THE ADJOINT REPRESENTATION L-FUNCTION FOR GL(n)

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Ideas underlying the proof of the “simple” trace formula are used to show the following. Let $F$ be a global field, and $\mathbb{A}$ its ring of adeles. Let $\pi$ be a cuspidal representation of $GL(n, \mathbb{A})$ which has a supercuspidal component, and $\omega$ a unitary character of $\mathbb{A}^\times/F^\times$. Let $s_0$ be a complex number such that for every separable extension $E$ of $F$ of degree $n$, the $L$-function $L(s, \omega \circ \text{Norm}_{E/F})$ over $E$ vanishes at $s = s_0$ to the order $m > 0$. Then the product $L$-function $L(s, \pi \otimes \omega \times \bar{\pi})$ vanishes at $s = s_0$ to the order $m$. This result is a reflection of the fact that the tensor product of a finite dimensional representation with its contragredient contains a copy of the trivial representation.

Let $F$ be a global field, $\mathbb{A}$ its ring of adeles and $\mathbb{A}^\times$ its group of ideles. Denote by $G$ the group scheme $GL(n)$ over $F$, and put $G = G(F)$, $G = G(\mathbb{A})$, and $Z \simeq F^\times$, $Z \simeq \mathbb{A}^\times$ for the corresponding centers. Fix a unitary character $\varepsilon$ of $Z/Z$, and signify by $\pi$ a cuspidal representation of $G$ whose central character is $\varepsilon$. For almost all $F$-places $v$ the component $\pi_v$ of $\pi$ at $v$ is unramified and is determined by a semi-simple conjugacy class $t(\pi_v)$ in $\hat{G} = G(\mathbb{C})$ with eigenvalues $(z_i(\pi_v); 1 \leq i \leq n)$. Given a finite dimensional representation $r$ of $\hat{G}$, and a finite set $V$ of $F$-places containing the archimedean places and those where $\pi_v$ is ramified, one has the $L$-function

$$L^V(s, \pi, r) = \prod_{v \in V} \det(I - q_v^{-s}r(t(\pi_v)))^{-1}$$

which converges absolutely in some right half plane $\text{Re}(s) >> 1$. Here $q_v$ is the cardinality of the residue field of the ring $R_v$ of integers in the completion $F_v$ of $F$ at $v$.

In this paper we consider the representation $r$ of $\hat{G}$ on the $(n^2 - 1)$-dimensional space $M$ of $n \times n$ complex matrices with trace zero, by the adjoint action $r(g)m = \text{Ad}(g)m = gmg^{-1} \ (m \in M, \ g \in \hat{G})$. More generally we can introduce the representation $\text{Adj}$ of $G \times \mathbb{C}^\times$ by $\text{Adj}((g, z)) = zr(g)$, and hence for any character $\omega$ of $Z/Z$ the
Here $V$ contains all places $v$ where $\pi_v$ or the component $\omega_v$ of $\omega$ is ramified, and $t(\omega_v) = \omega_v(\pi_v)$; $\pi_v$ is a generator of the maximal ideal in $R_v$.

In fact the full $L$-function is defined as a product over all $v$ of local $L$-functions. These are introduced in the $p$-adic case as (a quotient of) the "greatest common denominator" of a family of integrals whose definition is recalled from [JPS] after Proposition 3 below. The local $L$-functions in the archimedean case are introduced below as a quotient of the $L$-factors studied in [JS1]. We denote by $L(s, \pi, \ldots)$ the full $L$-function.

More precisely, we have

$$L^V(s, \pi, \omega, \text{Adj}) = \frac{L^V(s, \pi \otimes \omega \times \pi)}{L^V(s, \omega)},$$

where $L^V(s, \pi_1 \times \pi_2)$ denotes the partial $L$-function attached to the cuspidal $GL(n_i, A)$-modules $\pi_i$ ($i = 1, 2$) and the tensor product of the standard representation of $\widehat{G}_1 = GL(n_1, \mathbb{C})$ and $\widehat{G}_2 = GL(n_2, \mathbb{C})$. This provides a natural definition for the complete function $L(s, \pi, \omega, \text{Adj})$ globally, and also locally. This definition permits using the results of [JPS] and [JS1] mentioned above. In particular, for any cuspidal $G$-module $\pi$, the $L$-function $L(s, \pi, \omega, \text{Adj})$ has analytic continuation to the entire complex $s$-plane.

