problem of the second order studied below is

\[ -\text{div} \mathcal{A}_x'(x, \nabla u) = f(x, u), \quad u|_{\partial \Omega} = 0, \]

where \( \Omega \subset \mathbb{R}^n \) is an open bounded domain, \( \mathcal{A}(x, \xi): \Omega \times \mathbb{R}^n \to \mathbb{R} \) and “ellipticity” of \( \mathcal{A} \) is defined as:

\[ \mathcal{A}_x'(x, \xi) \cdot \xi \geq c|\xi|^p, \quad p > 1, \quad c > 0, \quad (x, \xi) \in \Omega \times \mathbb{R}^n. \]

When \( \mathcal{A} \) is a quadratic form of \( \xi \) and \( p = 2 \), (1.1) becomes a semilinear equation. The study extends also to equations of higher order and to systems.

Our approach follows the approach of Browder [2] and Berger [1] with a subsequent refinement due to [3] and [4]. Let \( X \) be a Banach space, \( A, B \in C^1(X \to \mathbb{R}) \). We consider a critical point equation

\[ \rho A'(x) = B'(x). \]

This equation might be associated with the maximum problem

\[ \sigma(t) = \sup_{A(x) = t} B(x). \]

Under general compactness conditions, the maximum in (1.4) is attained and provides (1.3). Remarkably the eigenvalues \( \rho \) are derivatives of the function of critical values \( \sigma(t) \). More precisely, since we prove that \( \sigma \) might have different right and left hand derivatives \( \sigma'_{\pm}(t) \), both of them are eigenvalues. Under additional conditions we prove that (1.3) is solvable for any \( \rho \) between \( \inf \sigma'_{\pm}(t) \) and
sup \sigma'_+(t). Thus, if the graph of \sigma(t) has a slope less than 1 and a slope greater than 1, then \( A'(u) = B'(u) \) will be solvable. Applications to quasilinear elliptic problems then follow.

2. Maxima on the level sets. Let \( X \) be a reflexive Banach space and let \( A, B \in C^1(X \to \mathbb{R}) \). Let us list several conditions to be used later. Let

\begin{align*}
(2.1) \quad & S_t = \{ u \in X : A(u) = t \}, \\
(2.2) \quad & \omega_t = \{ u \in X : A(u) \leq t \}
\end{align*}

and define for those \( t \) when \( S_t \neq \emptyset \),

\begin{align*}
(2.3) \quad & \sigma(t) = \sup_{u \in S_t} B(u), \\
(2.4) \quad & \beta(t) = \sup_{u \in \omega_t} B(u) = \sup_{\tau \leq t} \sigma(\tau).
\end{align*}

**Lemma 2.1.** Assume the following conditions:

(A1) \( A \) is coercive, i.e., \( \|u_j\| \to \infty \Rightarrow A(u_j) \to \infty \).

(A2) \( A \) is weakly lower semicontinuous.

(A3) \( \langle A'(u), u \rangle > 0 \) for \( u \in X \setminus \{0\} \).

(B1) \( B \) has no local maxima on \( X \).

(B2) \( B \) is weakly continuous.

Then

\begin{align*}
(2.5) \quad & \sigma(t) \text{ is increasing on } [0, \infty), \\
(2.6) \quad & \sigma(t) \text{ is continuous on } [0, \infty), \\
(2.7) \quad & \text{the maximum in } (2.3) \text{ is attained for every } t > 0.
\end{align*}

(An immediate consequence of (2.5) is that \( \sigma(t) = \beta(t) \).

Without loss of generality one can assume from now that

\begin{align*}
(2.8) \quad & A(0) = 0.
\end{align*}

Then, due to (A3)

\begin{align*}
(2.9) \quad & A(u) = \int_0^1 \frac{d}{dt} A(tu) \, dt = \int_0^1 \langle A'(tu), u \rangle \, dt \\
& = \int_0^1 t^{-1} \langle A'(tu), tu \rangle \, dt > 0, \quad \text{unless } u = 0.
\end{align*}

By (A1) the range of \( A \) is \([0, \infty)\).
Proof of Lemma 2.1. Consider a maximizing sequence \( u_j \in \omega_t \) in (2.4). By (A1) it is bounded in norm. Then there is a weakly convergent renamed subsequence \( u_j \xrightarrow{w} u_0 \). By (A2) \( u_0 \in \omega_t \). By (B2) \( B(u_j) \to B(u_0) = \beta(t) \). If \( u_0 \notin S_t \), then it is a point of interior maximum for \( B \) in \( \omega_t \). This contradicts (B1). Consequently, \( u_0 \in S_t \). This implies in turn that \( \sigma(t) = \beta(t) \) and it is a monotone increasing function. For the given \( t \), the point \( u_0 \) is a point of maximum over \( S_t \). Thus, (2.5) and (2.7) are proved.

