

# Pacific Journal of Mathematics

**ON THE REPRESENTATION OF THE DETERMINANT OF  
HARISH-CHANDRA'S  $C$ -FUNCTION OF  $SL(n, \mathbb{R})$**

SHOHEI TANAKA

ON THE REPRESENTATION  
 OF THE DETERMINANT OF HARISH-CHANDRA'S  
 C-FUNCTION OF  $SL(n, \mathbb{R})$

SHOHEI TANAKA

**This paper gives the explicit representation of the determinant of the Harish-Chandra  $C$ -function of  $SL(n, \mathbb{R})$  ( $n \geq 3$ ) and some application.**

**1. Introduction.** Let  $G$  be a semisimple Lie group with finite center,  $K$  a maximal compact subgroup of  $G$ . Let  $\theta$  be the Cartan involution of  $G$  fixing  $K$ . Let  $P$  be a cuspidal parabolic subgroup and  $P = MAN$  its Langlands decomposition. For  $\sigma$  in  $\widehat{M}_d$  and  $\gamma$  in  $\widehat{K}$ , we set  $\tau = (\gamma, \sigma)$  and denote the space of the  $\tau_M$ -spherical cusp forms on  $M$  by  ${}^0\mathfrak{C}_M(M, \tau_M)$ . The Harish-Chandra  $C$ -function  $C_{\overline{P}|P}(1 : \nu)$  has important information in the representation theory.

In the determinant of  $C_{\overline{P}|P}(1 : \nu)$ , L. Cohn has proved the following results.

**THEOREM** (see [2], p. 129). *There exist functions  $\mu_1, \dots, \mu_r \in \mathfrak{a}^*$  and constants  $p_{i,j}, q_{i,j}$  ( $i = 1, \dots, r, j = 1, \dots, j_i$ ) depending on  $\tau$  such that*

$$\det C_{\overline{P}|P}(1 : \nu) = \text{const} \cdot \prod_{i=1}^r \prod_{j=1}^{j_i} \frac{\Gamma(\frac{\langle \nu, \alpha_i \rangle}{2\langle \mu_i, \alpha_i \rangle} + q_{i,j})}{\Gamma(\frac{\langle \nu, \alpha_i \rangle}{2\langle \mu_i, \alpha_i \rangle} + p_{i,j})},$$

where  $\alpha_1, \dots, \alpha_r$  are reduced  $\mathfrak{a}$ -roots.

He gives a conjecture that the constants  $p_{i,j}$  and  $q_{i,j}$  are rational numbers and depending linearly on the highest weight of the irreducible components of  $\tau$ .

Let  $G$  be  $SL(n, \mathbb{R})$  and  $P$  the minimal parameter subgroup of  $G$ . In the case that  $n = 2$ , the Harish-Chandra  $C$ -function and determinant of it are well known explicitly. If  $n$  is 3 or 4, in [4] Eguchi and the author give the explicit formula of the determinant of Harish-Chandra's  $C$ -function of  $G$ , which solves Cohn's conjecture affirmatively. The purpose of this paper is to extend the result in [4]

to  $G$  and apply it to the study of the reducibility of  $\pi_{p,\sigma,\nu}$ . The application does not give any new result but it gives another proof of Speth-Vogan's reducibility condition ([12], [13]).

The author would like to thank Professor M. Eguchi and Professor K. Okamoto for their helpful suggestions and encouragement.

**2. Notation and preliminaries.** Let  $G$  be a semisimple Lie group with finite center and  $\mathfrak{g}$  its Lie algebra. Let  $\mathfrak{l}$  be a maximal compact subalgebra of  $\mathfrak{g}$ ,  $\mathfrak{g} = \mathfrak{l} + \mathfrak{p}$  the corresponding Cartan decomposition and  $\theta$  the Cartan involution defining the decomposition. We introduce an inner product  $B_\theta$  on  $\mathfrak{g}$  in the standard way such that  $B_\theta(X, Y) = -B(X, \theta Y)$ , where  $B$  is the Killing form on  $\mathfrak{g}$ . Let  $\mathfrak{a}_p$  be a maximal abelian subgroup of  $\mathfrak{p}$ . We fix an order in the dual space  $(\mathfrak{a}_p)^*$  of  $\mathfrak{a}_p$ , and put  $\mathfrak{n}_p = \sum_{\alpha > 0} \mathfrak{g}_\alpha$ , where  $\mathfrak{g}_\alpha$  denotes the root space of the  $\mathfrak{a}_p$ -root  $\alpha$ , and we let  $\mathfrak{v}_p = \theta \mathfrak{n}_p$ . Then we have an Iwasawa decomposition  $\mathfrak{g} = \mathfrak{l} + \mathfrak{a}_p + \mathfrak{n}_p$  of  $\mathfrak{g}$ . Let  $\mathfrak{m}_p = Z_{\mathfrak{l}}(\mathfrak{a}_p)$  the centralizer of  $\mathfrak{a}_p$  in  $\mathfrak{l}$ .

We now let  $K = N_G(\mathfrak{l})$  be the normalizer of  $\mathfrak{l}$  in  $G$ ,  $M_p = Z_K(\mathfrak{a}_p)$  the centralizer of  $\mathfrak{a}_p$  in  $K$  and  $M'_p = N_K(\mathfrak{a}_p)$  the normalizer of  $\mathfrak{a}_p$  in  $K$ . Let  $A_p$ ,  $N_p$  and  $V_p$  be the analytic subgroups of  $G$  corresponding to  $\mathfrak{a}_p$ ,  $\mathfrak{n}_p$  and  $\mathfrak{v}_p$  respectively.

Any conjugate of  $\mathfrak{m}_p \oplus \mathfrak{a}_p \oplus \mathfrak{n}_p$  is called a minimal parabolic subalgebra, and any Lie subalgebra  $\mathfrak{s}$  that contains a minimal parabolic subalgebra is called parabolic. Then  $\mathfrak{s}$  has a Langlands decomposition (relative to  $\theta$ )  $\mathfrak{s} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ . Here  $\mathfrak{m} \oplus \mathfrak{a} = Z_{\mathfrak{g}}(\mathfrak{a})$ , and we can impose an ordering on the  $\mathfrak{a}$ -roots so that  $\mathfrak{n}$  is built from the positive  $\mathfrak{a}$ -roots. Let  $\mathfrak{v} = \theta \mathfrak{n}$ . If  $\mathfrak{a}_M$  is a maximal abelian subspace of  $\mathfrak{m} \cap \mathfrak{p}$ , then  $\mathfrak{a} \oplus \mathfrak{a}_M$  is a maximal abelian subspace of  $\mathfrak{p}$  and can be taken as  $\mathfrak{a}_p$  in our theory. When we introduce an ordering on the  $\mathfrak{a}_p$ -roots so that  $\mathfrak{a}$  comes before  $\mathfrak{a}_M$ , then the positive  $\mathfrak{a}$ -roots are the nonzero restriction to  $\mathfrak{a}$  of the positive  $\mathfrak{a}_p$ -roots. The sum of the root spaces for the positive  $\mathfrak{a}_p$ -roots that vanish on  $\mathfrak{a}$  is denoted by  $\mathfrak{n}_M$ .

Let  $M_0$ ,  $A$ ,  $A_M$ ,  $N$ ,  $V$ ,  $N_M$  be analytic subgroups corresponding to  $\mathfrak{m}$ ,  $\mathfrak{a}$ ,  $\mathfrak{a}_M$ ,  $\mathfrak{n}$ ,  $\mathfrak{v}$ ,  $\mathfrak{n}_M$  respectively and put  $M = M_0 M_p$ . The group  $P = MAN$  is a parabolic subgroup. The subgroups in our discussion have the following properties (see e.g. [8]).

- (1.1) (1)  $MA = Z_G(\mathfrak{a})$ ,  $MAN = N_G(\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n})$ ,  $MAN$  is closed, and  $(m, a, n) \in M \times A \times N \rightarrow man \in MAN$  is a diffeomorphism onto,
- (2)  $\theta|_{\mathfrak{m}}$  is a Cartan involution of  $\mathfrak{m}$ , and  $K_M = K \cap M$  is the corresponding maximal compact subgroup of  $M$ ,

- (3)  $M = K_M A_M N_M$  is an Iwasawa decomposition of  $M$ ,
- (4)  $A_p = A_M A$  and  $N_p = N_M N$  diffeomorphically,
- (5)  $G = KMAN$  with the  $KM$ ,  $A$  and  $N$  components unique,
- (6)  $K \cap MA = K \cap M$ ,
- (7)  $V \cap MAN = \{1\}$ ,
- (8) the  $M_p$  group for  $M$  equals the  $M_p$  group for  $G$ .

Two parabolic subgroups with the same  $MA$  are associated. The choices for  $N$  are in obvious one-to-one correspondence with the Weyl chambers. Let  $M' = N_K(\mathfrak{a})M$  and  $W(\mathfrak{a}) = M'/M$ . If  $w$  is in  $M'$ , then  $w$  acts on characters of  $A$  and representations of  $M$  by

$$w \cdot \nu(\mathfrak{a}) = \nu(w^{-1}aw), \quad w \cdot \sigma(m) = \sigma(w^{-1}mw).$$

Then  $W(\mathfrak{a})$  acts on characters of  $A$  and classes of representations of  $M$ . An  $\mathfrak{a}$ -root is said to be reduced if  $r\alpha$  is not a root for  $0 < r < 1$  ( $r \in \mathbb{R}$ ). Let  $\beta$  be a reduced  $\mathfrak{a}$ -root in the dual  $\mathfrak{a}^*$ ,  $H_\beta$  the corresponding member of  $\mathfrak{a}$  under the identification set up by  $B_\theta$ , and  $(H_\beta)^\perp$  the orthogonal complement of  $\mathbb{R} \cdot H_\beta$  in  $\mathfrak{a}$ . We set  $\mathfrak{n}^{(\beta)} = \sum_{c>0} \mathfrak{g}_{c\beta}$ ,  $\mathfrak{v}^{(\beta)} = \theta \mathfrak{n}^{(\beta)} = \sum_{c<0} \mathfrak{g}_{c\beta}$  and let  $\mathfrak{g}^{(\beta)}$  be the subalgebra of  $\mathfrak{g}$  generated by  $\mathfrak{n}^{(\beta)}$  and  $\mathfrak{v}^{(\beta)}$ . Let  $N^{(\beta)}$ ,  $V^{(\beta)}$  and  $G^{(\beta)}$  be the analytic subgroups corresponding to  $\mathfrak{n}^{(\beta)}$ ,  $\mathfrak{v}^{(\beta)}$  and  $\mathfrak{g}^{(\beta)}$  respectively.