To simplify the notations we shall assume, when $\omega \neq 1$, that $\omega$ does not factorize through $z \mapsto \nu(z) = |z|$; this last case can easily be reduced to the case of $\omega = 1$. Indeed, $L(s, \pi, \omega \otimes \nu^{s'}, \text{Adj}) = L(s + s', \pi, \omega, \text{Adj})$. Our main result is the following.

1. **Theorem.** Suppose that the cuspidal $G$-module $\pi$ has a supercuspidal component, and $\omega$ is a character of $\mathbb{Z}/Z$ of finite order for which the assumption (Ass; $E$, $\omega$) below is satisfied for all separable field extensions $E$ of $F$ of degree $n$. Then the $L$-function $L(s, \pi, \omega, \text{Adj})$ is entire, unless $\omega \neq 1$ and $\pi \otimes \omega \simeq \pi$. In this last case the $L$-function is holomorphic outside $s = 0$ and $s = 1$. There it has simple poles.

To state (Ass; $E$, $\omega$) note that given any separable field extension $E$ of degree $n$ of $F$ there is a finite galois extension $K$ of $F$, containing $E$, such that $\omega$ corresponds by class field theory to a character, denoted again by $\omega$, of the galois group $J = \text{Gal}(K/F)$. 

Denote by \( H = \text{Gal}(K/E) \) the subgroup of \( J \) corresponding to \( E \), and by \( \omega|E \) the restriction of \( \omega \) to \( H \). It corresponds to a character, denoted again by \( \omega|E \), of the idele class group \( \mathbb{A}_E^x/E^x \) of \( E \).

When \( E/F \) is galois, and \( N_{E/F} \) is the norm map from \( E \) to \( F \), then \( \omega|E = \omega \circ N_{E/F} \). The adjoint representation \( L(s, \omega|E) \) is entire, except at \( s = 0 \) and \( s = 1 \) when \( \omega \neq 1 \) and \( \omega|E = 1 \).

If \( E/F \) is an abelian extension, \( \text{(Ass; } E, \omega \text{)} \) follows by the product decomposition \( L(s, \omega|E) = \prod_{\zeta} L(s, \omega \zeta) \), where \( \zeta \) runs through the set of characters of \( \text{Gal}(E/F) \). More generally, \( \text{(Ass; } E, \omega \text{)} \) is known when \( E/F \) is galois, and when the galois group of the galois closure of \( E \) over \( F \) is solvable, for \( \omega = 1 \) (see, e.g., [CF], p. 225, and the survey article [W]).

For a general \( E \) we have

\[
L(s, \omega|E) = L(s, \text{Ind}_H^J(\omega|E)) = L(s, \omega)L(s, \rho),
\]

where the representation \( \text{Ind}_H^J(\omega|E) \) of \( J = \text{Gal}(K/F) \) induced from the character \( \omega|E \) of \( H \), contains the character \( \omega \) with multiplicity one (by Frobenius reciprocity); \( \rho \) is the quotient by \( \omega \) of \( \text{Ind}_H^J(\omega|E) \).

Artin's conjecture for \( J \) now implies that \( L(s, \rho) \) is entire, unless \( \omega|E = 1 \) and \( \omega \neq 1 \), in which case \( L(s, \rho) \) is holomorphic except at \( s = 0,1 \), where it has a simple pole. When \( [E:F] = n, \omega = 1 \) and \( K \) is a galois closure of \( E/F \), then \( J = \text{Gal}(K/F) \) is a quotient of the symmetric group \( S_n \). Artin's conjecture is known to hold for \( S_3 \) and \( S_4 \), hence \( \text{(Ass; } E, 1 \text{)} \) holds for all \( E \) of degree 3 or 4 over \( F \), and Theorem 1 holds unconditionally (when \( \omega = 1 \)) for \( \text{GL}(3) \) and \( \text{GL}(4) \), as well as for \( \text{GL}(2) \).