Consider

\[
\begin{align*}
\sigma(t_0 + 0) &= \liminf_{t \to t_0, t > t_0} \sigma(t), \\
\sigma(t_0 - 0) &= \limsup_{t \to t_0, t < t_0} \sigma(t).
\end{align*}
\]

Let \( t_j > t_0, t_j \to t_0 \) and let \( \sigma(t_j) \) be attained an \( u_j \in S_t \). Then by (A1) \( u_j \) has a renamed weakly convergent subsequence: \( u_j \xrightarrow{w} u_0 \in \omega_{t_0} \). Therefore,

\[
\sigma(t_0 + 0) = B(u_0) \leq \sup_{u \in \omega_{t_0}} B(u) = \sigma(t_0).
\]

The converse inequality is true by monotonicity of \( \sigma \). Thus

\[
\sigma(t_0 + 0) = \sigma(t_0).
\]

Let now \( u_0 \) be a point of maximum in (2.1) at \( t = t_0 \). Then by (A3)

\[
t(s) := A(su_0)
\]

is a monotone increasing function and

\[t(s) \to t_0, \quad t(s) < t_0 \quad \text{as} \quad s \to 1, \quad s < 1.\]

Therefore,

\[
\sigma(t_0 - 0) \geq \limsup_{s \to 1, s < 1} B(su_0) = \sigma(t_0).
\]

The converse inequality is due to monotonicity of \( \sigma \). Thus \( \sigma(t_0 - 0) = \sigma(t_0) \) and (2.6) is proved. □

Lemma 2.2. Assume the conditions of Lemma 2.1. Let \( u_0 \) be a point of maximum in (2.3), \( t > 0 \). Then there is a \( \rho \geq 0 \), such that

\[
\rho A'(u_0) = B'(u_0).
\]

Proof. Let \( v \in X \) be such that

\[
\langle A'(u_0), v \rangle < 0.
\]
By Lemma 2.1 $u_0$ is a point of maximum of $B$ in $\omega_t$ and

\begin{equation}
B(u_0 + \theta v) \leq B(u_0)
\end{equation}

for $\theta > 0$ sufficiently small. Thus,

\begin{equation}
\langle B'(u_0), v \rangle \leq 0
\end{equation}

for any $v$ satisfying (2.17). From (2.17), (2.19) a routine argument shows that $A'(u_0)$ and $B'(u_0)$ are parallel. By (A3), $A'(u_0) \neq 0$ and (2.16) follows immediately. \qed

Let us define a set $\Lambda_t \subset S_t \times [0, \infty)$, $t \in (0, \infty)$:

\begin{equation}
\Lambda_t = \{(u, \rho) : B(u) = \sigma(t), \rho A'(u) = B'(u)\}.
\end{equation}

By Lemmas 2.1, 2.2 $\Lambda_t$ is nonempty. Let

\begin{equation}
\Lambda_t^{(\rho)} = \{\rho \geq 0 : \exists u \in S_t, (u, \rho) \in \Lambda_t\},
\end{equation}

\begin{equation}
\Lambda_t^{(u)} = \{u \in S_t : \exists \rho \geq 0, (u, \rho) \in \Lambda_t\}.
\end{equation}

By (2.20), one has also:

\begin{equation}
\Lambda_t^{(u)} = \{u \in S_t : B(u) = \sigma(t)\}
\end{equation}

and

\begin{equation}
\Lambda_t^{(\rho)} = \{\rho = \langle B'(u), u \rangle / \langle A'(u), u \rangle, u \in \Lambda_t^{(u)}\}.
\end{equation}

**Lemma 2.3.** Assume the conditions of Lemma 2.1 and in addition

(A4) If $u_j \xrightarrow{w} u_0$, then

\begin{equation}
A(u_j) \rightarrow A(u_0) \iff \langle A'(u_j), u_j \rangle \rightarrow \langle A'(u_0), u_0 \rangle.
\end{equation}

(B3) $\langle B'(u), u \rangle$ is weakly continuous.

Let $T \subset (0, \infty)$ and

\begin{equation}
\Lambda_T = \bigcup_{t \in T} \Lambda_t.
\end{equation}

If $T$ is compact in $(0, \infty)$, then $\Lambda_T$ is weakly compact in $X \times (0, \infty)$.

**Proof.** Let $(u_j, \rho_j) \in \Lambda_{t_j}, \ t_j \in T$. Consider a renamed convergent subsequence $t_j \rightarrow t_0 \in T$. Then by (A1), (A2) $u_j$ has a weakly convergent (renamed) subsequence $u_j \xrightarrow{w} u_0 \in \omega_0$. By (B2), $B(u_j) \rightarrow B(u_0)$, by Lemma 2.1 $B(u_0) = \sigma(t_0)$ and $u_0 \in S_{t_0}$, i.e.,
u_0 \in \Lambda^{(u)}_{t_0}. By Lemma 2.2 \ u_j satisfy (2.16) with eigenvalues, say, \( \rho_j \),

\( u_0 \) satisfies (2.16) with some \( \rho_0 \), and

\[ \rho_j = \langle B'(u_j), u_j \rangle / \langle A'(u_j), u_j \rangle, \]

\[ \rho_0 = \langle B'(u_0), u_0 \rangle / \langle A'(u_0), u_0 \rangle. \]

Note now that \( \langle B'(u_j), u_j \rangle \to \langle B'(u_0), u_0 \rangle \) by (B3) and \( \langle A'(u_j), u_j \rangle \to \langle A'(u_0), u_0 \rangle \) by (A4). Thus \( \rho_j \to \rho_0 \).

