A PHRAGMÉN-LINDELOF THEOREM

X. T. LIANG AND Y. W. LU
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Let $\Omega$ be an unbounded and connected domain in $E^n$. Consider on $\Omega \times (0, \infty)$ the parabolic equation

$$u_t - \text{div} A(x, t, u, \nabla u) = B(x, t, u, \nabla u).$$

Under proper conditions a theorem of Phragmén-Lindelöf type is proved for generalized solutions of the equation.

**Introduction.** The classical Phragmén-Lindelöf principle gives an important property of harmonic functions defined on a plane sector domain. That has been generalized not only to generalized solutions of quasi-linear elliptic equations in more general unbounded and connected domains (see [1]-[5]), but also to the ones of quasi-linear parabolic equations in divergence form which have their principal parts only [6]. In this paper the result is extended to generalized solutions of the equation (1). We prove the result by an argument based on the technique of Moser [7] and Ladyženskaja-Ural’ceva [8]. We have not seen any reference discussing such behavior for solutions of parabolic equations except [6] where the simpler situation of the equation (1), namely $B \equiv 0$, is considered.

The paper is organized as follows. In §1 the main result is mentioned and in §2 several lemmata are given as preliminaries. Finally, a full proof of our theorem is stated in §3.

1. **Main result.** Let $\Omega$ be an unbounded and connected domain in the $n$-dimensional Euclidean space $E^n$. Denote by $\partial \Omega$ the boundary of $\Omega$. On $\Omega \times (0, \infty)$ we consider the following equation:

$$u_t - \text{div} A(x, t, u, \nabla u) = B(x, t, u, \nabla u)$$

where $A(x, t, u, \xi)$ and $B(x, t, u, \xi)$ are defined on $\Omega \times (0, \infty) \times E^1 \times E^n$, continuous with respect to $u$ and $\xi$ for fixed $x$ and $t$, measurable with respect to $x$ and $t$ for fixed $u$ and $\xi$, and satisfying the following structural conditions:

$$\xi \cdot A(x, t, u, \xi) \geq \kappa_0 |\xi|^2,$$

$$|A(x, t, u, \xi)| \leq \kappa_1 |\xi|,$$

$$|B(x, t, u, \xi)| \leq b(x, t)|\xi|,$$

where $b(x, t)$ is a measurable function, and $\kappa_0$, $\kappa_1$, $\kappa_2$, and $\kappa_3$ are positive constants.

The theorem states that if $u \in C^{1, \lambda}(\Omega \times (0, \infty))$ (the Holder space), then $u$ satisfies certain properties which are typical of harmonic functions in a sector domain.
where \( \kappa_1 \geq \kappa_0 > 0 \), \( b(x, t) \in L_\infty(\Omega \times (0, \infty)) \) and
\[
(3) \quad |b(x, t)| = O(|x|^{-1}) \text{ (uniformly for } t) \quad \text{as } |x| \to \infty.
\]

We need the supposition on \( \Omega \): there exist some \( x_0 \in \partial \Omega \) and a \( \theta \in (0, 1) \) such that
\[
(4) \quad \text{meas}(\Omega \cap \{B(x_0, \rho_0) \setminus B(x_0, \rho_1)\}) \\
\quad \leq \theta \text{meas}\{B(x_0, \rho_0) \setminus B(x_0, \rho_1)\}
\]
for any \( \rho_0 > \rho_1 > 0 \), where \( \text{meas} \) denotes the Lebesgue measure of the set \( e \) in \( E^n \) and
\[
B(x_0, \rho) = \{x \in E^n, |x - x_0| < \rho\}.
\]

For \( G \subset E^n \), \( W^1_2(G) \) and \( \bar{W}^1_2(G) \) stand for the usual Sobolev spaces. Let \( X \) be a Banach space formed by measurable functions defined on \( G \) with respect to the norm \( \| \cdot \|_X \). Denote \( L_p(0, T, X) \) the Banach space formed by the mapping from \([0, T]\) into \( X \) with norm \( \|u\|_{L_p(0, T, X)} \) defined by
\[
\|u\|_{L_p(0, T, X)} = \left( \int_0^T \|u\|_X^p \, dx \right)^{1/p} = \text{ess sup}_{t \in (0, T)} \|u\|_X \text{ if } p = \infty.
\]

Similarly, the space \( C(0, T, X) \) etc. can also be defined.