The conclusion of Theorem 1 can be rephrased as asserting that \( L(s, \omega) \) divides \( L(s, \pi \otimes \omega \times \pi) \) when \( \pi \otimes \omega \neq \pi \) or \( \omega = 1 \), namely the quotient is entire, and that the quotient is holomorphic outside \( s = 0, 1 \), if \( \pi \otimes \omega \simeq \pi \) and \( \omega \neq 1 \); of course we assume \( \text{(Ass; } E, \omega \text{)} \) for all separable extensions \( E \) of \( F \) of degree \( n \). Note that the product \( L(s, \pi_1 \times \pi_2) \) has been shown in [JS], [JS1], [JPS] and (differently) in [MW] to be entire unless \( \pi_2 \simeq \pi_1 \). In this last case the \( L \)-function is holomorphic outside \( s = 0, 1 \), and has a simple pole at \( s = 0 \) and \( s = 1 \). This pole is matched by the simple pole of \( L(s, \omega) \) when \( \omega = 1 \). Hence \( L(s, \pi, 1, \text{Adj}) \) is also entire.

Another way to state the conclusion of Theorem 1 is that if \( L(s, \omega) \) vanishes at \( s = s_0 \) to the order \( m \geq 0 \), then so does \( L(s, \pi \otimes \omega \times \pi) \),
provided that \((\text{Ass}; E, \omega)\) is satisfied for all separable extensions \(E\) of \(F\) of degree \(n\). Note that \(L(s, \omega)\) does not vanish on \(|\text{Re } s - \frac{1}{2}| \geq \frac{1}{2}\).

Yet another restatement of the Theorem: Let \(\pi\) be a cuspidal \(G\)-module with a supercuspidal component, and \(\omega\) a unitary character of \(\mathbb{Z}/\mathbb{Z}\). Let \(s_0\) be a complex number such that for every separable extension \(E\) of \(F\) of degree \(n\), the \(L\)-function \(L(s, \omega|E)\) vanishes at \(s = s_0\) to the order \(m \geq 0\). Then \(L(s, \pi \otimes \omega \times \hat{\pi})\) vanishes at \(s = s_0\) to the order \(m\). This is the statement which is proven below. Note that the assumption that \(\omega\) is of finite order was put above only for convenience. Embedding \(A_E^\times\) as a torus in \(G\), the character \(\omega|E\) can be defined also by \((\omega|E)(x) = \omega(\text{det } x)\) on \(x \in A_E^\times \subset G\). In general \(\omega\) would be a character of a Weil group, and not a finite galois group.

When \(n = 2\) the three dimensional representation \(\text{Adj}\) of \(\text{GL}(2, \mathbb{C})\) is the symmetric square \(\text{Sym}^2\) representation, and the holomorphy of the \(L\)-function \(L(s, \omega \otimes \text{Sym}^2 \pi)\) \((s \neq 0, 1\) if \(\pi \otimes \omega \simeq \pi, \omega \neq 1\) is proven in [GJ] using the Rankin-Selberg technique of Shimura [Sh], and in [Fl] using a trace formula. Another proof was suggested by Zagier [Z] in the context of \(\text{SL}(2, \mathbb{R})\) and generalized by Jacquet-Zagier [JZ] to the context of \(\pi\) on \(\text{GL}(2, \mathbb{A})\). This last technique is the one extended to the context of cuspidal \(\pi\) with a supercuspidal component and arbitrary \(n \geq 2\), in the present paper.

The path followed in [Z] and [JZ] is to compute the integral

\[
\int K_\phi(x, x)E(x, \Phi, \omega, s)\, dx
\]

on \(x\) in \(\mathbb{Z}G \backslash G\), where \(E(x, \Phi, \omega, s)\) is an Eisenstein series, and \(K_\phi(x, y)\) the kernel representing the cuspidal spectrum in the trace formula. The computation shows that the integral is a sum of multiples of \(L(s, \omega|E)\) \((\text{with } [E : F] = 2\) in the case of [Z] and [JZ]), and on the other hand of \((\text{a sum of multiples of } L(s, \pi \otimes \omega \times \hat{\pi}))\), from which the conclusion is readily deduced. However, [Z] and [JZ] computed all terms in the integral, and reported about the complexity of the formulae. To generalize their computations to \(\text{GL}(n), n \geq 3\), considerable effort would be required.

To bypass these difficulties in this paper we use the ideas employed in [FK] and [F2] to establish various lifting theorems by means of a simple trace formula. In particular we use a special class of test functions \(\phi\), with one component supported on the elliptic regular set, and another component is chosen to be supercuspidal. The first choice reduces the conjugacy classes contributing to \(K_\phi(x, y)\) to elliptic ones only, while the second guarantees the vanishing of the non-cuspidal
terms in the spectral kernel. The first choice does not restrict the applicability of our formulae. Thus our Theorem 1 is offered as another example of the power and usefulness of the ideas underlying the simple trace formula.