3. Critical values and eigenvalues.

**Theorem 3.1.** Assume

(A1) \( A \) is coercive, i.e., \( \|u_j\| \to \infty \Rightarrow A(u_j) \to \infty \).

(A2) \( A \) is weakly lower semicontinuous.

(A3) \( \langle A'(u), u \rangle > 0 \) for \( u \in X \setminus \{0\} \).

(A4) If \( u_j \rightharpoonup u_0 \), then

\[ A(u_j) \to A(u_0) \Leftrightarrow \langle A'(u_j), u_j \rangle \to \langle A'(u_0), u_0 \rangle. \]

(B1) \( B \) has no local maxima on \( X \).

(B2) \( B \) is weakly continuous.

(B3) \( \langle B'(u), u \rangle \) is weakly continuous.

Then for every \( t > 0 \) there exist left and right derivatives \( \sigma'_\pm(t) \), \( \sigma''_\pm(t) \leq \sigma'_i(t) \). Moreover,

\[ \sigma'_+(t) = \sup \{ \rho \in \Lambda^{(\rho)}_t \}, \]

\[ \sigma'_-(t) = \inf \{ \rho \in \Lambda^{(\rho)}_t \}. \]

Before we prove the theorem, we wish to note that supremum in (3.1) and infimum in (3.2) are attained on some \( u_\pm \in \Lambda^{(u)}_t \) due to Lemma 2.3. As a result one has

**Theorem 3.2.** Under conditions of Theorem 3.2 for every \( t > 0 \) there exist \( u_\pm \in S_t \), such that

\[ \sigma'_+(t)A'(u_+) = B'(u_+), \]

\[ \sigma'_-(t)A'(u_-) = B'(u_-). \]

**Proof of Theorem 3.1.** 1. Let \( u_0 \in \Lambda^{(u)}_{t_0} \). Let \( \theta_j \to 1 \), \( t_j = A(\theta_j u_0) \). Then by continuity of \( A \), \( t_j \to A(u_0) = t_0 \). Moreover,

\[ t_j - t_0 = A(\theta_j u_0) - A(u_0) = \langle A'(u_0), u_0 \rangle (\theta_j - 1) + o(\theta_j - 1). \]
Consequently,

\[ (3.6) \quad \sigma(t_j) - \sigma(t_0) \geq B(\theta_j u_0) - B(u_0) \]
\[ = \langle B'(u_0), u_0 \rangle (\theta_j - 1) + o(\theta_j - 1) \]
\[ = ((B'(u_0), u_0)/(A'(u_0), u_0))(t_j - t_0) + o(t_j - t_0). \]

From (3.6) and (2.23) one has immediately,

\[ (3.7) \quad D^+ \sigma(t_0) \geq \sup \{ \rho \in \Lambda_{t_0}^{(\rho)} \}, \]
\[ (3.8) \quad D^- \sigma(t_0) \leq \inf \{ \rho \in \Lambda_{t_0}^{(\rho)} \}. \]

2. Let now \( u_j \in \Lambda_{t_j}^{(u)} \), \( t_i \to t_0 \). By Lemma 2.3 a renamed sequence \( u_j \) converges weakly to \( u_0 \in \Lambda_{t_0}^{(u)} \). Let us define \( \theta_j > 0 \) by

\[ (3.9) \quad A(\theta_j u_j) = t_0. \]

By (A3), the function \( \theta \to A(\theta u_j) \) is monotone for any \( u \neq 0 \) and by (A1) the range of it is \([0, \infty)\). Thus, for given \( t_0 > 0 \) and \( u_j \neq 0 \), (3.9) has a unique solution \( \theta_j > 0 \). Since \( A \in C^1 \), there exist \( \eta_j \in [\theta_j, 1] \), such that

\[ (3.10) \quad t_j - t_0 = A(u_j) - A(\theta_j u_j) = \langle A'(\eta_j u_j), u_j \rangle (1 - \theta_j). \]

From (3.9) it follows that \( \theta_j \) is a bounded sequence. Let us consider a renamed convergent subsequence: \( \theta_j \to \theta_0 \). Then \( \theta_j u_j \to \theta_0 u_0 \) and, necessarily, \( \theta_0 = 1 \). Therefore \( \eta_j \to 1 \) and

\[ (3.11) \quad t_j - t_0 = \langle A'(u_j), u_j \rangle (1 - \theta_j) + o(1 - \theta_j). \]

Similarly,

\[ (3.12) \quad B(u_j) - B(\theta_j u_j) = \langle B'(u_j), u_j \rangle (1 - \theta_j) + o(1 - \theta_j). \]

Therefore

\[ (3.13) \quad \sigma(t_j) - \sigma(t_0) \leq \langle B'(u_j), u_j \rangle (1 - \theta_j) + o(1 - \theta_j) \]
\[ = (\langle B'(u_j), u_j \rangle/(A'(u_j), u_j))(t_j - t_0) + o(t_j - t_0). \]

We have to note only that \( \langle A'(u_j), u_j \rangle \to \langle A'(u_0), u_0 \rangle \) by (A4) and \( \langle B'(u_j), u_j \rangle \to \langle B'(u_0), u_0 \rangle \) by (B2). Then from (3.13) follows:

\[ (3.14) \quad D^+ \sigma(t_0) \leq \sup \{ \rho \in \Lambda_{t_0}^{(\rho)} \}, \]
\[ (3.15) \quad D^- \sigma(t_0) \geq \inf \{ \rho \in \Lambda_{t_0}^{(\rho)} \}. \]

3. Let us combine (3.7) and (3.14). Then

\[ \sup \{ \rho \in \Lambda_{t_0}^{(\rho)} \} \leq D^+ \sigma(t_0) \leq D^+ \sigma(t_0) \leq \sup \{ \rho \in \Lambda_{t_0}^{(\rho)} \}, \]
i.e., $\sigma'_+(t_0)$ exists and satisfies (3.1). Similarly, (3.2) follows from (3.8) and (3.15).