Let  $\widehat{K}$  and  $\widehat{M}$  be the set of all equivalence classes of the irreducible unitary representations of  $K$  and  $M$  respectively. For each  $\sigma \in \widehat{M}$  we fix a representation  $(\tilde{\sigma}, H^{\tilde{\sigma}})$  in  $\sigma$  and, abusing notation, we use also  $\sigma$  for  $\tilde{\sigma}$ . For each  $\gamma$  in  $\widehat{K}$  we fix an element  $(\pi_\gamma, H^\gamma)$  in  $\gamma$ .

We recall the generalized principal series representations. Let  $P = MAN$  be a parabolic subgroup and  $\rho_P = \frac{1}{2} \cdot \sum_{\alpha>0} (\dim \mathfrak{g}_\alpha) \alpha$ . Let  $\sigma$  be in  $\widehat{M}$  and  $\nu$  in  $\mathfrak{a}_\mathbb{C}^*$  (the complexification of  $\mathfrak{a}^*$ ). Let  $C_{P,\sigma,\nu}(G)$  be the space of all continuous functions  $f$  from  $G$  to  $H^\sigma$  such that

$$f(xman) = e^{-(\nu+\rho_P)(\log a)} \sigma(m)^{-1} f(x) \quad (x \in G).$$

Let  $h^{P,\sigma,\nu}$  be the completion of  $C_{P,\sigma,\nu}(G)$  by the norm

$$\|f\|^2 = \int_K \|f(k)\|^2 dk \quad (f \in C_{P,\sigma,\nu}(G)).$$

The representation  $\pi_{P,\sigma,\nu}$  is given by

$$\pi_{P,\sigma,\nu}(g)f(x) = f(g^{-1}x) \quad (g \in G)'.$$

The compact picture is the restriction of the induced picture to  $K$ . Here the dense subspace  $C_\sigma(K)$  is

$$\{f: K \rightarrow H^\sigma \mid f \text{ is continuous and } f(km) = \sigma(m)^{-1}f(k)\}$$

and is independent of  $\nu$ . According to the decomposition  $G = KMAN$  of (1.1) each  $g \in G$  is written as

$$g = \kappa(g)\mu(g)(\exp H(g))n(g), \\ (\kappa(g) \in K, \mu(g) \in M, H(g) \in \mathfrak{a}, n(g) \in N).$$

Then representation is given by

$$\pi_{P,\sigma,\nu}(g)f(k) = e^{-(\nu+\rho_p)(H(g^{-1}k))}f(\kappa(g^{-1}k)).$$

If  $\gamma$  is in  $\widehat{K}$ , the projection operator  $E_\gamma$  defined by

$$E_\gamma = d(\gamma)\overline{\chi}_\gamma * f \quad (f \in C_\sigma(K)),$$

where  $d(\gamma)$  and  $\chi_\gamma$  denote the dimension and the character of  $\gamma$  respectively. For  $\gamma$  in  $\widehat{K}$ , we put

$$H^{P,\sigma,\nu} = \{f \in H^{P,\sigma,\nu} \mid E_\gamma f = f\}.$$

**3. Some lemmas for the intertwining operators.** Let  $P = MAN'$  and  $P' = M'AN'$  be associated parabolic subgroups and let  $\sigma$  be in  $\widehat{M}$  and  $\nu$  in  $\mathfrak{a}_\mathbb{C}^*$ . For  $f$  in  $C_{P,\sigma,\nu}(G)$  we set

$$A(P' : P : \sigma : \nu)f(x) = \int_{V \cap N'} f(xv) dv,$$

where  $V = \theta N$  and  $dv$  is the normalized Haar measure on  $V \cap N'$  by

$$\int_{V \cap N'} e^{-2\rho_p(H(v))} dv = 1.$$

The operator  $A(P' : P : \sigma : \nu)$  is called the intertwining operator. In this section we shall describe the properties of the intertwining operators, which are well known results (see e.g. [8]).

The inner product  $B_\theta$  on  $\mathfrak{g}$  induces an inner product on the dual  $\mathfrak{a}^*$  of  $\mathfrak{a}$ , which we denote by  $\langle \cdot, \cdot \rangle$ .

Let  $\rho_M$  be half the sum of the positive  $\mathfrak{a}_M$ -roots. Since the parabolic subgroup  $P = MAN$  contains the minimal parabolic subgroup  $P_p = M_p A_p N_p$  such that  $\mathfrak{a}_p = \mathfrak{a} \oplus \mathfrak{a}_M$ .

For each  $\mathfrak{a}$ -root  $\beta$ , set  $C_\beta = \max\{\rho_M(H_\alpha)\}$ , where the maximum is taken over all  $\mathfrak{a}_p$ -roots  $\alpha$  satisfying  $\alpha|_{\mathfrak{a}} = \beta$ .

LEMMA 3.1. *Let  $P = MAN$  and  $P' = M'AN'$  be associated parabolic subgroups and suppose that  $\langle \operatorname{Re} \nu, \beta \rangle > C_\beta$  for every  $\mathfrak{a}$ -root  $\beta$  such that  $\mathfrak{g}_\beta \subset \mathfrak{n} \cap \mathfrak{v}'$ . Then the integral  $A(P' : P : \sigma : \nu)f(x)$  ( $x \in G, f \in C_{P, \sigma, \nu}(G)$ ) is a convergent. Moreover, if  $f$  is a  $K$ -finite function in the compact picture of  $\pi_{P, \sigma, \nu}$  then the integral has an analytic continuation to a global meromorphic function in  $\nu$ .*

LEMMA 3.2. *If  $\sigma$  is in  $\widehat{M}$  and  $\nu$  in  $\mathfrak{a}_\mathbb{C}^*$ , then we have*

$$A(P' : P : \sigma : \nu)\pi_{P, \sigma, \nu}(g) = \pi_{P', \sigma, \nu}(g)A(P' : P : \sigma : \nu)$$

for all  $g$  in  $G$ .

For  $w$  in  $M'$ , let  $R(w)f(x) = f(xw)$ . Then it follows from Lemma 3.2 that

$$(3.1) \quad A_P(w, \sigma, \nu) = R(w)A(w^{-1}Pw : P : \sigma : \nu)$$

satisfies

$$\pi_{P, w\sigma, w\nu}(\cdot)A_P(w, \sigma, \nu) = A_P(w, \sigma, \nu)\pi_{P, \sigma, \nu}(\cdot).$$

LEMMA 3.3. *Let  $P = MAN$  and  $P' = M'AN'$  be associated parabolic subgroups. Then there exists a scalar-valued function  $\eta(P' : P : \sigma : \nu)$  meromorphic in  $\nu$  such that*

$$(3.2) \quad A(P : P' : \sigma : \nu)A(P' : P : \sigma : \nu) = \eta(P' : P : \sigma : \nu)I.$$

Let  $P = MAN$  and  $P' = M'AN'$  be as in Lemma 3.3. A sequence  $P_i = MAN_i$  ( $0 \leq i \leq r$ ) is called a string from  $P$  to  $P'$  if there are  $P$ -positive reduced  $\mathfrak{a}$ -roots  $\beta_i$  ( $1 \leq i \leq r$ ) such that

$$V_{i-1} \cap N_i = V^{(\beta_i)} \text{ or } N^{(\beta_i)} \quad (1 \leq i \leq r),$$

$$P_0 = P \quad \text{and} \quad P_r = P'.$$

The string  $P_i$  from  $P$  to  $P'$  is called minimal if we have

$$V_{i-1} \cap N_i = V^{(\beta_i)} \quad (1 \leq i \leq r),$$

$$P_0 = P \quad \text{and} \quad P_r = P'.$$

LEMMA 3.4. *Suppose that  $P = MAN$  and  $P' = MAN'$  are associated parabolic subgroups and  $P_i = MAN_i$  ( $0 \leq i \leq r$ ) is a minimal string from  $P$  to  $P'$ , with associated  $P$ -positive reduced  $\mathfrak{a}$ -roots  $\{\beta_i\}$ . Then*

- (1) *the set  $\{\beta_i\}$  is characterized as the set of reduced  $\mathfrak{a}$ -roots  $\alpha$  that are positive for  $P$  and negative for  $P'$ .*
- (2)  *$r$  is characterized as the number of  $\mathfrak{a}$ -roots described in (1).*
- (3) *the intertwining operators satisfy*

$$A(P' : P : \sigma : \nu) = A(P_r : P_{r-1} : \sigma : \nu) \cdots A(P_1 : P_0 : \sigma : \nu).$$

LEMMA 3.5. *Let  $P = MAN$  be a parabolic subgroup, let  $\sigma$  be in  $\widehat{M}$  and  $\nu$  in  $\mathfrak{a}_{\mathbb{C}}^*$  such that  $\text{Re } \nu$  is in the open positive Weyl chamber. Then  $\pi_{P, \sigma, \nu}$  has a unique irreducible quotient  $J(p, \sigma, \nu)$  and  $J(P, \sigma, \nu)$  is isomorphic with the image of the intertwining operator  $A(\bar{P} : P : \sigma : \nu)$  on  $H^{P, \sigma, \nu}$ , where  $\bar{P} = MAV$ .*

**4. The  $B_{\gamma}^{\sigma}$ -functions.** In this section we shall work only with minimal parabolic subgroups and omit the subscripts  $p$ . Let  $P, P'$  be associated minimal parabolic subgroups and let  $\gamma$  be in  $\widehat{K}$ ,  $\sigma$  in  $\widehat{M}$  and  $A$  in  $\text{Hom}_M(V^{\gamma}, H^{\sigma})$ , where  $V^{\gamma}$  denotes the representation space of  $\gamma$ . For  $\nu$  in  $\mathfrak{a}_{\mathbb{C}}^*$ ,  $v$  in  $V^{\gamma}$ , let

$$L_P(A, v, \nu)(\text{kan}) = e^{-(\nu + \rho_p)(\log a)} A(\pi_{\gamma}(k^{-1})v)$$

for  $k$  in  $K$ ,  $a$  in  $A$ ,  $n$  in  $N$ . Then an easy computation shows that  $L_P(A, v, \nu)$  is in  $H_{\gamma}^{P, \sigma, \nu}$ . Furthermore the map

$$V^{\gamma} \otimes \text{Hom}_M(V^{\gamma}, H^{\sigma}) \rightarrow H_{\gamma}^{P, \sigma, \nu},$$

given by  $v \otimes A \rightarrow L_P(A, v, \nu)$  is a bijective  $K$ -intertwining operator. Set

$$A_{\gamma}(P' : P : \sigma : \nu) = A(P' : P : \sigma : \nu)|_{H_{\gamma}^{P, \sigma, \nu}}.$$

Then we have  $A_{\gamma}(P' : P : \sigma : \nu)$  is in  $\text{Hom}_K(H_{\gamma}^{P', \sigma, \nu}, H_{\gamma}^{P, \sigma, \nu})$ .