For a "twisted tensor" analogue of this paper see [F4].

We shall work with the space $L(G)$ of smooth complex valued functions $\phi$ on $G\setminus G$ which satisfy

1. $\phi(zg) = \varepsilon(z)\phi(g)$ ($z \in \mathbb{Z}$, $g \in G$),
2. $\phi$ is absolutely square integrable on $\mathbb{Z}G\setminus G$.

The group $G$ acts on $L(G)$ by right translation: $(r(g)\phi)(h) = \phi(hg)$. The action is unitary since $\varepsilon$ is. The function $\phi \in L(G)$ is called cuspidal if for each proper parabolic subgroup $P$ of $G$ over $F$ with unipotent radical $N$ we have $\int \phi(n)dn = 0$ ($n \in N\setminus N$) for all $g \in G$. Let $r_0$ be the restriction of $r$ to the space $L_0(G)$ of cusp forms in $L(G)$. The space $L_0(G)$ decomposes as a direct sum with finite multiplicities of invariant irreducible unitary $G$-modules called cuspidal $G$-modules.

Let $\phi$ be a complex valued function on $G$ with $\phi(g) = \varepsilon(z)\phi(zg)$ ($z \in \mathbb{Z}$), compactly supported modulo $\mathbb{Z}$, smooth as a function on the archimedean part $G(F_\infty)$ of $G$, and bi-invariant by an open compact subgroup of $G(A_f)$; here $A_f$ is the ring of adeles without archimedean components, and $F_\infty$ is the product of $F_v$ over the archimedean places. Fix Haar measures $dg_v$ on $G_v/Z_v$ ($G_v = G(F_v)$, $Z_v$ its center) for all $v$ such that the product of the volumes $|K_v/Z_v \cap K_v|$ converges; $K_v$ is a maximal compact subgroup of $G_v$, chosen to be $K_v = G(R_v)$ at the finite places. Then $dg = \otimes dg_v$ is a measure on $G/Z$. The convolution operator $r(\phi) = \int_{G/Z} \phi(g) r(g)dg$ is an integral operator on $L(G)$ with the kernel $K_\phi(x, y) = \sum \phi(x^{-1}y^{-1}y)$ ($y \in G/Z$). In this paper we work only with discrete functions $\phi$.

**Definition.** The function $\phi$ is called discrete if for every $x \in G$ and $\gamma \in G$ we have $\phi(x^{-1}\gamma x) = 0$ unless $\gamma$ is elliptic regular.

Recall that $\gamma$ is called regular if its centralizer $Z_\gamma(G)$ is a torus, and elliptic if it is semi-simple and $Z_\gamma(G)/Z_\gamma(G)Z$ has finite volume. The centralizer $Z_\gamma(G)$ of an elliptic regular $\gamma \in G$ is the multiplicative group of a field extension $E$ of $F$ of degree $n$. For a general elliptic $\gamma$, we have that $Z_\gamma(G)$ is $GL(m, F')$ with $n = m[F': F]$.

The proof of Theorem 1 is based on integrating the kernel $K_\phi(x, y)$ on $x = y$ against an Eisenstein series, as in [Z] and [JZ].

Identify $GL(n-1)$ with a subgroup of $GL(n)$ via $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. Let $U$ be the unipotent radical of the upper triangular parabolic subgroup of type $(n-1, 1)$. Put $Q = GL(n-1)U$. Given a local field $F$,
let $S(F^n)$ be the space of smooth and rapidly decreasing (if $F$ is archimedean), or locally constant compactly supported (if $F$ is non-archimedean) complex valued functions on $F^n$. Denote by $\Phi^0$ the characteristic function of $R^n$ in $F^n$ if $F$ is non-archimedean. For a global field $F$ let $S(A^n)$ be the linear span of the functions $\Phi = \otimes \Phi_v$, $\Phi_v \in S(F_v^n)$ for all $v$, $\Phi_v$ is $\Phi^0$ for almost all $v$. Put $\mathbf{e} = (0, \ldots, 0, 1) \in A^n$. The integral of

(1.1) \[ f(g, s) = \omega(\det g)|\det g|^s \int_{A^n} \Phi(a \mathbf{e} g)|a|^{ns}\omega^n(a) \, d\mathbf{a} \]

converges absolutely, uniformly in compact subsets of $\text{Res} > \frac{1}{n}$. The absolute value is normalized as usual, and $\omega$ is a character of $A^\times/F^\times$.