The function \( u \) is called a generalized solution of the equation (1) if for any \( T > 0 \) and for arbitrary \( G \subset \Omega \) and \( G \subset E^n \),
\[
(5) \quad u \in C(0, T, L_2(G)) \cap L_2(0, T, W^1_2(G))
\]
and the following holds:
\[
(1)' \quad \int_0^t \int_G \{ -v_t u + \nabla v \cdot A(x, t, u, \nabla u) - v B(x, t, u, \nabla u) \} \, dx \, dt \\
\quad + \int_G v(x, t) u(x, t) \bigg|_{t=0}^{t=t} \, dx = 0, \\
\quad \forall t \in (0, T), \quad v \in W^1_2(0, T, L_2(G)) \cap L_2(0, T, \bar{W}^1_2(G))
\]
where \( u(x, 0) \) is a given initial value of \( u \).

As the main result we have

**Theorem.** Suppose that the conditions (2)–(4) are satisfied and the generalized solution \( u \) of the equation (1) satisfies
\[
(6) \quad u^+ = \max(u, 0) = 0 \text{ on } \partial \Omega \times (0, \infty) \text{ and } u^+|_{t=0} = 0.
\]
If there exists an $R > 0$ such that $M(R) > 0$, then

$$M(p) \to \infty \text{ as } p \to \infty$$

where

$$M(p) = \text{ess sup} u(x, t), \quad Q(p) = \{\Omega \cap B(x_0, p)\} \times (0, p^2).$$

As an immediate consequence we have

COROLLARY. If the $u$ in the theorem is bounded from above, then $u \leq 0$ on $\Omega \times (0, \infty)$.

REMARK. The results of the theorem and corollary and the proof given in §3 below are also true for subsolutions of the equation (1). As the definition $u$ is a subsolution if besides (5) it satisfies the following:

$$\int_{t'}^{t''} \int_G \{-v_t u + \nabla v \cdot A(x, t, u, \nabla u) - v B(x, t, u, \nabla u) - v B(x, t, u, \nabla w) - v B(x, t, u, \nabla w)\} \, dx \, dt$$

$$+ \int_G v(x, t)u(x, t) \bigg|_{t=t''} \, dx \leq 0,$$

$$\forall (t', t'') \subset (0, T), \quad v \in W^{1,1}_2(0, T, L^2(G)) \cap L^2(0, T, W^{1,1}_2(G))$$

and $v \geq 0$.

2. Preliminaries.

LEMMA 1. Suppose $G$ is a bounded domain in $E^n$, $T > 0$ is a definite value and $u$ satisfies (5) and (1)'. If there exists a constant $M > 0$ such that

$$u-M)^+ \in L^2(0, T, W^{1,1}_2(G)) \quad \text{and} \quad (u-M)^+|_{t=0} = 0$$

then

$$\text{ess sup} u(x, t) \leq M.$$

Proof. If the statement were not true, there would be a

$$M' = \text{ess sup} u > M \quad (M' = \infty \text{ is not exclusive}).$$

By (7), we have for any $k \in (M, M')$

$$(u-k)^+ \in L^2(0, T, W^{1,1}_2(G)) \quad \text{and} \quad (u-k)^+|_{t=0} = 0.$$
Hence it follows by the imbedding inequality in $L^2(0, T, \dot{W}^1_2(G))$ that

$$\left( \int_0^T \int_G |(u - k)^+|^q \, dx \, dt \right)^{2/q} \leq C(n) \| (u - k)^+ \|_{G \times (0, T)}$$

where $q = 2(1 + 2/n)$ and

$$\| (u - k)^+ \|_{G \times (0, T)} = \text{ess sup}_{G \times (0, T)} \int_G |(u - k)^+|^2 \, dx + \int_0^T \int_G |\nabla (u - k)^+|^2 \, dx \, dt.$$  