LEMMA 4.1. (See [4], [15].) *If  $\nu$  is in  $\mathfrak{a}_{\mathbb{C}}^*$  and  $\langle \text{Re } \nu, \alpha \rangle > 0$  for all  $P$ -positive roots  $\alpha$  then we have*

$$A_{\gamma}(P' : P : \sigma : \nu)L_P(A, v, \nu) = L_P(A \circ B_{\gamma}(P' : P : \nu), v, \nu),$$

where

$$B_{\gamma}(P' : P : \nu) = \int_{V \cap N'} \pi_{\gamma}(\kappa(v))^{-1} e^{-(\nu + \rho_p)(H(v))} dv.$$

Furthermore  $B_\gamma(P' : P : \nu)$  satisfies the following conditions,

- (1)  $B_\gamma(P' : P : \nu)$  is absolutely convergent.
- (2)  $B_\gamma(P' : P : \nu)$  is in  $\text{End}(V^\gamma)$  and satisfies

$$B_\gamma(P' : P : \nu)\pi_\gamma(m)B_\gamma(P' : P : \nu) \quad (m \in M).$$

Now we define  $B_\gamma^\sigma$ -functions. If  $\sigma$  is in  $\widehat{M}$ , we denote the  $\sigma$ -component of  $V^\gamma$  by  $V_\sigma^\gamma$ . Let

$$B_\gamma^\sigma(P' : P : \nu) = B_\gamma(P' : P : \nu)|_{V_\sigma^\gamma}.$$

Then  $B_\gamma^\sigma(P' : P : \nu)$  is in  $\text{End}(V_\sigma^\gamma)$  and from Lemma 3.1 it has an analytic continuation to a global meromorphic function in  $\nu$ . Particularly,  $B_\gamma(\overline{P} : P : \nu)$  is called Harish-Chandra's  $C$ -function.

**COROLLARY 4.2.** *If  $w$  is in  $M'$ ,  $\nu$  is in  $\mathfrak{a}_C^*$  such that  $\langle \text{Re } \nu, \alpha \rangle > 0$  for all  $P$ -positive roots  $\alpha$ , then we have*

$$A_P(w, \sigma, \nu)L_P(A, v, \nu) = L_{P'}(A \circ B_\gamma(P, w, \nu) \circ \pi_\gamma(w)^{-1}, v, w\nu),$$

where

$$B_\gamma(P, w, \nu) = B_\gamma(w^{-1}Pw : P : \nu).$$

Let  $w$  be in  $M'$  such that

$$(4.1) \quad w^{-1}Pw = \overline{P} \quad \text{and} \quad w = w_r w_{r-1} \cdots w_1,$$

where each  $w_i$  ( $1 \leq i \leq r$ ) is the reflection with respect to the  $P$ -simple  $\alpha$ -root  $\gamma_i$  and  $r$  is the length of  $w$ . Then by the relation

$$(4.2) \quad A_P(w, \sigma, \nu) = A_P(w_r, w_{r-1} \cdots w_1 \sigma, w_{r-1} \cdots w_1 \nu) \cdots A_P(w_1, \sigma, \nu)$$

and Corollary 4.2, we have

$$(4.3) \quad B_\gamma^\sigma(\overline{P} : P : \nu) = B_\gamma^\sigma(P, w_1, \nu)\pi_\gamma^\sigma(w_1)B_\gamma^{w_1\sigma}(P, w_2, w_1\nu) \cdots B_\gamma^{w_{r-1}\cdots w_1\sigma}(P, w_r, w_{r-1} \cdots w_1\nu) \cdot \pi_\gamma^{w_{r-1}\cdots w_1\sigma}(w_r)\pi_\gamma^{w\sigma}(w).$$

In connection with Lemma 4.1 we have the following proposition.

**PROPOSITION 4.3.** *Let  $w$  be as above. We set*

$$P_i = (w_i w_{i-1} \cdots w_1)^{-1} P (w_i w_{i-1} \cdots w_1) \quad (0 \leq i \leq r)$$

and

$$\beta_i = (w_{i-1} \cdots w_1)^{-1} \gamma_i \quad (1 \leq i \leq r).$$

Then  $P_i$  ( $0 \leq i \leq r$ ) is a minimal string  $P$  to  $\bar{P}$ , with associated reduced  $P$ -positive  $\mathfrak{a}$ -roots  $\{\beta_i\}$  and we have

$$\begin{aligned} A(\bar{P} : P : \sigma : \nu) \\ = A(P_r : P_{r-1} : \sigma : \nu) A(P_{r-1} : P_{r-2} : \sigma : \nu) \cdots A(P_1 : P_0 : \sigma : \nu). \end{aligned}$$

*Proof.* By an easy computation, we have

$$(4.4) \quad V_{i-1} \cap N_i = V^{(\beta_i)} \quad (1 \leq i \leq r).$$

We shall prove reduced  $\mathfrak{a}$ -roots  $\beta_i$  ( $1 \leq i \leq r$ ) are  $P$ -positive. For an integer  $i$  such that  $1 \leq i \leq r$  we set

$$[N_i] = \{\alpha \mid \alpha \text{ is a } P\text{-positive and } P_i\text{-positive reduced } \mathfrak{a}\text{-root}\}$$

and denote the cardinality of  $[N_i]$  by  $n_i$ . Since  $r$  is  $n_0$ , we have

$$(4.5) \quad n_{i-1} - n_i = 1 \quad (1 \leq i \leq r).$$

From (4.4) and (4.5),  $\beta_i$  ( $1 \leq i \leq r$ ) are  $P$ -positive. Therefore  $P_i$  ( $1 \leq i \leq r$ ) is the minimal string with associated  $P$ -positive reduced  $\mathfrak{a}$ -roots  $\{\beta_i\}$ . The other assertion follows from Lemma 3.4(3).

**5. The  $B_\gamma$ -function in the  $SL(n, \mathbb{R})$  case.** We shall specialize to  $SL(n, R)$  the notation described in the previous sections. Our notation is as follows. Let  $G$  be in  $SL(n, R)$ , the group of  $n$ -by- $n$  real matrices  $g$  of determinant one. Let

$$\begin{aligned} \theta &= - \text{ transpose,} \\ K &= SO(n), \\ \mathfrak{a} &= \text{ the vector space of the diagonal matrices of trace } 0, \\ M &= \{m \in G \mid m = \text{diag}(m_1, \dots, m_n) \text{ and } m_i = \pm 1 \ (1 \leq i \leq n)\}, \\ A &= \exp \mathfrak{a}, \\ N &= \{n \in G \mid n \text{ is the sum of the identity and strictly upper} \\ &\quad \text{triangular matrices}\}, \\ P &= MAN. \end{aligned}$$

Then  $P$  is a minimal parabolic subgroup of  $G$ . Let  $e_j$  ( $1 \leq j \leq n$ ) be the linear functional on  $\mathfrak{a}_\mathbb{C}$  that picks out the  $j$ th diagonal entry and set  $\alpha_j = e_j - e_{j+1}$  ( $1 \leq j \leq n - 1$ ). Then simple  $\mathfrak{a}$ -roots are  $\alpha_j$  ( $1 \leq j \leq n - 1$ ). We denote the simple reflection with respect to  $\alpha_j$  by  $s_{\alpha_j}$ .



and

$$B_\gamma(\bar{P}, s_{\alpha_j}, \nu)v_\mu = \text{Const} \cdot \alpha(-\nu_j, \sqrt{-1}\mu(H_{[(j+1)/2]}))v_\mu,$$

where

$$\alpha(s, n) = \frac{\gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})}{\Gamma(\frac{s+1-n}{2})\Gamma(\frac{s+1+n}{2})} \quad (s \in \mathbb{C}, n \in \mathbb{Z}).$$

*Proof.* From Lemma 5.1, we have

$$(5.1) \quad \begin{aligned} B_\gamma(P, s_{\alpha_j}, \nu)v_\mu &= \text{Const} \cdot \int_{-\infty}^{\infty} f(x)^{-(\nu_j+1)} \pi_\gamma(f(x)^{-1}k_j(x))^{-1}v_\mu dx. \end{aligned}$$

We note that

$$\pi_\gamma(\exp tH_{[(j+1)/2]})v_\mu = e^{t\mu(H_{[(j+1)/2]})}v_\mu \quad (t \in \mathbb{R}).$$

Putting  $\cos t = f(x)^{-1}$ ,  $\sin t = x/f(x)$ , we obtain that

$$\pi_\gamma(f(x)^{-1}k_j(x))^{-1}v_\mu = \left(\frac{1 + \sqrt{-1}x}{f(x)}\right)^{-\sqrt{-1}\mu(H_{[(j+1)/2]})}v_\mu.$$

Thus (5.1) is equal to

$$\text{Const} \cdot \int_{-\infty}^{\infty} f(x)^{-(\nu_j+1)} \left(\frac{1 + \sqrt{-1}x}{f(x)}\right)^{-\sqrt{-1}\mu(H_{[(j+1)/2]})} dx v_\mu.$$

Therefore, the assertion of the lemma follows from the next proposition.

**PROPOSITION 5.3** (cf. A.3 in [3]). *Suppose that  $s$  is a complex number and  $n$  an integer. Then we have*

$$\int_{-\infty}^{\infty} (1 + x^2)^{-(s+1)/2} \left(\frac{1 - \sqrt{-1}x}{(1 + x^2)^{1/2}}\right)^n dx = \frac{\sqrt{-1}\Gamma(\frac{s}{2})\Gamma(-\frac{s+1}{2})}{\Gamma(\frac{s+1-n}{2})\Gamma(\frac{s+1+n}{2})}.$$



Therefore (5.3) is equal to

$$\begin{aligned} &= \text{Const} \cdot \int_{-\infty}^{\infty} f(x)^{-((-C_{j/2} \cdot \nu))_{j-1} + 1} \pi_{\gamma}(f(x)^{-1} k_{j-1}(x))^{-1} dx \\ &= B_{\gamma}(P, s_{\alpha_{j-1}}, -(C_{j/2} \cdot \nu)). \end{aligned}$$

This proves the lemma.