It follows from Lemmas (11.5), (11.6) of [GoJ] that the Eisenstein series

$$ E(g, \Phi, \omega, s) = \sum f(\gamma g, s) \quad (\gamma \in \mathbb{Z}Q\backslash G) $$

converges absolutely in $\text{Res} > 1$. In [JS], (4.2), p. 545, and [JS2], (3.5), p. 7, it is shown (with a slight modification caused by the presence of $\omega$ here) that $E(g, \Phi, \omega, s)$ extends to a meromorphic function on $\text{Res} > 0$, in fact to the entire complex $s$-plane with a functional equation $E(g, \Phi, \omega, s) = E(\mathbf{t}^{-1} g, \hat{\Phi}, \omega^{-1}, 1 - s)$; here $\mathbf{t} g$ is the transpose of $g$ and $\hat{\Phi}$ is the Fourier transform of $\Phi$. Moreover, $E(g, \Phi, \omega, s)$ is slowly increasing in $g \in G\backslash G$, and it is holomorphic except for a possible simple pole at $s = 1$ and $0$. Note that $f(g)$ and $E(g, s)$ are $Z$-invariant.

2. Proposition. For any character $\omega$ of $A^\times/F^\times$, Schwartz function $\Phi$ in $S(A^n)$, and discrete function $\varphi$ on $G$, for each extension $E$ of degree $n$ of $F$ there is an entire holomorphic function $A(\Phi, \varphi, E, s)$ in $s$ such that

(2.1) \[ \int_{ZG\backslash G} K_{\varphi}(x, x) E(x, \Phi, \omega, s) \, dx = \sum_E A(\Phi, \varphi, \omega, E, s) L(s, \omega|E) \]

on $\text{Res} > 1$. The sum over $E$ ranges over a finite set depending on the support of $\varphi$.

Proof. Since the function $\varphi$ is discrete the sum in $K_{\varphi}(x, x) = \sum \varphi(x^{-1}\gamma x)$ ranges only over the elliptic regular elements $\gamma$ in $G/Z$.
It can be expressed as

\[ K_\phi(x, x) = \sum_T [W(T)]^{-1} \sum_{\gamma \in T/Z} \sum_{\delta \in G/T} \phi(x^{-1}\delta^{-1}\gamma\delta x). \]

Here \( T \) ranges over a set of representatives for the conjugacy classes in \( G \) of elliptic tori (\( T \) is isomorphic over \( F \) to the multiplicative group of a field extension \( E \) of degree \( n \) of \( F \); \( T \) is uniquely determined by such \( E \), and each such \( E \) is so obtained). The cardinality of the Weyl group (normalizer/centralizer) \( W(T) \) of \( T \) in \( G \) is denoted by \( |W(T)| \). It is easy to check that for any elliptic \( T \) we have \( G = TQ \), and \( T \cap Q = \{1\} \). Hence the sum over \( \delta \) can be taken to range over \( Q \).

The left side of (2.1) is equal, in the domain of absolute convergence of the series which defines the Eisenstein series, to

\[ \int_{G} K_\phi(x, x) \sum_{\gamma \in Q \setminus G} f(\gamma x, s) \, dx = \int_{Q \setminus G} K_\phi(x, x) f(x, s) \, dx, \]

since \( x \mapsto K_\phi(x, x) \) is left \( G \)-invariant. Substituting (2.2) this is equal to

\[ \int_{Q \setminus G} \sum_T [W(T)]^{-1} \sum_{\gamma \in T/Z} \sum_{\delta \in Q} \phi(x^{-1}\delta^{-1}\gamma\delta x) f(x, s) \, dx \]

\[ = \sum_T [W(T)]^{-1} \sum_{\gamma \in T/Z} \int_{Q \setminus G} \phi(x^{-1}\gamma x) f(x, s) \, dx; \]

note that \( x \mapsto f(x, s) \) is left \( Q \)-invariant.