\section{Continua of solutions. Range of solvability.}

In this section we assume the conditions of Theorem 3.1.

**Proposition 4.1.** Assume that for every $t \in (s_1, s_2) \subset (0, \infty)$ the set $\Lambda_t^{(u)}$ consists of a single element $u_t$. Then the map $t \to u_t$ is weakly continuous on $(s_1, s_2)$.

**Proof.** Let $t_0 \to t_0 \in (s_1, s_2)$. Then $\{t_j\}$ is compact in $(0, \infty)$ and by Lemma 2.3 $u_{t_j}$ has a (renamed) weakly convergent subsequence $u_{t_j} \overset{w}{\to} u_0 \in \Lambda_{t_0}^{(u)}$. Since $\Lambda_{t_0}^{(u)} = \{u_{t_0}\}$, the original sequence $u_{t_j}$ must be weakly convergent to the same element $u_{t_0}$.

**Corollary 4.2.** Under conditions of Proposition 4.1 the problem

\begin{equation}
\rho A'(u) = B'(u)
\end{equation}

possesses a weakly continuous family of eigenfunctions $t \to u_t$ corresponding to eigenvalues $\rho_t = \langle B'(u_t), u_t \rangle / \langle A'(u_t), u_t \rangle$ continuous in $t \in (s_1, s_2)$. The function (2.3) has a continuous derivative on $(s_1, s_2)$ and $\rho_t = \sigma'(t)$.

The proof follows from Theorems 3.1 and 3.2.

**Proposition 4.3.** Let $h : X^2 \to X$ be a map, such that

(Ah) $A(h(u, v)) > \max\{A(u), A(v)\}$ for $u \neq v$.

(Bh) $B(h(u, v)) \geq \min\{B(u), B(v)\}$ for $u, v \in X$.

Then the assertions of Proposition 4.1 and Corollary 4.2 hold for any $t > 0$.

**Proof.** Let $u, v \in \Lambda_t^{(u)} u \neq v$. Then by (Ah), $h(u, v) \in \omega_{t'}, t' < t$, but by (Bh) $B(h(u, v)) \geq B(u) = \sigma(t)$. This contradicts Lemma 2.1. Thus the conditions of Lemma 4.1 are satisfied at any $t > 0$.

A simple example when (Ah), (Bh) are satisfied, can be provided by

**Corollary 4.4.** Let $A$ be strictly convex and $B$ be concave on $X$. Then the assertions of Proposition 4.1 and Corollary 4.2 hold for $t \in (0, \infty)$.

**Proof.** Take $h(u, v) = \lambda u + (1 - \lambda)v$, $\lambda \in (0, 1)$. 

\qedsymbol
If $\sigma \in C_{\text{loc}}^1(0, \infty)$, then (4.1) is solvable (with $u \neq 0$) for any $p \in I(A, B)$, where

$$I(A, B) = \left( \inf_t \sigma_-'(t), \sup_t \sigma_+'(t) \right).$$

However, $\sigma$ does not generally have a continuous derivative (cf. [4]). Thus we wish to answer the question, for what subsets of $I(a, b)$ does (4.1) still have a non-zero solution.

**Proposition 4.5.** Let the function $pt - \sigma(t)$ have a local minimum on $(0, \infty)$. Then (4.1) has a non-zero solution.

**Proof.** Let $pt - \sigma(t)$ have a local minimum at $t_0 > 0$. Then $\sigma_-'(t_0) \geq p \geq \sigma_+'(t_0)$. By Theorem 3.1, $\sigma_-'(t_0) \leq \sigma_+'(t_0)$. Thus, $\sigma$ is differentiable at $t_0$ and $\sigma'(t_0) = p$. Then by Theorem 3.2, (4.1) has a solution with $A(u) = t_0$. $\square$

In order to get a more extensive result we use a mountain pass theorem from [5].

**Theorem 4.6.** Let $G \in C^1(X \to \mathbb{R})$ and let $u_0 \in X \setminus \{0\}$. Let $N \subset X$ be an open bounded set, such that $0 \in N$ but $u_0 \notin \overline{N}$. Assume that

$$G(u) \geq 0, \quad u \in \partial N,$$

$$G(0) \leq 0, \quad G(u_0) \leq 0.$$

Then there is a sequence $u_k \in X$ and $\gamma \geq 0$ such that

$$G(u_k) \to \gamma,$$

$$\|G'(u_k)\|_{X^*} \|u_k\|_X \to 0.$$

Let now

$$A_\theta = \theta \langle A'(u), u \rangle - A(u),$$

$$B_\theta = \theta \langle B'(u), u \rangle - B(u),$$

$$\Phi = \{ \theta \in \mathbb{R} : |A_\theta| \text{ is coercive} \},$$

and let

$$Q(\theta) \text{ be set of limit points for } B_\theta(u)/A_\theta(u)$$

when $\|u\| \to \infty$, $\theta \in \Phi$.