**6.  $M$ -isotypic components of  $\gamma$ .** In this section we shall describe the  $M$ -isotypic components of  $\gamma$  in  $\widehat{K}$ . We fix  $\gamma$  in  $\widehat{K}$ . Let  $\sigma$  be in  $\widehat{M}$  and denote the  $\sigma$ -isotypic component by  $V_{\sigma}^{\gamma}$ . Then we have

$$V_{\gamma} = \sum_{\sigma \in \widehat{M}} V_{\sigma}^{\gamma} \quad (\text{direct sum}).$$

Let  $P_{\sigma}$  be the projection map  $V^{\gamma} \rightarrow V_{\sigma}^{\gamma}$ . From Lemma 4.1(2), for  $P, P'$  in  $\mathcal{P}(A)$  and  $\nu$  in  $\mathfrak{a}_{\mathbb{C}}^*$  we have

$$(6.1) \quad B_{\gamma}(P' : P : \nu) P_{\sigma} = P_{\sigma} B_{\gamma}(P' : P : \nu).$$

Let  $\mu$  be a weight of  $V^{\gamma}$  and let  $[\mu]$  denote the equivalence class of  $\mu$ , which is defined as follows;  $\mu'$  is in  $[\mu]$  if and only if  $\mu(H_l)$  is equal to  $\pm \mu'(H_l)$  for any integer  $l$  such that  $1 \leq l \leq [n/2]$ . Let  $\check{\gamma}$  be the set of the equivalence classes  $[\mu]$  and  $V^{\gamma, \mu}$  the  $\mu$ -weight space of  $V^{\gamma}$ . Set

$$V_{\sigma}^{\gamma, \mu} = P_{\sigma}(V^{\gamma, \mu}) \quad \text{and} \quad V_{\sigma}^{\gamma, [\mu]} = \sum_{\mu' \in [\mu]} V_{\sigma}^{\gamma, \mu'}.$$

**LEMMA 6.1.** *In the above situation we have*

$$V_{\sigma}^{\gamma} = \sum_{[\mu] \in \check{\gamma}} V_{\sigma}^{\gamma, [\mu]} \quad (\text{direct sum}).$$

*Proof.* Let  $m$  be a positive integer and  $\mu_k$  ( $1 \leq k \leq m$ ) a weight of  $V^{\gamma}$  such that  $\mu_k$  is not equivalent to  $\mu_{k'}$ , if  $k \neq k'$ . Suppose  $v_{[\mu_k]}$  ( $1 \leq k \leq m$ ) are in  $V_{\sigma}^{\gamma, [\mu_k]}$  which satisfy the following relation,

$$\sum_{k=1}^m v_{[\mu_k]} = 0.$$

To prove the lemma, it is enough to show that

$$v_{[\mu_k]} = 0 \quad (1 \leq k \leq m).$$

We shall prove by induction on  $m$ . If  $m = 1$  it is clear. Suppose the assertion is true for  $1 \leq m < t$ . We check the case that  $m = t$ . Suppose that

$$(6.2) \quad \sum_{k=1}^t v_{[\mu_k]} = 0.$$

Then for an integer  $i$  such that  $1 \leq i \leq l$  we have

$$0 = (B_\gamma(P, w_{2i-1}, \nu) - \alpha(\nu_{2i-1}, \sqrt{-1}\mu_1(H_i))) \left( \sum_{k=1}^t v_{[\mu_k]} \right),$$

by Lemma 5.2 and (6.1)

$$= \sum_{k=2}^t (\alpha(\nu_{2i-1}, \sqrt{-1}\mu_k(H_i)) - \alpha(\nu_{2i-1}, \sqrt{-1}\mu_1(H_i))) v_{[\mu_k]}.$$

Applying the inductive hypothesis, we have

$$(\alpha(\nu_{2i-1}, \sqrt{-1}\mu_k(H_i)) - \alpha(\nu_{2i-1}, \sqrt{-1}\mu_1(H_i))) v_{[\mu_k]} = 0 \quad (2 \leq k \leq t).$$

Since  $[\mu_k] \neq [\mu_1]$  ( $2 \leq k \leq t$ ), we obtain

$$v_{[\mu_k]} = 0 \quad (2 \leq k \leq t).$$

From (6.2) we have

$$v_{[\mu_k]} = 0 \quad (1 \leq k \leq t).$$

This proves the lemma.

LEMMA 6.2. Suppose  $\nu$  is in  $\alpha_{\mathbb{C}}^*$  and  $j$  an integer such that  $1 \leq j \leq n - 1$ . Then  $B_\gamma^\sigma(P, s_{\alpha_j}, \nu)$  are diagonalizable and

(1) if  $j \equiv 1 \pmod{2}$ , we have

$$\begin{aligned} & \deg(B_\gamma^\sigma(P, \alpha_j, \nu)) \\ &= \text{Const} \cdot \prod_{[\mu] \in \check{\gamma}} \alpha(\nu_j, \sqrt{-1}\mu(H_{[(j+1)/2]}))^{d(\gamma, \sigma, [\mu])}, \end{aligned}$$

(2) if  $j \equiv 0 \pmod{2}$ , we have

$$\begin{aligned} & \det(B_\gamma^\sigma(P, \alpha_j, \nu)) \\ &= \text{Const} \cdot \prod_{[\mu] \in \check{\gamma}} \alpha(\nu_j, \sqrt{-1}\mu(H_{[(j+1)/2]}))^{d(\gamma, C_{[(j+1)/2]} \cdot \sigma, [\mu])}, \end{aligned}$$

where  $d(\gamma, \sigma, [\mu])$  is the dimension of the space  $V_\sigma^{\gamma, [\mu]}$  and  $C_{j/2} \cdot \sigma$  ( $1 \leq j \leq n-1, j \equiv 0 \pmod{2}$ ) are defined by

$$C_{j/2} \cdot \sigma(m) = \sigma(C_{j/2}^{-1} \cdot m \cdot C_{j/2}) \quad (m \in M).$$

*Proof.* The relation (1) follows immediately from Lemma 5.2, Lemma 6.1 and (6.2). The relation (2) follows from Lemma 5.4 and (1). The first assertion is obvious.

**7. The determinant of the C-function.** Let  $w$  be in  $W$  and satisfy that

$$w^{-1}Pw = \bar{P} \quad \text{and} \quad w = w_r w_{r-1} \cdots w_1,$$

where each  $w_i$  ( $1 \leq i \leq r$ ) is the reflection with respect to the simple  $\alpha$ -root  $\alpha_{j_i}$  and  $r$  is the length of  $w$ . Then we have

$$A(\bar{P} : P : \sigma : \nu) = R(w)A_P(w, \sigma, \nu).$$

By the relation

$$(7.1) \quad A_P(w, \sigma, \nu) = A_P(w_r, w_{r-1} \cdots w_1 \sigma, w_{r-1} \cdots w_1 \nu) \cdots A_P(w_2, w_1 \sigma, w_1 \nu) \cdot A_P(w_1, \sigma, \nu)$$

and by Corollary 4.2, we have for  $\gamma$  in  $\widehat{K}$

$$(7.2) \quad B_\gamma(\bar{P} : P : \nu) = B_\gamma(P, w_1, \nu) \pi_\gamma(w_1) B_\gamma(P, w_2, w_1 \nu) \cdots B_\gamma(P, w_r, w_{r-1} \cdots w_1 \nu) \cdot \pi_\gamma(w_r) \pi_\gamma(w).$$

For each integer  $j$  such that  $1 \leq j \leq n-1$ , we define  $\tilde{C} \cdot \sigma$  ( $\in \widehat{M}$ ) as follows:

if  $j \equiv 0 \pmod{2}$ ,

$$\tilde{C}_j \cdot \sigma = C_j \cdot (w_{j-1} \cdots w_1 \sigma),$$

if  $j \equiv 1 \pmod{2}$ ,

$$\tilde{C}_j \cdot \sigma = w_{j-1} \cdots w_1 \sigma.$$

**THEOREM 7.1.** *Suppose  $\nu$  is in  $\mathfrak{a}_\mathbb{C}^*$ ,  $\gamma$  in  $\widehat{K}$  and  $\sigma$  in  $\widehat{M}$ . Then we have*

$$\begin{aligned} & \det(B_\gamma^\sigma(\bar{P} : P : \nu)) \\ &= \text{Const} \cdot \prod_{i=1}^r \prod_{[\mu] \in \tilde{\gamma}} \alpha(2 \cdot \langle \nu, \beta_i \rangle \cdot \langle \beta_i, \beta_i \rangle^{-1}, \sqrt{-1} \mu(H_{[(j_i+1)/2]}))^{d_{i, [\mu]}} \end{aligned}$$

where  $\beta_i$  ( $1 \leq i \leq r$ ) are as in Corollary 3.3 and

$$d_{i, [\mu]} = d(\gamma, \tilde{C}_{j_i} \cdot \sigma, [\mu]).$$

*Proof.* From (7.2), we have

$$\begin{aligned} B_\gamma^\sigma(\bar{P} : P : \nu) &= B_\gamma^\sigma(P, w_1, \nu) \pi_\gamma^\sigma(w_1) B_\gamma^{w_1\sigma}(P, w_2, w_1\nu) \\ &\quad \dots B_\gamma^{w_{r-1}\dots w_1\sigma}(P, w_r, w_{r-1} \dots w_1\nu) \\ &\quad \cdot \pi_\gamma^{w_{r-1}\dots w_1\sigma}(w_r) \pi_\gamma^{w\sigma}(w), \end{aligned}$$

where  $\rho_\gamma^\sigma(w')$  ( $w' \in W$ ) is  $\pi_\gamma(w')|_{V_\sigma^\gamma}$ .