We assume temporarily that $(u - k)^+ \in W^1_2(0, T, L^2(G))$; then $v = (u - k)^+$ can be taken as a test function. Substituting $v$ into (1)' and integrating by parts with respect to $t$, we have by the use of (2) that

$$\int_G |(u - k)^+|^2 \, dx + \int_0^T \int_G |\nabla (u - k)^+|^2 \, dx \, dt \leq C \int_0^t \int_G b(x, t)(u - k)^+ |\nabla (u - k)^+| \, dx \, dt,$$

where the constant $C > 0$ depends only on $n$ and $\kappa_0$. However, we cannot guarantee $(u - k)^+ \in W^1_2(0, T, L^2(G))$ when $u$ is the function in Lemma 1. What we have to do now is to extend $(u - k)^+$ to $G \times (-\infty, 0)$ by letting $(u - k)^+ = 0$ and instead of $v$ we take

$$v' = \frac{1}{h} \int_t^{t+h} (u - k)^+ \, d\tau$$

as the test function. Repeating the above process again we obtain (9) by letting $h \to 0$ in the last result.

Since the two terms on the left-hand side of (9) are all non-negative, each of them does not exceed that on the right-hand side. Taking their supremums for $t \in (0, T)$, we have

$$\| (u - k)^+ \|_{G \times (0, T)} \leq C \int_0^T \int_G (u - k)^+ |\nabla (u - k)^+| \, dx \, dt,$$

where we absorb the $\| b(x, t) \|_{L^\infty}$ into the constant $C$. Considering that the effective integral domain in (10) is only $\{G \times (0, T)\} \cap$
{k < u < M'}}, we then have by Hölder inequality that

\[ \int_0^T \int_G (u - k)^+ |\nabla(u - k)^+| \, dx \, dt \]

\[ \leq \varepsilon(k, M') \left( \int_0^T \int_G |(u - k)^+|^q \, dx \, dt \right)^{1/q} \]

\[ \cdot \left( \int_0^T \int_G |\nabla(u - k)^+|^2 \, dx \, dt \right)^{1/2} \]

\[ \leq C(n) \varepsilon(k, M') \||u - k||_{G \times (0, T)} \]

where

\[ \varepsilon(k, M') = \left( \int_0^T \int_{G \cap \{k < u < M'\}} dx \, dt \right)^{1/(n+2)} \]

Combining (10) with (11) we get

\[ 1 \leq C(n) \varepsilon(k, M'), \]

where the constant \( C(n) > 0 \) is independent of \( k \). So, we have \( \varepsilon(k, M') \to 0 \) as \( k \to M' \) because

\[ \int \int_{\{G \times (0, T)\} \cap \{k < u < M'\}} dx \, dt \to 0 \quad \text{as} \quad k \to M'. \]

Hence, the contradiction is obtained by (12). \( \square \)

For simplicity we write \( B(\rho) = B(0, \rho) \).

**Lemma 2.** Suppose \( \rho_0 > \rho_1 > 0 \), \( S \subset B(\rho_0) \setminus B(\rho_1) \) and

\[ \operatorname{meas} S \geq \theta \operatorname{meas} \{B(\rho_0) \setminus B(\rho_1)\}, \quad \theta \in (0, 1). \]

Suppose \( u \in W^1_p(B(\rho_0) \setminus B(\rho_1)), p \geq 1 \) and \( u = 0 \) on \( S \). Then

\[ \int_{B(\rho_0) \setminus B(\rho_1)} |u|^p \, dx \leq C \left( n, p, \theta, \frac{\rho_0}{\rho_1} \right) \rho_0^p \int_{B(\rho_0) \setminus B(\rho_1)} |\nabla u|^p \, dx. \]