Set now

$$Q_\ast = \bigcap_{\theta \in \Phi} Q(\theta).$$
THEOREM 4.7. Assume, in addition to conditions of Theorem 3.1, that:

(ABw) $A', B'$ are continuous from $X$ to $X^*$ with regard to respective weak topologies.

Then for every $p \in I(A, B) \backslash Q_*$ the equation (4.1) has a nonzero solution.

Proof. 1. Let $p \in I(A, B) \backslash Q_*$, i.e., $p$ is a slope of a secant to the graph of $\sigma(t)$, the functional $|A_\theta|$ is coercive for some $\theta$ and $p$ is not a limit point of $B_\theta/A_\theta$ at infinity. If $pt - \sigma(t)$ has a local minimum at $t \neq 0$, then $p$ is an eigenvalue (Proposition 4.5). Thus we would consider the case when $pt - \sigma(t)$ has no local minimum. If $p = \sigma'_+(t_0)$ or $p = \sigma'_-(t_0)$ for some $t_0 > \theta$, then it is an eigenvalue by Theorem 3.2. The remaining case is: for some $t_0 > 0$, $\sigma'(t_0) < p_\sigma(t_0)$. This implies that $t_0$ is a point of local maximum of $pt - \sigma(t)$. Since we assume that $pt - \sigma(t)$ has no local minimum, $t_0$ is a point of strict global maximum. In particular, $\delta := \rho t_0 - \sigma(t_0) > 0 - \sigma(0) = 0$, and there exists $t_1 > t_0$, such that $\delta := \rho t_1 - \sigma(t_1) < \delta$.

2. Let

\begin{equation}
G(u) = pA(u) - B(u) - \delta_1.
\end{equation}

Then all the conditions of Theorem 4.6 are fulfilled with $N = \{u \in X, A(u) \leq t_0\}$ and $u_0 \in S_{t_1}$. Let $u_k \in X$, $\gamma \geq 0$ satisfy

\begin{align}
\rho A(u_k) - B(u_k) &\to \delta_1 + \gamma \geq \delta_1, \\
\|\rho A'(u_k) - B'(u_k)\|_{X^*} \|u_k\|_X &\to 0.
\end{align}

Then

\begin{equation}
\rho \langle A(u_k), u_k \rangle - \langle B'(u_k), u_k \rangle \to 0
\end{equation}

and, consequently,

\begin{equation}
\rho A_\theta(u_k) - B_\theta(u_k) \to -\gamma - \delta.
\end{equation}

If $\|u_k\| \to \infty$, then $|A_\theta(u_k)| \to \infty$, $\rho = \lim B_\theta(u_k)/A_\theta(u_k)$, which contradicts the assumptions. Thus the sequence $u_k$ is bounded in norm. Let now $u_k$ be a renamed weakly convergent sequence, and $u_0 = w\text{-}\lim u_k$. Then by (ABw) from (4.14) follows:

\begin{equation}
\rho A'(u_0) = B'(u_0).
\end{equation}

Moreover, by (B3), (4.14), (4.17)

\begin{equation}
\lim \rho \langle A'(u_k), u_k \rangle = \lim \langle B'(u_k), u_k \rangle = \langle B'(u_0), u_0 \rangle = \rho \langle A'(u_0), u_0 \rangle.
\end{equation}
Then by (A4)
\[(4.19)\quad \lim A(u_k) = A(u_0).\]
Thus by (4.13), (B2),
\[
0 < \delta_1 + \gamma = \lim \rho A(u_k) - B(u_k) = \rho A(u_0) - B(u_0)
\]
which proves that $u_0 \neq 0$. \hfill \Box

**Corollary 4.8.** Let $\rho_* = \inf_{\theta \in \Phi} \lim \sup_{\|u\| \to \infty} B_\theta(u)/A_\theta(u)$. Then (4.1) has a non-zero solution for
\[
\rho \in I(A, B) \cap (\rho_*, \infty).
\]