Let  $i$  be an integer such that  $0 \leq i \leq n-1$  and  $\sigma'$  in  $\widehat{M}$  such that  $V_{\sigma'}^\gamma \neq \{0\}$ . We extend  $B_\gamma^{\sigma'}(w_i, \cdot)$  to an operator  $\tilde{B}_\gamma^{\sigma'}(w_i, \cdot)$  of  $V^\gamma$  by

$$(7.3) \quad \tilde{B}_\gamma^{\sigma'}(w_i, \cdot) = \begin{cases} B_\gamma^{\sigma'}(w_i, \cdot) & \text{on } V_{\sigma'}^\gamma, \\ \text{identity} & \text{on } V_{\sigma''}^\gamma \ (\sigma'' \neq \sigma') \end{cases}$$

and define

$$(7.4) \quad \begin{aligned} \tilde{B}_\gamma^\sigma(\bar{P} : P : \nu) &= \tilde{B}_\gamma^\sigma(P, w_1, \nu) \pi_\gamma^\sigma(w_1) \tilde{B}_\gamma^{w_1\sigma}(P, w_2, w_1\nu) \\ &\quad \dots \tilde{B}_\gamma^{w_{r-1}\dots w_1\sigma}(P, w_r, w_{r-1} \dots w_1\nu) \\ &\quad \cdot \pi_\gamma^{w_{r-1}\dots w_1\sigma}(w_r) \pi_\gamma^{w\sigma}(w). \end{aligned}$$

Then we have

$$(7.5) \quad \tilde{B}_\gamma^\sigma(\bar{P} : P : \nu)|_{V_\sigma^\gamma} = B_\gamma^\sigma(\bar{P} : P : \nu)$$

and

$$(7.6) \quad \det(\tilde{B}_\gamma^\sigma(\bar{P} : P : \nu)) = d_1 \cdot \det(B_\gamma^\sigma(\bar{P} : P : \nu)),$$

where  $d_1$  is a nonzero constant which is independent of  $\nu$ . On the other hand, from (7.3) and (7.4) we have

$$(7.7) \quad \begin{aligned} \det(\tilde{B}_\gamma^\sigma(\bar{P} : P : \nu)) &= d_2 \cdot \det(B_\gamma^\sigma(P, w_1, \nu)) \\ &\quad \dots \det(B_\gamma^{w_{r-1}\dots w_1\sigma}(P, w_r, w_{r-1} \dots w_1\nu)), \end{aligned}$$

where  $d_2$  is a constant such that  $|d_2| = 1$ . Therefore, from (7.6) and (7.7) we have

$$\begin{aligned} \det(B_\gamma^\sigma(\bar{P} : P : \nu)) &= \text{Const} \cdot \det(B_\gamma^\sigma(P, w_1, \nu)) \\ &\quad \dots \det(B_\gamma^{w_{r-1}\dots w_1\sigma}(P, w_r, w_{r-1} \dots w_1\nu)) \end{aligned}$$

by Lemma 6.2

$$= \text{Const} \cdot \prod_{i=1}^r \prod_{[\mu] \in \check{\gamma}} \alpha((w_{i-1} \cdots w_1 \nu)_{j_i}, \sqrt{-1} \mu(H_{[(j+1)/2]}))^{d_{i, [\mu]}}$$

by Proposition 4.3

$$= \text{Const} \cdot \prod_{i=1}^r \prod_{[\mu] \in \check{\gamma}} \alpha(2 \cdot \langle \nu, \beta_i \rangle \cdot \langle \beta_i, \beta_i \rangle^{-1}, \sqrt{-1} \mu(H_{[(j+1)/2]}))^{d_{i, [\mu]}}.$$

This proves the theorem.

**8. The reducibility of  $\pi_{P, \sigma, \nu}$  in the nonsingular case.** Let  $\nu$  be in  $\mathfrak{a}_\mathbb{C}^*$  such that  $\langle \text{Re } \nu, \alpha \rangle \neq 0$  for all  $P$ -positive roots. In this section we shall describe a necessary and sufficient condition for that  $\pi_{P, \sigma, \nu}$  is reducible.

Let  $\beta$  be a reduced  $P$ -positive  $\mathfrak{a}$ -root and  $G^{(\beta)}$  as in §1. In this case  $G^{(\beta)}$  is isomorphic to  $\text{SL}(2, \mathbb{R})$  and we can put

$$M \cap G^{(\beta)} = \{e, m_\beta\},$$

where  $e$  is the identity matrix. Let  $\sigma$  be in  $\widehat{M}$ . Since  $M$  is abelian and any element of  $M$  is of order two,  $\sigma(m)$  ( $m \in M$ ) is a scalar operator and the scalar is  $\pm 1$ . We define integers  $\sigma_\beta$  such that  $0 \leq \sigma_\beta \leq 1$  by

$$\sigma(m_\beta) = (-1)^{\sigma_\beta} \cdot I,$$

where  $I$  is the identity operator.

LEMMA 8.1. *Let  $\sigma$  be in  $\widehat{M}$ ,  $\gamma$  in  $\widehat{K}$  and  $\mu$  a weight of  $V^\gamma$ . Let  $j$  be an integer such that  $0 \leq j \leq n - 1$  and  $j \equiv 1 \pmod{2}$ . Suppose that*

$$(8.1) \quad \sqrt{-1} \mu(H_{[(j+1)/2]}) - \sigma_{\alpha_j} \equiv 1 \pmod{2}.$$

Then we have

$$V_\sigma^{\gamma, [\mu]} = \{0\}.$$

*Proof.* Let  $v$  be in  $V_\sigma^{\gamma, [\mu]}$ . By an easy computation, we have

$$\pi_\gamma(m_{\alpha_j})v = \sqrt{-1} \mu(H_{[(j+1)/2]})v.$$

On the other hand, we have

$$\pi_\gamma(m_{\alpha_j})v = \sigma_{\alpha_j} v.$$

Therefore, from (8.1) the element  $\nu$  must be zero. This proves the lemma.

LEMMA 8.2. *Let  $\gamma$  be in  $\widehat{K}$ ,  $\sigma$  in  $\widehat{M}$  and let  $j$  be an integer such that  $1 \leq j \leq n - 1$  and  $j \equiv 1 \pmod{2}$ . If  $\nu$  is in  $\mathfrak{a}_{\mathbb{C}}^*$  such that  $\langle \text{Re } \nu, \alpha_j \rangle > 0$ , then the operator  $B_{\gamma}^{\sigma}(P, s_{\alpha_j}, \nu)$  has a nontrivial kernel if and only if*

(c1)  $\nu_j$  is an integer and  $\nu_j + 1 \equiv \sigma_{\alpha_j} \pmod{2}$ .

(c2) there exists a weight  $\mu$  of  $V^{\gamma}$  such that

$$|\sqrt{-1}\mu(H_{[(j+1)/2]})| \geq \nu_j + 1 \quad \text{and} \quad V_{\sigma}^{\gamma, [\mu]} \neq \{0\},$$

(c3) there exists a weight  $\mu'$  of  $V^{\gamma}$  such that

$$|\sqrt{-1}\mu'(H_{[(j+1)/2]})| < \nu_j + 1 \quad \text{and} \quad V_{\sigma}^{\gamma, [\mu']} \neq \{0\},$$

where  $\nu_j$  are as in §5.

*Proof.* Suppose that  $B_{\gamma}^{\sigma}(P, s_{\alpha_j}, \nu)$  has the nontrivial kernel. By Lemma 5.4, the conditions (c2), (c3) are obvious and  $\nu_j$  is an integral. Moreover, we have

$$(8.2) \quad \nu_j + 1 + \sqrt{-1}\mu(H_{[(j+1)/2]}) \equiv 0 \pmod{2}.$$

Therefore, by Lemma 8.1, we have

$$\nu_j + 1 \equiv \sigma_{\alpha_j} \pmod{2}.$$

Conversely, suppose that (c1), (c2) and (c3) are satisfied. Then from Lemma 8.1 and (c1), it follows that any weight  $\mu$  of  $V^{\gamma}$  such that  $V_{\sigma}^{\gamma, [\mu]} \neq \{0\}$  satisfies (8.2). Therefore, from Lemma 5.1, (c2) and (c3) it follows that  $B_{\gamma}^{\sigma}(P, s_{\alpha_j}, \nu)$  has the nontrivial kernel.

COROLLARY 8.3. *Let  $\gamma$  be in  $\widehat{K}$ ,  $\sigma$  in  $\widehat{M}$  and let  $j$  be an integer such that  $1 \leq j \leq n - 1$ . If  $\nu$  is in  $\mathfrak{a}_{\mathbb{C}}^*$ , such that  $\langle \text{Re } \nu, \alpha_j \rangle > 0$  then the operator  $B_{\gamma}^{\sigma}(P, s_{\alpha_j}, \nu)$  has the nontrivial kernel if and only if*

(c1)  $\nu_j$  is an integer and  $\nu_j + 1 \equiv \sigma_{\alpha_j} \pmod{2}$ ,

(c2) there exists a weight  $\mu$  of  $V^{\gamma}$  such that

$$|\sqrt{-1}\mu(H_{[(j+1)/2]})| \geq \nu_j + 1 \quad \text{and} \quad V_{\sigma}^{\gamma, [\mu]} \neq \{0\},$$

(c3) there exists a weight  $\mu'$  of  $V^{\gamma}$  such that

$$|\sqrt{-1}\mu'(H_{[(j+1)/2]})| < \nu_j + 1 \quad \text{and} \quad V_{\sigma}^{\gamma, [\mu']} \neq \{0\},$$

where  $\nu_j$  ( $1 \leq j \leq n - 1$ ) are as in §5.

*Proof.* If the integer  $j$  is odd, then the assertion is that of Lemma 6.2. Thus we may assume that  $j$  is even. By Lemma 5.4, the operator  $B_\gamma^\infty(P_p, s_{\alpha_j}, \nu)$  has the nontrivial kernel if and only if the operator  $B_\gamma^{C_{j/2} \cdot \sigma}(P, s_{\alpha_{j-1}}, -(C_{j/2} \cdot \nu))$  does also. Since

$$\langle \text{Re}(-(C_{j/2} \cdot \nu)), \alpha_j \rangle = \langle \text{Re } \nu, \alpha_{j-1} \rangle > 0,$$

we can apply Lemma 8.2 to the operator  $B_\gamma^{C_{j/2} \cdot \sigma}(P, s_{\alpha_{j-1}}, -(C_{j/2} \cdot \nu))$ .

We note that

$$(8.3) \quad (C_{j/2} \cdot \sigma)_{\alpha_j} = \sigma_{\alpha_{j-1}} \quad \text{and} \quad -(C_{j/2} \cdot \nu)_j = \nu_{j-1}.$$

Combining Lemma 8.2 and the relations (8.3) we have the assertion of the corollary.