*Lemma 2 is a variety of Theorem 3.6.5, in Morrey [9] and it can be proved by the same method.*

**Lemma 3 [10].** Let \( f(t) \) be a non-negative bounded function defined for \( 0 \leq r' \leq t \leq r \). If

\[ f(t) \leq A(s - t)^{-\alpha} + B + \theta f(s), \quad \forall r' \leq t < s \leq r \]
where $A, B, \alpha, \theta$ are non-negative constants and $\theta \in (0, 1)$, then there exists a constant $C$ depending only on $\alpha$ and $\theta$ such that
\[
f(r) \leq C(A(R - r)^{-\alpha} + B), \quad \forall r' \leq r < R \leq r.
\]

3. Proof of the theorem. Without loss of generality, let $x_0$ be the origin. We can rewrite the condition (3) as
\[(3)' \quad |b(x, t)| \leq K|x|^{-1} \quad \text{as } |x| \geq 1,
\]
where $K$ is a positive constant.

Let $\rho \geq \max(R, 1)$, $0 \leq \rho_2 < \rho_1 < \rho_0 \leq \rho$ and let $\zeta(x) = \zeta(|x|)$ be a piecewise linear and continuous function of $|x|$ satisfying
\[
\zeta(x) = \begin{cases} 
0, & \text{as } |x| \leq 2\rho - \rho_1 \text{ or } |x| \geq 4\rho + \rho_1, \\
1, & \text{as } 2\rho - \rho_2 \leq |x| \leq 4\rho + \rho_2.
\end{cases}
\]
Then
\[
|\nabla \zeta(x)| \leq (\rho_1 - \rho_2)^{-1}.
\]
The function $u$ in the theorem as the generalized solution satisfying (5) and (6) is locally bounded from above on $(\Omega \cup \partial \Omega) \times (0, \infty)$ [11]. Therefore
\[
M(\rho) = \operatorname{ess} \sup_{Q(\rho)} u(x, t) < \infty, \quad Q(\rho) = \{\Omega \cap B(\rho)\} \times (0, \rho^2).
\]
On $Q(5\rho)$ let
\[
\begin{align*}
w(x, t) &= \ln \frac{M(5\rho) + \varepsilon}{M(5\rho) + \varepsilon - u^+}, \quad \varepsilon > 0, \\
v(x, t) &= \frac{\zeta^2(x)(w - k)^+}{M(5\rho) + \varepsilon - u^+}, \quad k \geq 0.
\end{align*}
\]
Because of the boundedness of $u$ on $Q(5\rho)$, we have
\[
\begin{align*}
w &\in L_2(0, 25\rho^2, W_{2,1}^1(\Omega \cap B(5\rho))) \cap L_{\infty}(Q(5\rho)), \\
w &= 0 \quad \text{on } \{\partial \Omega \cap B(5\rho)\} \times (0, 25\rho^2) \cup \{t = 0\}
\end{align*}
\]
and
\[
\begin{align*}
v &\in L_2(0, 25\rho^2, W_{2,1}^1(\Omega \cap B(5\rho))) \cap L_{\infty}(Q(5\rho)), \quad v|_{t=0} = 0.
\end{align*}
\]
Suppose $v \in W_{2,1}^1(0, 25\rho^2, L_2(\Omega \cap B(5\rho)))$ (otherwise, we add a limit process to arrive at the same result). Such $v$ can be taken as a test
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Substituting it into (1)' yields

\[ 0 = \int_0^t \int_{\Omega \cap B(5\rho)} \left\{ \zeta^2 \left( \frac{1}{2} [(w - k)^+]^2 \right) t \right. \]
\[ + \left[ \frac{\zeta^2 \nabla (w - k)^+}{M(5\rho) + \varepsilon - u^+} \right. \]
\[ + \left. \frac{\zeta^2 (w - k)^+ \nabla u^+}{(M(5\rho) + \varepsilon - u^+)^2} + \frac{(w - k)^+ \zeta \nabla \zeta}{M(5\rho) + \varepsilon - u^+} \right] \cdot A \]
\[ + \left. \frac{\zeta^2 (w - k)^+ B}{M(5\rho) + \varepsilon - u^+} \right\} dx \, dt, \quad t \in (0, 25\rho^2). \]