To justify the change of summation and integration note that given \( \phi \), the sums over \( T \) and \( \gamma \) are finite. Indeed, the coefficients of the characteristic polynomial of \( \gamma \) are rational, and lie in a compact set depending on the support of \( \phi \) (and a discrete subset of a compact is finite). This explains also the finiteness assertion at the end of the proposition.

Substituting now the expression (1.1) for \( f(x, s) \) we obtain a sum over \( T \) and \( \gamma \) of

\[ \int_{Q \setminus G} \phi(x^{-1}\gamma x) f(x, s) \, dx = \int_G \phi(x^{-1}\gamma x) \omega(\det x) |\det x|^s \Phi(\text{e}^x) \, dx \]

\[ = \int_T \phi(x^{-1}\gamma x) \int_T \Phi(\text{e}^tx) \omega(\det tx) |\det tx|^s \, dt \, dx. \]

Here \( T = T(A) \cong A_E^\times \), where \( T \) is the centralizer of \( \gamma \) in \( G \), and \( T(F) = T \). The inner integral, over \( T \), is a “Tate integral” for
$L(s, \omega|E)$; it is a multiple of $L(s, \omega|E)$ by a function which is holomorphic in $s$ in $C$ and smooth in $x$, depending on $\Phi$, $\omega$ and $E$. The integral over $x$ ranges over a compact in $T\backslash G$, since $\varphi$ is compactly supported modulo $\mathbb{Z}$. The proposition follows.

We now turn to the spectral expression for the kernel $K_\varphi(x, y)$.

**Definition.** The function $\varphi$ on $G$ is called cuspidal if for every $x, y$ in $G$ and every proper $F$-parabolic subgroup $P$ of $G$, we have $\int_N \varphi(xny)dn = 0$, where $N = N(A)$ is the unipotent radical of $P = P(A)$.

When $\varphi$ is cuspidal, the convolution operator $r(\varphi)$ factorizes through the projection on $L^2(G)$. Then $r(\varphi)$ is an integral operator whose kernel has the form

$$K_\varphi(x, y) = \sum_\pi K^\pi_\varphi(x, y), \quad \text{where} \quad K^\pi_\varphi(x, y) = \sum_{\phi^\pi} (r(\varphi)\phi^\pi)(x)\overline{\phi^\pi}(y).$$

The sum over $\pi$ ranges over all cuspidal $G$-modules in $L^2(G)$. The $\phi^\pi$ range over an orthonormal basis consisting of $K = \prod_\nu K_\nu$-finite vectors in $\pi$. The $\phi^\pi$ are rapidly decreasing functions and the sum over $\phi^\pi$ is finite for each $\varphi$ (uniformly in $x$ and $y$) since $\varphi$ is $K$-finite. The sum over $\pi$ converges in $L^2$, and hence also in a space of rapidly decreasing functions. Hence $K_\varphi(x, y)$ is rapidly decreasing in $x$ and $y$, and the product of $K_\varphi(x, x)$ with the slowly increasing functions $E(x, \Phi, \omega, s)$, is integrable over $ZG\backslash G$. The resulting integral, which is equal to (2.1), can also be expressed then in the form

$$\sum_\pi \sum_{\phi^\pi} \int_{ZG\backslash G} (r(\varphi)\phi^\pi)(x)\overline{\phi^\pi}(x)E(x, \Phi, \omega, s)dx.$$

To prove Theorem 1 we now assume that $L(s, \omega)$ is zero at $s = s_0$. It is well known then that $|\text{Re} s_0 - \frac{1}{2}| < \frac{1}{2}$, hence $s_0 \neq 0, 1$. If $s_0$ is a zero of order $m$ of $L(s, \omega)$, then by (Ass $E, \omega$) the function $L(s, \omega|E)$ vanishes at $s_0$ to the order $m$. Making this assumption for every separable field extension $E$ of degree $n$ of $F$ we conclude that (2.1) vanishes at $s = s_0$ to the order $m$, and that for all $j (0 \leq j \leq m)$ we have

$$(2.3)j \sum_\pi \sum_{\phi^\pi} \int_{ZG\backslash G} (\pi(\varphi)\phi^\pi)(x)\overline{\phi^\pi}(x)E^{(j)}(x, \Phi, \omega, s_0)dx = 0.$$

Here $E^{(j)}(\ast, s_0) = \frac{d^j}{ds^j}E(\ast, s)|_{s=s_0}$.