**LEMMA 8.4.** *Let  $\nu$  be in  $\mathfrak{a}_\mathbb{C}^*$  such that  $\langle \text{Re } \nu, \alpha \rangle > 0$  for all  $P$ -positive roots  $\alpha$  and  $\sigma$  in  $\widehat{M}$ . Then  $A(\overline{P} : P : \sigma : \nu)$  has the nontrivial kernel if and only if there exists a reduced  $P$ -positive  $\mathfrak{a}$ -root  $\beta$  satisfying the following conditions:*

$$(*) \quad 2\langle \nu, \beta \rangle \cdot \langle \beta, \beta \rangle^{-1} \text{ is an integer and } 2\langle \nu, \beta \rangle \cdot \langle \beta, \beta \rangle^{-1} + 1 \equiv \sigma_\beta, \pmod{2}.$$

*Proof.* Let  $w$  be in  $M'$  such that

$$w^{-1}Pw = \overline{P} \quad \text{and} \quad w = w_r w_{r-1} \cdots w_1,$$

where each  $w_i$  ( $1 \leq i \leq r$ ) is the reflection with respect to the  $P$ -simple  $\mathfrak{a}$ -root  $\alpha_{k_i}$  ( $1 \leq k_i \neq n-1$ ) and  $r$  is the length of  $w$ . Let  $P_i$  ( $1 \leq i \leq r$ ) be the minimal string  $P$  to  $\overline{P}$ , which is described in Proposition 4.3. From Lemma 4.1 it follows that  $A(\overline{P} : P : \sigma : \nu)$  has the nontrivial kernel if and only if

(c1) there exists  $\gamma$  in  $\widehat{K}$  such that  $B_\gamma^\sigma(\overline{P} : P : \nu)$  has the nontrivial kernel.

Moreover, the condition (c1) is equivalent to

(c2) there exist  $\gamma$  in  $\widehat{K}$  and an integer  $j$  ( $1 \leq j \leq r$ ) such that  $B_\gamma^{w_{j-1} \cdots w_1 \sigma}(P, w_j, w_{j-1} \cdots w_1 \nu)$  has the nontrivial kernel.

Since we have

$$\langle w_{j-1} \cdots w_1 \nu, \alpha_j \rangle = \langle \nu, \beta_j \rangle > 0,$$

from Corollary 6.3 the condition (c2) is equivalent to

(c3) there exist  $\gamma$  in  $\widehat{K}$ , weights of  $V^\gamma \mu$ ,  $\mu'$  and an integer  $j$  ( $1 \leq j \leq r$ ) satisfying the following relations:

$$\begin{aligned}
 (8.4) \quad & 2 \cdot \langle \nu, \beta_j \rangle \cdot \langle \beta_j, \beta_j \rangle^{-1} \in \mathbb{Z}, \\
 & V_\sigma^{\gamma, [\mu]} \neq \{0\}, \quad V_\sigma^{\gamma, [\mu']} \neq \{0\}, \\
 & 2 \cdot \langle \nu, \beta_j \rangle \cdot \langle \beta_j, \beta_j \rangle^{-1} + 1 \equiv \sigma_{\alpha_{k_j}} \pmod{2}, \\
 & |\sqrt{-1}\mu(H_{k_j})| \geq 1 + 2 \cdot \langle \nu, \beta_j \rangle \cdot \langle \beta_j, \beta_j \rangle^{-1}, \\
 & |\sqrt{-1}\mu'(H_{k_j})| < 1 + 2 \cdot \langle \nu, \beta_j \rangle \cdot \langle \beta_j, \beta_j \rangle^{-1}.
 \end{aligned}$$

From Proposition 8.5, the condition (c3) is equivalent to

(c3') there exists an integer  $j$  ( $1 \leq j \leq r$ ) such that  $2 \cdot \langle \nu, \beta_j \rangle \cdot \langle \beta_j, \beta_j \rangle^{-1}$  is an integer and satisfies the relation (8.4).

Since  $\beta_j = \alpha_{k_j}$ , the assertion of the lemma follows from the condition (c3').

**PROPOSITION 8.5.** *Let  $\sigma$  be in  $\widehat{M}$  and  $k$  an integer such that  $1 \leq k \leq n - 1$  and  $k \equiv 1 \pmod{2}$ . Then for any positive integer  $l$  which satisfies (6.1), there exists  $\gamma$  in  $\widehat{K}$  such that*

$$V_\sigma^{\gamma, [\bar{\mu}]} \neq \{0\} \quad \text{and} \quad \bar{\mu}(H_{[(k+1)/2]}) = l,$$

where  $\bar{\mu}$  is the highest weight of  $V^\gamma$ .

*Proof.* Let  $\gamma$  be an element in  $\widehat{K}$  such that the highest weight of  $V^\gamma$  is  $\bar{\mu}$ . We put

$$n_j = \sqrt{-1}\bar{\mu}(H_{[(j+1)/2]}) \quad (1 \leq j \leq n - 1, j \equiv 1 \pmod{2}).$$

Then each  $n_j$  is an integer. By the representation theory of compact groups, we can choose  $\gamma$  in  $\widehat{K}$  satisfying the following conditions;

$$\begin{aligned}
 & n_k = n, \\
 & n_j \neq 0 \quad \text{and} \quad n_j - \sigma_j \equiv 0 \pmod{2} \quad (1 \leq j \leq n - 1, j \equiv 1 \pmod{2}).
 \end{aligned}$$

Let  $v_{\bar{\mu}}$  be a  $\bar{\mu}$ -weight vector. We shall prove that  $P_\sigma(v_{\bar{\mu}}) \neq 0$ . We can easily see that

$$P_\sigma(v_{\bar{\mu}}) = \prod_{\substack{1 \leq i \leq n-1 \\ i \equiv 0 \pmod{2}}} \frac{1}{2} (I + \sigma_{\alpha_i} \cdot \pi_\gamma(m_{\alpha_i}))(v_{\bar{\mu}}),$$

where  $I$  is the identity operator on  $V^\gamma$ . On the other hand, for integers  $i, j$  such that  $1 \leq i, j \leq n - 1$ ,  $i \equiv 0 \pmod{2}$  and  $j \equiv 1 \pmod{2}$  we have

$$\sqrt{-1}m_{\alpha_i} \cdot \bar{\mu}(H_{[(j+1)/2]}) = \begin{cases} -n_j & (i \leq 1 \leq j \leq i + 1), \\ n_j & \text{otherwise.} \end{cases}$$

Therefore,  $P_\sigma(v_{\bar{\mu}}) \neq 0$ . This proves the assertion of the lemma.

**THEOREM 8.7.** *Let  $\nu$  be an element in  $\mathfrak{a}_\mathbb{C}^*$  such that  $\langle \text{Re } \nu, \alpha \rangle \neq 0$  for all  $P$ -positive roots  $\alpha$  and  $\sigma$  in  $\widehat{M}$ . Then  $\pi_{P, \sigma, \nu}$  is reducible if and only if there exists a reduced  $P$ -positive  $\mathfrak{a}$ -root  $\beta$  satisfying the following conditions:*

(\*)  $2\langle \nu, \beta \rangle \cdot \langle \beta, \beta \rangle^{-1}$  is an integer and  $2\langle \nu, \beta \rangle \cdot \langle \beta, \beta \rangle^{-1} + 1 \equiv \sigma_\beta \pmod{2}$ .

*Proof.* Suppose that  $\langle \text{Re } \nu, \alpha \rangle > 0$  for all  $P$ -positive  $\mathfrak{a}$ -roots  $\alpha$ . Then by Lemma 3.5  $\pi_{P, \sigma, \nu}$  is reducible if and only if  $A(\bar{P} : P : \sigma : \nu)$  has the nontrivial kernel. Thus in this case, the assertion of the theorem follows from Lemma 8.4. In general, there exists  $w$  in  $W(\mathfrak{a})$  such that  $\langle \text{Re } w\nu, \alpha \rangle > 0$  for all  $P$ -positive  $\mathfrak{a}$ -roots. Since  $\pi_{P, \sigma, \nu}$  and  $\pi_{P, w\sigma, w\nu}$  have equivalent composition series,  $\pi_{P, \sigma, \nu}$  is reducible if and only if there exists a reduced  $P$ -positive  $\mathfrak{a}$ -root  $\beta$  such that  $w\beta$  satisfies the condition (\*). Since the inner product  $\langle \cdot, \cdot \rangle$  is  $W(\mathfrak{a})$ -invariant and  $\sigma_{w\beta} = \sigma_\beta$ , Theorem 8.6 is proved.

**9. The reducibility of  $\pi_{P, \sigma, \nu}$  in the singular cases.** Let  $\nu_0$  be in  $\mathfrak{a}_\mathbb{C}^*$  such that  $\langle \text{Re } \nu_0, \alpha \rangle \geq 0$  for all  $P$ -positive  $\mathfrak{a}$ -roots. Set

$$\Delta_{\nu_0}^+(P) = \{i \in \mathbb{N} \mid 1 \leq i \leq n - 1 \text{ and } \langle \text{Re } \nu_0, \alpha_i \rangle \neq 0\}.$$

Then we have

$$\text{Re } \nu_0 = \sum_{j \in \Delta_{\nu_0}^+(P)} b_j \omega_j,$$

where  $b_j$  ( $j \in \Delta_{\nu_0}^+(P)$ ) are positive real numbers and  $\omega_j$  ( $1 \leq j \leq n - 1$ ) in  $\mathfrak{a}_\mathbb{C}^*$  are defined by

$$\langle \alpha_i, \omega_j \rangle = \delta_{ij} \quad (1 \leq i, j \leq n - 1).$$

We take

$$\begin{aligned} \mathfrak{a}_1 &= \sum_{j \in \Delta_{\nu_0}^+(P)} \mathbb{R} \cdot H_{\omega_j}, & \mathfrak{a}_2 &= \sum_{j \in \Delta_{\nu_0}^+(P)} \mathbb{R} \cdot H_{\alpha_j}, \\ \mathfrak{n}_1 &= \sum_{\substack{\beta \in \Sigma^+ \\ \beta|_{\mathfrak{a}} \neq 0}} \mathfrak{g}_\beta, & \mathfrak{n}_2 &= \sum_{\substack{\beta \in \Sigma^+ \\ \beta|_{\mathfrak{a}} = 0}} \mathfrak{g}_\beta, \\ \mathfrak{m}_1 &= \mathfrak{m} \oplus \mathfrak{a}_2 \oplus \mathfrak{n}_2 \oplus \mathfrak{v}_2, & M_1 &= Z_K(\mathfrak{a})(M_1)_0, \\ P_1 &= M_1 A_1 N_1, & P_2 &= M A_2 N_2, \end{aligned}$$

where  $\Sigma^+$  is the set of  $P$ -positive  $\mathfrak{a}$ -roots. Then  $P_1$  is a parabolic subgroup of  $G$  and  $P_2$  is a minimal parabolic subgroup of  $M_1$ . Let us write  $\nu_0 = \nu_0^1 + \nu_0^2$  correspondingly, with  $\nu_0^1 = \nu_0|_{\mathfrak{a}_1}$  and  $\nu_0^2 = \nu_0|_{\mathfrak{a}_2}$ . From the double induction formula (see [8], p. 170),  $\text{ind}_P^G \sigma \otimes \nu_0 \otimes 1$  and  $\text{ind}_{P_1}^G (\text{ind}_{P_2}^{M_1} \sigma \otimes \nu_0^2 \otimes 1) \otimes \nu_0^1 \otimes 1$  are infinitesimally equivalent.  $\text{ind}_{P_2}^{M_1} \sigma \otimes \nu_0^2 \otimes 1$  is a tempered unitary representation of  $M_1$  and we denote it by  $\xi$ .