By virtue of the appearance of \( \zeta(x) \) and \( (w - k)^+ \) in (16) the effective integral domain is only

\[ \{ \Omega \cap (B(4\rho + \rho_1) \setminus B(2\rho - \rho_1)) \times (0, t) \} \cap \{ w > k \}, \]

on which \( u^+ > 0 \) because of (14). By the use of (2) it follows from (16) that

\[ \frac{1}{2} \int_{\Omega \cap B(5\rho)} \zeta^2 [(w - k)^+]^2 \, dx \]
\[ + \kappa_0 \int_0^t \int_{\Omega \cap B(5\rho)} \right( \zeta^2 |\nabla (w - k)^+|^2 + \zeta^2 (w - k)^+ |\nabla (w - k)^+|^2 \right) \, dx \, dt \]
\[ \leq \int_0^t \int_{\Omega \cap B(5\rho)} (w - k)^+ [2\zeta |\nabla \zeta| \kappa_1 + \zeta^2 b(x, t)] |\nabla (w - k)^+| \, dx \, dt. \]

With the aid of Young’s inequality it follows from the inequality above that

\[ \int_{\Omega \cap B(5\rho)} \zeta^2 [(w - k)^+]^2 \, dx + \int_0^t \int_{\Omega \cap B(5\rho)} \zeta^2 |\nabla (w - k)^+|^2 \, dx \, dt \]
\[ \leq C \int_0^t \int_{\Omega \cap B(4\rho + \rho_1) \setminus B(2\rho - \rho_1)} (w - k)^+ [|\nabla \zeta|^2 + \zeta^2 |b(x, t)|^2] \, dx \, dt \]
\[ \leq C \left( \frac{1}{(\rho_1 - \rho_2)^2} + \frac{1}{\rho^2} \right) \int_0^t \int_{\Omega \cap B(4\rho + \rho_1) \setminus B(2\rho - \rho_1)} (w - k)^+ \, dx \, dt, \]

where the last inequality in (18) is obtained by the fact that (3)' holds on the effective integral domain (17) and the constant \( C > 0 \) depends
only on $n$, $\kappa_0$, $\kappa_1$ and $K$. Extend $w$ by taking $w(x, t) = 0$ as $x \not\in \Omega$. We have from (4)

$$\text{meas}\{B(4\rho + \rho_1)\setminus B(2\rho - \rho_1)\} \cap \{(w - k)^+ = 0\} \geq (1 - \theta) \text{meas}\{B(4\rho + \rho_1)\setminus B(2\rho - \rho_1)\}.$$ 

For $p = 1, 2$ applying Lemma 2 to $(w - k)^+$ on $B(4\rho + \rho_1)\setminus B(2\rho - \rho_1)$, we obtain

$$(19)' \quad \int_{\Omega \cap B(4\rho + \rho_1)\setminus B(2\rho - \rho_1)} (w - k)^+ \, dx \leq C(n, \theta) \rho \int_{\Omega \cap B(4\rho + \rho_1)\setminus B(2\rho - \rho_1)} |\nabla (w - k)^+| \, dx$$

and

$$(19)'' \quad \int_{\Omega \cap B(4\rho + \rho_1)\setminus B(2\rho - \rho_1)} [(w - k)^+]^2 \, dx \leq C(n, \theta) \rho^2 \int_{\Omega \cap B(4\rho + \rho_1)\setminus B(2\rho - \rho_1)} |\nabla (w - k)^+|^2 \, dx$$