5. Applications to quasilinear elliptic problems. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set $X = W_0^{l, p}(\Omega)^k$, $p > 1$, $l \in \mathbb{N}$, $k \in \mathbb{N}$. Let $\nu(l)$ be the number of multi-indices of length not exceeding $l$. Assume that $A(x, \{y_\alpha\}_{|\alpha| \leq l})$ and $B(x, \{y_\alpha\}_{|\alpha| \leq l-1})$ are $C^1$ real valued functions of $\{y_\alpha\}$ whose derivatives are Carathéodory functions of $(x, \{y_\alpha\})$. Without loss of generality we assume that
\[(5.1)\quad A(x, 0) = 0, \quad B(x, 0) = 0.
\]
We require for the function $A$ the following ellipticity condition:
\[(5.2)\quad A(x, \cdot) \text{ is convex for almost every } x \in \Omega
\]
and the following coercivity condition:
\[(5.3)\quad \frac{d}{dt} A(x, \{ty_\alpha\})_{|t|=1} \geq c \sum_{|\alpha|=l} |y_\alpha|^p, \quad c > 0, \quad x \in \Omega.
\]
We also require the following upper bounds for $A'$ and $B'$:
\[(5.4)\quad |A'_y(x, \{y_\alpha\})| \leq c \left(1 + \sum_{|\alpha| \leq l} |y_\alpha|^{p-1}\right), \quad c > 0,
\]
\[(5.5)\quad |B'_y(x, \{y_\alpha\})| \leq C \sum_{|\beta| \leq l-1} |y_\beta|^q_{\beta}
+ V_\alpha(\{y_\beta\}_{|\beta| \leq l-n/p}) + W_\alpha(x),
\]
where
\[(5.6)\quad C > 0, \quad W_\alpha \in L^r, \quad r_\alpha = 1 \text{ if } l - |\alpha| > n/p, \quad r_\alpha > pn/(pn - n - p(l - |\alpha|)) \text{ if } l - |\alpha| \leq n/p, \quad V_\alpha \text{ is a continuous function,}
\]
PROPOSITION 5.1. Assume (5.2)–(5.6). Then the functionals
\begin{align}
A(u) &= \int_\Omega \mathcal{A}(x, \{\partial^\alpha u\}) \, dx, \\
B(u) &= \int_\Omega \mathcal{B}(x, \{\partial^\alpha u\}) \, dx
\end{align}

satisfy (A1–A4), (B2), (B3), (ABw) on \( X = W_0^{1,p}(\Omega)^k \).

Proof. The verification of continuity and differentiability properties is standard and based on compactness in the Sobolev embedding theorem. We wish to make remarks on only a few details.

1. Relation (5.3) implies (A3) and also (A1), since it immediately gives
\begin{equation}
\frac{d}{dt} \mathcal{A}(x, \{ty_\alpha\}) \geq ct^{p-1} \sum_{|\alpha|=l} |y_\alpha|^p, \quad t > 0.
\end{equation}

2. Relation (5.2) implies that the set (2.2) is convex. Thus it is weakly closed and \( A \) is lower semicontinuous.

3. Due to (5.9), (5.4) weak convergence of a sequence \( u_j \) together with convergence of either \( A(u_j) \) or of \( \langle A'(u_j), u_j \rangle \) is equivalent to convergence in norm.

To verify the condition (B1) in most of the applications it suffices to prove that \( B'(u) \neq 0 \) unless \( u = 0 \) and that \( u = 0 \) is not a point of maximum. Two particular cases are given below.

**Lemma 5.2.** Let
\begin{equation}
\sum_\alpha \mathcal{B}_y(x, \{y_\alpha\})y_\alpha > 0 \quad \text{for} \ \{y_\alpha\} \neq 0.
\end{equation}

Then (B1) holds.

Proof. From (5.10) it follows that \( \langle B'(u), u \rangle > 0 \) unless \( u = 0 \). The point \( u = 0 \) is not a point of maximum, but rather of minimum, since for every \( u \in X \setminus \{0\} \), \( B \) increases along the line \( t \to tu \), \( t > 0 \).
**Lemma 5.3.** Let $k = 1$, $b(x, \{y_\alpha\}) = F(y_0)$ and assume that with some $\varepsilon > 0$,

\begin{align}
(5.11) \quad & F'(y_0) > 0 \quad \text{for} \quad y_0 \in (0, \varepsilon) \quad \text{and} \\
& F'(y_0) \neq 0 \quad \text{for} \quad y_0 \in (-\varepsilon, 0).
\end{align}

Then (B1) holds.

**Proof.** Let $u$ be the point of maximum of $b$ and let $\bar{u}$ be a decreasing spherical rearrangement for $u$. Then $\bar{u}$ is a $W_0^{1,p}$-function dependent on the radial variable only. Therefore $\bar{u}$ is continuous away from the origin, the range of $\bar{u}$ is a closed interval $I$ containing zero. Moreover, the range of $u$ is dense in $I$, for if $(s, t)$ is not in the range of $u$, $(s, t)$ is not in the range of $\bar{u}$. If $u$ is a point of maximum for $B$, then $F' = 0$ on the range of $u$, and since $F'$ is continuous $F' = 0$ on $I$. By (5.11), therefore, $I \cap (-\varepsilon, \varepsilon) \subset \{0\}$. However, $u = 0$ is not a maximum: one can perturb 0 by a function $v \geq 0$ of an arbitrarily small norm so that $B(v) > B(0) = 0$. \hfill \Box

The following statement is now an immediate corollary of Theorems 3.1, 3.2.