Set  $P' = M A \overline{N}_2 N_1$  and let  $w', w''$  be elements in  $W(\mathfrak{a})$  such that

$$(w')^{-1} P w' = P', \quad (w'')^{-1} P' w'' = \overline{P},$$

respectively. Suppose that  $w' = w'_s \cdot w'_{s-1} \cdots w'_1$  and  $w'' = w''_t \cdot w''_{t-1} \cdots w''_1$  are the minimal expressions, respectively. Let  $w = w'' \cdot w'$ . Then we have

$$w^{-1} P w = \overline{P}.$$

By Lemma 3.4, the length of  $w$  is equal to  $r + s$  and

$$w = w''_t \cdot w''_{t-1} \cdots w''_1 \cdot w'_s \cdot w'_{s-1} \cdots w'_1$$

is the minimal expression. Let  $P_i$  ( $1 \leq i \leq s+t$ ) be the minimal string  $P$  to  $\overline{P}$  with associated reduced  $P$ -positive  $\mathfrak{a}$ -roots  $\{\beta_i\}$ , which are described in Proposition 4.3.

LEMMA 9.1. *Let  $\beta_i$  ( $1 \leq i \leq s+t$ ) be defined as above. We have*

$$n_2 = \sum_{\substack{1 \leq i \leq s \\ c > 0}} \mathfrak{g}_c \beta_i.$$

Therefore, we have

$$(9.2) \quad \langle \text{Re } \nu_0, \beta_i \rangle = 0 \quad (1 \leq i \leq s),$$

$$(9.3) \quad \langle \text{Re } \nu_0, \beta_j \rangle = 0 \quad (s+1 \leq j \leq s+t).$$

Since the proof is easy, it is left to the reader.

For  $\sigma$  in  $\widehat{M}$  and  $\gamma$  in  $\widehat{K}$ , we set

$$F_{\sigma, \gamma, \nu_0} = \{i \in \mathbb{N} \mid 1 \leq i \leq s \text{ and } B_\gamma^{w'_{i-1} \cdots w'_1 \sigma}(P, w'_i, w'_{i-1} \cdots w'_1 \nu) \text{ has a singularity at } \nu_0\}.$$

LEMMA 9.2. *Set  $F_{\sigma, \nu_0} = F_{\sigma, \gamma, \nu_0}$ . Then we have*

$$F_{\sigma, \nu_0} = F_{\sigma, \gamma, \nu_0}.$$

*Proof.* The assertion of the lemma follows from Lemma 6.2 and Lemma 8.1.

LEMMA 9.3. *Let  $\nu$  be in  $\mathfrak{a}_{\mathbb{C}}^*$ ,  $\sigma$  in  $\widehat{M}$  and  $\gamma$  in  $\widehat{K}$ . Then the function*

$$\prod_{i \in F_{\sigma, \nu_0}} -\langle \nu, \beta_i \rangle^2 B_{\gamma}^{\sigma}(\overline{P} : P : \nu) B_{\gamma}^{\sigma}((\overline{P}') : \overline{P} : \nu)$$

*has no singularity at  $\nu_0$ .*

*Proof.* For any  $u$  in  $W$ , we define  $\pi_{\gamma}^{\sigma}(u)$  by  $\pi_{\gamma}(u)|_{V_{\gamma}^{\sigma}}$ . By the relation (4.3), we have

$$\begin{aligned} & B_{\gamma}^{\sigma}(\overline{P} : P : \nu) \\ &= B_{\gamma}^{\sigma}(P, w'_1, \nu) \rho_{\gamma}^{w'_1 \sigma}(w'_1) \\ & \quad \dots B_{\gamma}^{w'_{s-1} \dots w'_1 \sigma}(P, w'_s, w'_{s-1} \dots w'_1 \nu) \pi_{\gamma}^{w'_s \sigma}(w'_s) \\ & \quad \cdot B_{\gamma}^{w'_1 \sigma}(P, w''_1, w'_1 \nu) \pi_{\gamma}^{w''_1 \sigma}(w''_1) \\ & \quad \dots B_{\gamma}^{w''_{t-1} \dots w''_1 w'_1 \sigma}(P, w''_t, w''_{t-1} \dots w''_1 w'_1 \nu) \\ & \quad \cdot \pi_{\gamma}^{w''_t \sigma}(w''_t) \pi_{\gamma}^{\sigma}(w), \\ &= B_{\gamma}^{\sigma}(P, w', \nu) \pi_{\gamma}^{w' \sigma}(w') B_{\gamma}^{w' \sigma}(P, w''_1, w'_1 \nu) \pi_{\gamma}^{w''_1 \sigma}(w''_1) \\ & \quad \dots B_{\gamma}^{w''_{t-1} \dots w''_1 w' \sigma}(P, w''_t, w''_{t-1} \dots w''_1 w' \nu) \pi_{\gamma}^{w''_t \sigma}(w''_t) \pi_{\gamma}^{\sigma}(w). \end{aligned}$$

Thus we have

$$\begin{aligned} & B_{\gamma}^{\sigma}(\overline{P} : P : \nu) \\ &= B_{\gamma}^{\sigma}(P, w, \nu) \pi_{\gamma}^{w \sigma}(w) B_{\gamma}^{w \sigma}(P, w''_1, w'_1 \nu) \pi_{\gamma}^{w''_1 \sigma}(w''_1) \\ & \quad \dots B_{\gamma}^{w''_{t-1} \dots w''_1 w \sigma}(P, w''_t, w''_{t-1} \dots w''_1 w' \nu) \pi_{\gamma}^{w''_t \sigma}(w''_t) \pi_{\gamma}^{\sigma}(w) \\ & \quad \cdot B_{\gamma}^{\sigma}((\overline{P}') : \overline{P} : \nu). \end{aligned}$$

From Lemma 6.2 and Lemma 9.1, the functions

$$\begin{aligned} & B_{\gamma}^{w''_1 w' \sigma}(P, w''_1, w'_1 \nu) \\ & \quad \dots B_{\gamma}^{w''_{t-1} \dots w''_1 w' \sigma}(P, w''_t, w''_{t-1} \dots w''_1 w' \nu) \pi_{\gamma}^{w''_t \sigma}(w''_t) \pi_{\gamma}^{\sigma}(w) \end{aligned}$$

and

$$\prod_{i \in F_{\sigma, \nu_0}} \langle \nu, \beta_i \rangle B_{\gamma}^{\sigma}(P, w', \nu)$$

have no singularity at  $\nu_0$ . On the other hand, we have

$$\begin{aligned} B_\gamma^\sigma((\overline{P}') : P' : \nu) &= B_\gamma(\overline{P}, w', \nu) \\ &= B_\gamma^\sigma(\overline{P}, w'_1, \nu) \pi_\gamma^\sigma(w'_1) \cdots B_\gamma^{w'_{s-1} \cdots w'_1 \sigma}(\overline{P}, w'_s, w'_{s-1} \cdots w'_1 \nu) \\ &\quad \cdot \pi_\gamma^{w' \sigma}(w'_s) \pi_\gamma^\sigma(w') \end{aligned}$$

by Lemma 5.2,

$$\begin{aligned} (9.6) &= B_\gamma^\sigma(\overline{P}, w'_1, -\nu) \pi_\gamma^\sigma(w'_1) \cdots B_\gamma^{w'_{s-1} \cdots w'_1 \sigma}(\overline{P}, w'_s, -w'_{s-1} \cdots w'_1 \nu) \\ &\quad \cdot \pi_\gamma^{w' \sigma}(w'_s) \pi_\gamma^\sigma(w') \\ &= B_\gamma^\sigma(P, w', -\nu). \end{aligned}$$

Then the function  $\prod_{i \in F_{\sigma, \nu_0}} \langle \nu, \beta_i \rangle B_\gamma^\sigma((\overline{P}') : \overline{P} : \nu)$  also has no singularity at  $\nu_0$ . Therefore, from the relation (9.5), the function

$$\prod_{i \in F_{\sigma, \nu_0}} -\langle \nu, \beta_i \rangle^2 B_\gamma^\sigma(\overline{P} : P : \nu) B_\gamma^\sigma((\overline{P}') : \overline{P} : \nu)$$

has no singularity at  $\nu_0$ .

**COROLLARY 9.4.** *Let  $\nu$  be in  $\mathfrak{a}_\mathbb{C}^*$  and  $\sigma$  in  $\widehat{M}$ . Then the operator*

$$\prod_{i \in F_{\sigma, \nu_0}} -\langle \nu, \beta_i \rangle^2 A((\overline{P}') : \overline{P} : \sigma : \nu) A(\overline{P} : P : \sigma : \nu)$$

*has no singularity at  $\nu_0$ .*

**LEMMA 9.5.** *Let  $\nu$  be in  $\mathfrak{a}_\mathbb{C}^*$  and  $\sigma$  in  $\widehat{M}$ . Then the kernel of the operator*

$$\lim_{\nu \rightarrow \nu_0} \prod_{i \in F_{\sigma, \nu_0}} \langle \nu, \beta_i \rangle A((\overline{P}') : \overline{P} : \sigma : \nu)$$

*is equal to  $\{0\}$ .*

*Proof.* It is enough to show that for any  $\gamma$  in  $\widehat{K}$ , the kernel of the operator

$$\lim_{\nu \rightarrow \nu_0} \prod_{i \in F_{\sigma, \nu_0}} \langle \nu, \beta_i \rangle^2 B_\gamma^\sigma((\overline{P}') : \overline{P} : \nu)$$

is equal to  $\{0\}$ . The assertion of the lemma follows from Lemma 6.2 and (9.6).

THEOREM 9.6. Let  $\nu$  be in  $\mathfrak{a}_{\mathbb{C}}^*$ ,  $\sigma$  in  $\widehat{M}$ . Then we have

$$\mathrm{Im} \left( \lim_{\nu \rightarrow \nu_0} \prod_{i \in F_{\sigma, \nu_0}} \langle -\nu, \beta_i \rangle A(\overline{P} : P : \sigma : \nu) \right) \simeq \mathrm{Im}(A(\overline{P}_1 : P_1 : \xi : \nu_0^1)),$$

(infinitesimally equivalent).