respectively. It follows from (18) and (19)' that

$$(20) \int_{\Omega \cap B(5\rho)} \xi^2[(w - k)^+]^2 \, dx + \int_0^t \int_{\Omega \cap B(5\rho)} \xi^2 |\nabla (w - k)^+|^2 \, dx \, dt \leq C \left[ \frac{1}{(\rho_1 - \rho_2)^2} + \frac{1}{\rho^2} \right] \rho \int_{\Omega \cap B(4\rho + \rho_1)\setminus B(2\rho - \rho_1)} |\nabla (w - k)^+| \, dx \, dt$$

$$\leq C \left[ \frac{1}{(\rho_1 - \rho_2)^2} + \frac{1}{\rho^2} \right]^2 \rho^2 \int_{\Omega \cap B(4\rho + \rho_1)\setminus B(2\rho - \rho_1)} \chi(k) \, dx \, dt$$

$$+ \frac{1}{4} \int_{\Omega \cap B(4\rho + \rho_1)\setminus B(2\rho - \rho_1)} |\nabla (w - k)^+|^2 \, dx \, dt$$

where the constant $C > 0$ depends only on $n$, $\kappa_0$, $\kappa_1$, $K$ and $\theta$, and $\chi(k)$ is the characteristic function of the set $\{w > k\}$. Taking the supremum in (20) for $t \in (0, \rho^2)$ we get
(21) \[
\operatorname{ess\sup}_{x \in (0, \rho^2)} \int_{\Omega \cap B(5\rho)} \zeta^2 [(w - k)^+]^2 \, dx \\
+ \int_0^{\rho^2} \int_{\Omega \cap B(5\rho)} \zeta^2 |\nabla (w - k)^+|^2 \, dx \, dt \\
\leq C \left[ \frac{1}{(\rho_1 - \rho_2)^2} + \frac{1}{\rho^2} \right] \rho^2 \\
\cdot \int_0^{\rho^2} \int_{\Omega \cap B(4\rho + \rho_1) \setminus B(2\rho - \rho)} \chi(k) \, dx \, dt \\
+ \frac{1}{2} \int_0^{\rho^2} \int_{\Omega \cap B(4\rho + \rho_1) \setminus B(2\rho - \rho)} |\nabla (w - k)^+|^2 \, dx \, dt.
\]

According to the definition of \( \zeta(x) \) it is obvious that

(22) \[
\int_0^{\rho^2} \int_{\Omega \cap B(4\rho + \rho_2) \setminus B(2\rho - \rho_2)} |\nabla (w - k)^+|^2 \, dx \, dt \\
\leq \int_0^{\rho^2} \int_{\Omega \cap B(5\rho)} \zeta^2 |\nabla (w - k)^+|^2 \, dx \, dt.
\]

On account of \( C \) being independent of \( \rho_1 \) and \( \rho_2 \) and the arbitrariness of \( \rho_1 \) and \( \rho_2 \) in \( 0 \leq \rho_2 < \rho_1 \leq \rho \), combining (22) with (21) and applying Lemma 3 we obtain

(23) \[
\int_0^{\rho^2} \int_{\Omega \cap B(4\rho + \rho_2) \setminus B(2\rho - \rho_2)} |\nabla (w - k)^+|^2 \, dx \, dt \\
\leq C \left[ \frac{1}{(\rho_1 - \rho_2)^2} + \frac{1}{\rho^2} \right] \rho^2 \\
\cdot \int_0^{\rho^2} \int_{\Omega \cap B(4\rho + \rho_1) \setminus B(2\rho - \rho)} \chi(k) \, dx \, dt,
\]

where the constant \( C > 0 \) is independent of \( \rho_1 \), \( \rho_2 \) and \( \rho \). Therefore, if \( 0 \leq \rho_1 < \rho_0 \leq \rho \), it follows from (23) by replacing \( \rho_1 \) and \( \rho_2 \) by \( \rho_0 \) and \( \rho_1 \) respectively that