**Theorem 5.4.** Assume (5.1–5.6) and (B1). Then for every $t > 0$ there exists a semistrong solution $u_t^\pm \in W_0^{1,p}(\Omega)^k$ satisfying the respective equations:

\begin{align}
(5.12) \quad & \sigma'_\pm(t) \sum_{|\alpha| \leq l} (-1)^{|\alpha|} \partial^\alpha \mathcal{A}_\alpha'(x, \{\partial^\gamma u(x)\}) \\
& \quad = \sum_{|\beta| \leq l-1} (-1)^{|\beta|} \partial^\beta \mathcal{B}_\beta'(x, \{\partial^\gamma u(x)\}).
\end{align}

Moreover,

\begin{align}
(5.13) \quad & A(u_t^\pm) = t \\
\text{and the function } & \sigma \text{ is given by} \\
(5.14) \quad & \sigma(t) = \sup_{A(u) = t} B(u).
\end{align}

We now will look for realization of conditions (Ah), (Bh) to get continuous curves of eigenfunctions.
THEOREM 5.5. Assume (5.1-5.6) and (B1). Let \( \mathcal{B}(x, \cdot) \) be concave for a.e. \( x \in \Omega \). Then there is a continuous family \( t \rightarrow (\rho_t, u_t) \in (0, \infty) \times W_0^{1,p}(\Omega)^k \), such that

\[
\rho_t \sum_{|\alpha| \leq l} (-1)^{|\alpha|} \partial^\alpha \mathcal{B}'_{\nu}(x, \{\partial^\gamma u_t(x)\}) = \sum_{|\alpha| \leq l-1} (-1)^{|\alpha|} \partial^\alpha \mathcal{B}'_{\nu}(x, \{\partial^\gamma u_t(x)\}).
\]

Proof. Apply Corollary 4.4. Then \( u_t \) satisfies (5.15) and \( t \rightarrow u_t \) is weakly continuous. However, by (5.4), (5.9) convergence of \( t_j = A(u_{t_j}) \) together with weak convergence of \( u_j \) is equivalent to convergence in norm and the family \( (\rho_t, u_t) \) is continuous.

THEOREM 5.6. Let \( \mathcal{A}(x, \{\partial^\alpha u\}) = |\nabla u|^p \), \( \mathcal{B}(x, \{y_\alpha\}) = F(y_0) \), \( k = 1 \). Assume also that \( F \in C^1 \),

\[
F'(s) > 0 \quad \text{for } s > 0 \quad \text{and} \quad F'(s) = 0 \quad \text{for } s \leq 0,
\]

and that the map

\[
s \rightarrow F(s^{1/p}) \text{ is concave for } s > 0.
\]

Then there is a continuous family \( t \rightarrow (\rho_t, u_t) \) satisfying

\[
-\rho_t \text{div}(\nabla u_t)^{p-2} \nabla u_t) = F'(u_t), \quad u_t \geq 0.
\]

Proof. Let

\[
h_\theta(u_1, u_2) = (\theta|u_1|^p + (1 - \theta)|u_2|^p)^{1/p}, \quad 0 \in (0, 1).
\]

Then, applying Hölder inequality with \( \frac{1}{p} + \frac{1}{p'} = 1 \),

\[
|\nabla(\theta|u_1|^p + (1 - \theta)|u_2|^p)|^{1/p'} = \left| \frac{\theta \nabla|u_1|^p + (1 - \theta) \nabla|u_2|^p}{\theta|u_1|^p + (1 - \theta)|u_2|^p} \right| \leq \theta|\nabla u_1|^p + (1 - \theta)|\nabla u_2|^p.
\]

The relation (5.20) makes sense a.e. when \( u_1, u_2 \in C^1_\Omega \) and the equality holds only if \( u_1 = u_2 \) or one of them vanishes. Then the following will be true on \( W_0^{1,p}(\Omega) \):

\[
A(h_\theta(u_1, u_2)) \leq \theta A(u_1) + (1 - \theta)A(u_2)
\]
with the same remark on equality. By (5.16), (5.17),

\[(5.22) \quad B(h_\theta(u_1, u_2)) \leq \theta B(u_1) + (1 - \theta) B(u_2).\]

Thus (Ah), (Bh) are satisfied and the assertion follows from Theorem 5.11. One has only to note that \( t_j \to t_0 \) and \( u_{t_j} \rightharpoonup u_{t_0} \) implies \( u_{t_j} \to u_{t_0} \). \( \square \)

Now we wish to find some realizations of Theorem 4.7.

**Lemma 5.7.** Assume (5.1-5.6) and (B1). Let

\[(5.23) \quad B_\mu(x, \{y_\alpha\}) = o \left( \sum_{|\alpha| \leq l-1} |y_\alpha|^p \right) \quad \text{uniformly in } \Omega.\]

Then

\[(5.24) \quad Q_* \subset \{0\}.\]

**Proof.** By (5.3), (5.4), \( A_\theta \geq \|u\|^p \) with \( \theta \) sufficiently large. By (5.23), \( B_\theta(u) = o(\|u\|^p) \). Thus \( Q_* \subset \{0\}. \) \( \square \)

**Theorem 5.8.** Assume (5.1-5.6), (B1) and (5.23). Then for every

\[(5.25) \quad \rho \in \left(0, \sup_{t} \sigma'_+(t)\right)\]

there is a non-zero solution of

\[(5.26) \quad \rho \sum_{|\alpha| \leq l} (-1)^{|\alpha|} \partial^\alpha \mathcal{A}(x, \{\partial_\gamma u(x)\})
\quad = \sum_{|\beta| \leq l-1} (-1)^{|\beta|} \partial^\beta \mathcal{B}(x, \{\partial_\gamma u(x)\}).\]

Moreover,

\[(5.27) \quad \sup_{t} \sigma'_+(t) \geq \sup_{u \neq 0} B(u)/A(u).\]