*Proof.* We have

$$\begin{aligned} & \lim_{\nu \rightarrow \nu_0} \prod_{i \in F_{\sigma, \nu_0}} -\langle \nu, \beta_i \rangle A((\overline{P}') : \overline{P} : \sigma : \nu) \lim_{\nu' \rightarrow \nu_0} \prod_{i \in F_{\sigma, \nu_0}} -\langle \nu', \beta_i \rangle \\ & \quad \cdot A(\overline{P} : P : \sigma : \nu') \\ &= \lim_{\nu \rightarrow \nu_0} \prod_{i \in F_{\sigma, \nu_0}} -\langle \nu, \beta_i \rangle^2 A((\overline{P}') : \overline{P} : \sigma : \nu) A(\overline{P} : P : \sigma : \nu) \\ &= \lim_{\nu \rightarrow \nu_0} \prod_{i \in F_{\sigma, \nu_0}} -\langle \nu, \beta_i \rangle^2 \eta(\overline{P} : (\overline{P}') : \sigma : \nu) A((\overline{P}') : P : \sigma : \nu_0). \end{aligned}$$

Thus, from Lemma 9.5 we have

$$(9.7) \quad \mathrm{Im} \left( \lim_{\nu \rightarrow \nu_0} \prod_{i \in F_{\sigma, \nu_0}} -\langle \nu, \beta_i \rangle^2 A(\overline{P} : P : \sigma : \nu) \right) \\ \simeq \lim_{\nu \rightarrow \nu_0} \prod_{i \in F_{\sigma, \nu_0}} -\langle \nu, \beta_i \rangle^2 \eta(\overline{P} : (\overline{P}') : \sigma : \nu) A((\overline{P}') : P : \sigma : \nu_0).$$

Since we have for any  $\gamma$  in  $\widehat{K}$

$$\eta(\overline{P} : (\overline{P}') : \sigma : \nu) = B_{\gamma}^{\sigma}(\overline{P} : (\overline{P}') : \nu) B_{\gamma}^{\sigma}((\overline{P}') : \overline{P} : \nu)$$

and

$$B_{\gamma}^{\sigma}(\overline{P} : (\overline{P}') : \nu) = B_{\gamma}^{\sigma}(P' : P : \nu),$$

we obtain

$$\eta(\overline{P} : (\overline{P}') : \sigma : \nu) = B_{\gamma}^{\sigma}(P, w', \nu) B_{\gamma}^{\sigma}(\overline{P}', w', \nu).$$

Thus by Lemma 5.2, we have

$$\lim_{\nu \rightarrow \nu_0} \prod_{i \in F_{\sigma, \nu_0}} -\langle \nu, \beta_i \rangle^2 \eta(\overline{P} : (\overline{P}') : \sigma : \nu) \neq 0,$$

and (9.7) is infinitesimally equivalent to  $\mathrm{Im}(A(\overline{P}') : P : \sigma : \nu_0)$ . From the double induction formula we have

$$\mathrm{Im}(A((\overline{P}') : P : \sigma : \nu)) \simeq \mathrm{Im}(A(\overline{P} : P : \xi : \nu^1)).$$

Therefore, we have

$$\operatorname{Im} \left( \lim_{\nu \rightarrow \nu_0} \prod_{i \in F_{\sigma, \nu_0}} -\langle \nu, \beta_i \rangle A(\bar{P} : P : \sigma : \nu) \right) \simeq \operatorname{Im}(A(\bar{P} : P : \xi : \nu^1)).$$

**THEOREM 9.7.** *The representation  $\pi_{P, \sigma, \nu_0}$  is reducible if and only if the tempered unitary representation  $\xi$  of  $M$  is reducible or there exists a  $P$ -positive reduced  $\alpha$ -root  $\beta$  satisfying the following conditions:*

- (\*)  $2\langle \nu, \beta \rangle \cdot \langle \beta, \beta \rangle^{-1}$  is an integer and  $2\langle \nu, \beta \rangle \cdot \langle \beta, \beta \rangle^{-1} + 1 \equiv \sigma_\beta \pmod{2}$ ,
- (\*\*)  $\beta|_{\alpha_1} \neq 0$ .

*Proof.* According to Lemma 3.4,  $\pi_{P, \sigma, \nu_0}$  is reducible if and only if  $A(\bar{P} : P : \xi : \nu_0^1)$  has the nontrivial kernel or  $\xi$  is reducible. By Theorem 9.6 or the double induction formula,  $A(\bar{P} : P : \xi : \nu_0^1)$  has the nontrivial kernel if and only if  $A((\bar{P}') : P : \nu_0)$  does so. Thus by similar argument to that in §8, we can prove the assertion of the theorem.

#### REFERENCES

- [1] A. Borel and N. R. Wallach, *Continuous Cohomology, Discrete Subgroups and Representations of Reductive groups*, Princeton Univ. Press, Princeton, N. J., 1980.
- [2] L. Cohn, *Analytic Theory of the Harish-Chandra C-function*, Lecture Notes in Math., vol. 429, Springer-Verlag, Berlin and New York, 1974.
- [3] —, *Some theorems on C-functions*, Bull. Amer. Math. Soc., **80** (1974), 921–926.
- [4] M. Eguchi and S. Tanaka, *The explicit representation of the determinant of Harish-Chandra's C-function in  $SL(3, \mathbb{R})$  and  $SL(4, \mathbb{R})$  cases*, preprint.
- [5] Harish-Chandra, *Harmonic analysis on real reductive groups I. The theory of the constant term*, J. Funct. Anal., **19** (1975), 104–204.
- [6] —, *Harmonic analysis on real reductive groups II. Wave packets in the Schwartz space*, Invent. Math., **36** (1976), 1–55.
- [7] —, *Harmonic analysis on real reductive groups III. The Maass-Selberg relations and the Plancherel formula*, Ann. of Math., **104** (1976), 117–201.
- [8] A. W. Knap, *Representation Theory of Semisimple Groups: An Overview Based on Examples*, Princeton Univ. Press, Princeton, N. J., 1986.
- [9] A. W. Knap and E. M. Stein, *Intertwining operators for semisimple groups*, Ann. of Math., **93** (1971), 489–578.
- [10] —, *Intertwining operators for semisimple groups II*, Invent. Math., **60** (1980), 9–84.
- [11] R. P. Langlands, *On the classification of irreducible representations of real algebraic groups*, mimeographed notes, Institute for Advanced Study, 1973.

- [12] B. Speh, *The unitary dual of  $GL(3, \mathbb{R})$  and  $GL(4, \mathbb{R})$* , *Math. Ann.*, **258** (1981), 113–133.
- [13] B. Speh and D. A. Vogan, *Reducibility of generalized principal series representations*, *Acta Math.*, **145** (1980), 227–299.
- [14] D. A. Vogan, *Representations of Real Reductive Lie Groups*, Birkhäuser, Boston, 1981.
- [15] N. R. Wallach, *Harmonic Analysis on Homogeneous Spaces*, Marcel Dekker, New York, 1973.
- [16] ———, *On Harish-Chandra's generalized  $C$ -functions*, *Amer. J. Math.*, **97** (1975), 385–403.
- [17] G. Warner, *Harmonic Analysis on Semi-Simple Lie Groups I, II*, Springer-Verlag, New York, 1972.

Received June 11, 1990 and in revised form April 15, 1991.

FUKUOKA UNIVERSITY OF EDUCATION  
729 AKAMA  
MUNAKATA CITY FUKUOKA PREF, 811-41 JAPAN

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

V. S. VARADARAJAN  
(Managing Editor)  
University of California  
Los Angeles, CA 90024-1555-05

HERBERT CLEMENS  
University of Utah  
Salt Lake City, UT 84112

F. MICHAEL CHRIST  
University of California  
Los Angeles, CA 90024-1555

THOMAS ENRIGHT  
University of California, San Diego  
La Jolla, CA 92093

NICHOLAS ERCOLANI  
University of Arizona  
Tucson, AZ 85721

R. FINN  
Stanford University  
Stanford, CA 94305

VAUGHAN F. R. JONES  
University of California  
Berkeley, CA 94720

STEVEN KERCKHOFF  
Stanford University  
Stanford, CA 94305

C. C. MOORE  
University of California  
Berkeley, CA 94720

MARTIN SCHARLEMANN  
University of California  
Santa Barbara, CA 93106

HAROLD STARK  
University of California, San Diego  
La Jolla, CA 92093

## ASSOCIATE EDITORS

R. ARENS

E. F. BECKENBACH  
(1906–1982)

B. H. NEUMANN

F. WOLF  
(1904–1989)

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA  
UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA, RENO  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON  
UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF HAWAII  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the 1991 *Mathematics Subject Classification* scheme which can be found in the December index volumes of *Mathematical Reviews*. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024-1555-05.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* (ISSN 0030-8730) is published monthly except for July and August. Regular subscription rate: \$190.00 a year (10 issues). Special rate: \$95.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

---

The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) is published monthly except for July and August. Second-class postage paid at Carmel Valley, California 93924, and additional mailing offices. Postmaster: send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Copyright © 1992 by Pacific Journal of Mathematics

# Pacific Journal of Mathematics

Vol. 153, No. 2

April, 1992

<b>R. Ayala, Eladio Domínguez Murillo, Alberto Márquez Pérez and A. Quintero</b> , Lusternik-Schnirelmann invariants in proper homotopy theory .....	201
<b>Hari Bercovici and Dan-Virgil Voiculescu</b> , Lévy-Hinčin type theorems for multiplicative and additive free convolution .....	217
<b>L. J. Bunce and Cho-Ho Chu</b> , Compact operations, multipliers and Radon-Nikodým property in $JB^*$ -triples .....	249
<b>Marius Dadarlat, Gabriel Nagy, András Némethi and Cornel Pasnicu</b> , Reduction of topological stable rank in inductive limits of $C^*$ -algebras .....	267
<b>François Dumas and Robert Vidal</b> , Dérivations, et hautes dérivations, dans certains corps gauches de series de Laurent .....	277
<b>Mourad Ismail and Xin Li</b> , On sieved orthogonal polynomials. IX: Orthogonality on the unit circle .....	289
<b>X. T. Liang and Y. W. Lu</b> , A Phragmén-Lindelöf theorem .....	299
<b>Mark Stephen Reeder</b> , On certain Iwahori invariants in the unramified principal series .....	313
<b>Shohei Tanaka</b> , On the representation of the determinant of Harish-Chandra's $C$ -function of $SL(n, \mathbb{R})$ .....	343
<b>Fritz von Haeseler and Guentcho Svetoslavov Skordev</b> , Borsuk-Ulam theorem, fixed point index and chain approximations for maps with multiplicity .....	369