(24) \[
\int_0^{\rho^2} \int_{\Omega \cap B(4\rho + \rho_1) \setminus B(2\rho - \rho)} |\nabla (w - k)^+|^2 \, dx \, dt \\
\leq C \left[ \frac{1}{(\rho_0 - \rho_1)^2} + \frac{1}{\rho^2} \right] \rho^2 \\
\cdot \int_0^{\rho^2} \int_{\Omega \cap B(4\rho + \rho_0) \setminus B(2\rho - \rho_0)} \chi(k) \, dx \, dt.
\]
From (15) we have

\[
\left( \int_0^{\rho^2} \int_{\Omega \cap B(4\rho + \rho_0) \setminus B(2\rho - \rho_0)} |(w - k)^+|^q \, dx \, dt \right)^{2/q} 
\leq C(n) \left\{ \text{ess sup} \int_{t \in (0, \rho^0)} \int_{\Omega \cap B(5\rho)} \zeta^2 [(w - k)^+]^2 \, dx \right.
\]
\[
+ \int_0^{\rho^2} \int_{\Omega \cap B(5\rho)} \zeta^2 |\nabla (w - k)^+|^2 \, dx \, dt
\]
\[
+ \int_0^{\rho^2} \int_{\Omega \cap B(4\rho + \rho_1) \setminus B(2\rho - \rho_1)} |\nabla \zeta|^2 [(w - k)^+]^2 \, dx \, dt \right\}.
\]

Collecting (19)', (21), (24) and (25), it follows that

\[
\left( \int_0^{\rho^2} \int_{\Omega \cap B(4\rho + \rho_0) \setminus B(2\rho - \rho_0)} |(w - k)^+|^q \, dx \, dt \right)^{2/q} 
\leq C \left[ \frac{1}{(\rho_1 - \rho_2)^2} + \frac{1}{\rho^2} \right]^2 \rho^2 \int_0^{\rho^2} \int_{\Omega \cap B(4\rho + \rho_1) \setminus B(2\rho - \rho_1)} \chi(k) \, dx \, dt
\]
\[
+ C \left[ \frac{1}{(\rho_0 - \rho_1)^2} + \frac{1}{\rho^2} \right]^2 \rho^2 \int_0^{\rho^2} \int_{\Omega \cap B(4\rho + \rho_0) \setminus B(2\rho - \rho_0)} \chi(k) \, dx \, dt,
\]

where \( C > 0 \) depends only on \( n, \kappa_0, \kappa_1, K \) and \( \theta \). In particular, let \( 0 \leq \rho'' = \rho_2 < \rho_0 = \rho' < \rho \) and \( \rho_1 = \frac{1}{2}(\rho' + \rho'') \). The inequality above can be rewritten as follows:

\[
\left( \int_0^{\rho^2} \int_{\Omega \cap B(4\rho + \rho'') \setminus B(2\rho - \rho'')} |(w - k)^+|^q \, dx \, dt \right)^{2/q} 
\leq C \left[ \frac{1}{(\rho' - \rho'')^2} + \frac{1}{\rho^2} \right]^2 \rho^2 \int_0^{\rho^2} \int_{\Omega \cap B(4\rho + \rho') \setminus B(2\rho - \rho')} \chi(k) \, dx \, dt.
\]

Take for \( \nu = 0, 1, 2, \ldots \)

\[
\rho_{\nu} = \rho / 2^\nu, \quad k_{\nu} = H - H / 2^\nu \quad (H > 0 \text{ will be special}),
\]

\[
A_{\nu} = \int_0^{\rho^2} \int_{\Omega \cap B(4\rho + \rho_{\nu}) \setminus B(2\rho - \rho_{\nu})} \chi(k_{\nu}) \, dx \, dt.
\]
Since the constant $C$ in (26) is independent of $\rho'$, $\rho''$ and $k$, replace $\rho'$, $\rho''$ by $\rho_\nu$, $\rho_{\nu+1}$, and $k$ by $k_\nu$, it follows from (26) that

$$(k_{\nu+1} - k_\nu)^2 A_{\nu+1}^{2/q}$$

$$\leq \left( \int_0^\rho \int_{\Omega \cap B(4\rho + \rho_{\nu+1}) \setminus B(2\rho - \rho_{\nu+1})} |(w - k_\nu)^+|^q \, dx \, dt \right)^{2/q}$$