**Proof.** By Theorem 4.7, (5.26) is solvable for \( \rho \in I(a, b) \). The lower bound in \( I(a, b) \) is less than \( \sigma(t)/t \) which goes to zero when \( t \) tends to \( \infty \). The upper bound of \( I(a, b) \) is greater or equal to any given slope of a secant line to the graph of \( \sigma \), e.g. \( \sigma(t)/t \), which implies (5.27). \( \square \)

**Theorem 5.9.** Assume that \( k = 1 \),

\[ \mathcal{B}(x, \{y_\alpha\}) = F(y_0), \quad \mathcal{A}(x, \{\partial_\alpha u\}) = |\nabla u|^p. \]
Let \( F \in C^1 \) and if \( n \geq p, \) let \( F'(s) = o(|s|^{q-1}), \) \( q < pn/(n-p). \)
Assume that for some \( \varepsilon > 0 \)
\[
F'(s)/s^{p-1+\varepsilon} \text{ is an increasing function in a neighbourhood of } +\infty,
\]
\[F'(s) = o(|s|^{p-1}) \text{ as } s \to -\infty.
\]
Then for every
\[
\rho \in \left( \inf \sigma_-(t), \infty \right)
\]
there is a solution \( u \neq 0 \) for
\[
-\rho p \text{ div}(|\nabla u|^{p-2} \nabla u) = F'(u).
\]

Proof. Note that \( A, B \) satisfy (5.1-5.6) and B1.

From (5.28) it follows that \( B_\theta(u) \leq o(||u||^p) \) for \( \theta > 1/(p+\varepsilon). \)
Thus \( Q_\ast \cap (0, \infty) = \emptyset, \) and one can apply Theorem 4.7. By (5.8)
\[
\sup \sigma_+(t) = \sup \sigma(t)/t = \infty.
\]

As a general realization of 4.7 we state:

**Theorem 5.10.** Assume (5.1-5.6) and (B1). Then (5.26) has a non-zero solution \( u \) for
\[
\rho \in \left( \inf \sigma_-(t), \sup \sigma_+(t) \right) \backslash Q_\ast.
\]

6. Examples. The following examples illustrate the solvability results of this paper.

**Example 6.1.** Let \( \Omega \subset \mathbb{R}^n \) be an open bounded set. Consider
\[
\begin{cases}
-\rho \text{ div}(|\nabla|^{p-2} \nabla u) = u^\alpha + u^\beta, & u \geq 0, \ p > 1, \\
u_{|\partial\Omega} = 0.
\end{cases}
\]

**Case 1.** \( 0 < \alpha < \beta < p - 1. \) Then by Theorem 5.6 there is a continuous family \((\rho_t, u_t)\) of eigenfunctions. By Theorem 5.4, the range of eigenvalues \( \rho \) is \( I(A, B) \) which is here \( (0, \infty). \)

**Case 2.** \( 0 < \alpha < p - 1 < \beta. \) If \( n > p \) assume also \( \beta < np/(n-p) - 1. \) Then (6.1) has a solution for every \( \rho \in (\rho_0, \infty), \) \( \rho_0 > 0 \) and
\[
\rho_0 \leq \inf \sup \int_{\|u\|_{1,p} = 1} t^{-1} \int_{\Omega} \left( \frac{p}{\alpha+1} u^{\alpha+1} t^{(\alpha+1)/p} + \frac{p}{\beta+1} u^{\beta+1} t^{(\beta+1)/p} \right) dx.
\]
Solvability for $\rho \in I(A, B)$ is provided by Theorem 5.9.

**Case 3.** $p - 1 < \alpha < \beta$ and if $n > p$, $\beta < np/(n-p) - 1$. Similarly, (6.1) is solvable with $\rho \in (0, \infty)$ by Theorem 5.9.

**Case 4.** $0 < \alpha < \beta = p - 1$. The argument is like in Case 1, only $I(A, B) = (0, \rho_0)$,

$$(6.3) \quad \rho_0 = \sup_{\|u\|_{I(A,B)} = 1} \frac{1}{p} \int |u|^p.$$

**Case 5.** $0 < \alpha = p - 1 < \beta$, if $n > p$, $\beta < np/(n-p) - 1$. The argument follows one of Case 3, but $I(a, b) = (\rho_0, \infty)$, where $\rho_0$ is like in (6.3).

**EXAMPLE 6.2.** Let $\Omega \subset \mathbb{R}^n$, $n < 6$, be an open bounded domain. Consider a system:

$$\begin{cases}
-3\rho \text{div}(|\nabla u| + \frac{1}{2}|\nabla v|^{3/2}|\nabla u|^{-1/2})\nabla u = 5u^4, \\
-3\rho \text{div}(|\nabla v| + \frac{1}{2}|\nabla u|^{3/2}|\nabla v|^{-1/2})\nabla v = 4v^3, \\
u, v|_{\partial \Omega} = 0.
\end{cases}$$

This system corresponds to

$$A(u, v) = \int_{\Omega} (|\nabla u|^3 + |\nabla v|^3 + |\nabla u|^{3/2} |\nabla v|^{3/2}) \, dx,$$

$$B(u, v) = \int_{\Omega} (u^5 + v^4) \, dx.$$ 

By Theorem 5.9, it is solvable for $\rho \in I(A, B)$ and $I(A, B) = (0, \infty)$.

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