$$\leq C \left[ \frac{1}{(\rho_\nu - \rho_{\nu+1})^2} + \frac{1}{\rho^2} \right]^2 \rho^2 A_\nu, \quad \nu = 0, 1, 2, \ldots,$$

namely,

$$(27) \quad A_{\nu+1}^{2/q} \leq C \left( \frac{2^{\nu+1}}{H} \right)^2 \left[ \left( \frac{2^{\nu+1}}{\rho} \right)^2 + \frac{1}{\rho^2} \right] \rho^2 A_\nu$$

$$\leq C 2^8 \cdot 2^{6\nu} (H \rho)^{-2} A_\nu, \quad \nu = 0, 1, 2, \ldots.$$  

For $\nu = 0$ we have

$$(28) \quad A_0 = \int_0^\rho \int_{\Omega \cap B(5\rho) \setminus B(\rho)} \chi(0) \, dx \, dt \leq \text{meas } B(5) \rho^{n+2}.$$  

As long as we assume $H > 0$ so large that

$$(29) \quad \left( \frac{C \cdot 2^8}{H} \right)^{1+2/(n+2)} \left[ \text{meas } B(5) \right]^{2/(n+2)} \leq \delta,$$

$$2^{6(1+2/(n+2))} \delta^{2/(n+2)} = 1,$$

from (27), (28) and (29) it can be shown by induction that

$$A_\nu \leq \delta^\nu A_0, \quad \nu = 1, 2, \ldots.$$  

Let $\nu \to \infty$; then

$$\int_0^\rho \int_{\Omega \cap B(4\rho) \setminus B(2\rho)} \chi(H) \, dx \, dt = 0,$$

which implies

$$\text{ess sup}_{\{\Omega \cap B(4\rho) \setminus B(2\rho)\} \times (0, \rho^2)} w \leq H.$$  

According to the definition of $w$ we have

$$\text{ess sup}_{\{\Omega \cap B(4\rho) \setminus B(2\rho)\} \times (0, \rho^2)} u^+ \leq [M(5\rho) + \varepsilon](1 - e^{-H}).$$  

Let $\varepsilon \to 0$; then

$$\text{ess sup}_{\{\Omega \cap B(4\rho) \setminus B(2\rho)\} \times (0, \rho^2)} u^+ \leq M(5\rho)(1 - e^{-H}).$$
It follows from Lemma 1 that

\[(30) \quad M(p) = \operatorname{ess sup} u < \operatorname{ess sup} u \in \Omega \cap B(p) \times (0, p^2) \quad \Omega \cap B(3p) \times (0, p^2) \]
\[\leq \operatorname{ess sup} u^+ \leq M(5p)(1 - e^{-H}).\]

We see from (29) that \(H\) is determined by constants \(C\) and \(n\); hence, \(H\) is independent of \(p\).

Now, suppose \(\rho_0 = \max(R, 1)\). For any \(\rho \geq \rho_0\) there exists an integer \(\nu\) such that \(5^\nu \rho_0 \leq \rho < 5^{\nu+1} \rho_0\). Iterating by (30) we get

\[M(\rho) \geq M(5^\nu \rho_0) \geq (1 - e^{-H})^{-\nu} M(\rho_0) \]
\[\geq (1 - e^{-H})M(\rho_0)(1 - e^{-H})^{-\log_5(\rho/\rho_0)} \]
\[= (1 - e^{-H})M(\rho_0)(\rho/\rho_0)^\lambda \geq (1 - e^{-H})M(R)(\rho/\rho_0)^\lambda,\]
\[\lambda = \log_5(1 - e^{-H})^{-1} > 0, \quad \rho \geq \rho_0.\]

Thus, \(M(\rho) \to \infty\) as \(\rho \to \infty\) whenever \(M(R) > 0\). The proof of the theorem is completed.

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