At our disposal we have all cuspidal discrete functions $\varphi$ on $G$, and our aim is to show the vanishing of some summands in the last
double sum over \( \pi \) and \( \phi^\pi \). In fact, fix a \( \pi \) for which Theorem 1 will now be proven. Let \( V \) be a finite set of \( F \)-primes, containing the archimedean primes and those where \( \pi \) or \( \omega \) ramify. Consider 
\[ \varphi = \bigotimes_v \varphi_v \text{ (product over all } F\text{-places } v) \text{ where each } \varphi_v \text{ is a smooth compactly supported modulo } \mathbb{Z}_v \text{ function on } G_v \text{ which transforms under } \mathbb{Z}_v \text{ via } \varepsilon_v^{-1}. \]
For almost all \( v \) the function \( \varphi_v \) is the unit element \( \varphi_v^0 \) in the Hecke algebra \( \mathbb{H}_v \) of \( K_v \)-biinvariant (compactly supported modulo \( \mathbb{Z}_v \) transforming under \( \mathbb{Z}_v \) via \( \varepsilon_v^{-1} \)) functions on \( G_v \). For all \( v \notin V \) the component \( \varphi_v \) is taken to be spherical, namely in \( \mathbb{H}_v \).

Each of the operators \( \pi_v(\varphi_v) \) for \( v \notin V \) factorizes through the projection on the subspace \( \pi_v^{K_v} \) of \( K_v \)-fixed vectors in \( \pi_v \). This subspace is zero unless \( \pi_v \) is unramified, in which case \( \pi_v^{K_v} \) is one-dimensional. On this \( K_v \)-fixed vector, the operator \( \pi_v(\varphi_v) \) acts as the scalar \( \varphi_v^\varepsilon(t(\pi_v)) \), where \( \varphi_v^\varepsilon \) denotes the Satake transform of \( \varphi_v \). Put \( \varphi^\varepsilon(t(\pi_v^\varepsilon)) \) for the product over \( v \notin V \) of \( \varphi_v^\varepsilon(t(\pi_v)) \), and \( \pi_v(\varphi_v) = \bigotimes_{v \in V} \pi_v(\varphi_v) \). Then (2.3) \( j \) takes the form

\[
(2.4)_j \sum_{\{ \pi; \pi^{K_v} \neq 0 \}} \varphi^\varepsilon(t(\pi^\varepsilon)) a(\pi, \varphi, j, \Phi, \omega, s_0) = 0,
\]

where

\[
(2.5)_j \quad a(\pi, \varphi, j, \Phi, \omega, s) = \sum_{\phi^\pi} \int_{G \setminus G} (\pi_v(\varphi_v)\phi^\pi(x)\overline{\phi^\pi}(x))E^{(j)}(x, \Phi, \omega, s) \, dx.
\]

The sum over \( \pi \) ranges over the cuspidal \( G \)-modules \( \pi = \bigotimes \pi_v \) with \( \pi_v^{K_v} \neq \{0\} \) for all \( v \notin V \); \( \pi^{K_v} \) denotes the space of \( \prod_{v \notin V} K_v \)-fixed vectors in \( \pi \). The sum over \( \phi^\pi \) ranges over those elements in the orthonormal basis of \( \pi \) which appears in (2.3) \( j \), which, for any \( v \notin V \), as functions in \( x \in G_v \), are \( K_v \)-invariant and eigenfunctions of \( \pi_v(\varphi_v) \), \( \varphi_v \in \mathbb{H}_v \), with eigenvalues \( t(\pi_v) \). In particular \( \phi^\pi(x) = \phi_v^\pi(x_v) \prod_{v \notin V} \phi_v^\pi(x_v) \), for such \( \phi_v^\pi(v \notin V) \).

A standard argument (see, e.g., Theorem 2 in [FK] in a more elaborate situation), based on the absolute convergence of the sum over \( \pi \) in (2.4) \( j \), standard estimates on the Hecke parameter \( t(\pi_v) \) of the unitary unramified \( \pi_v \) \( (v \notin V) \), and the Stone-Weierstrass theorem, implies the following.

3. Proposition. Let \( \pi \) be a cuspidal \( G \)-module which has a supercuspidal component. Let \( \omega \) be a character of \( \mathbb{Z}/\mathbb{Z} \). Suppose that
\( L(s, \omega|E) \) vanishes at \( s = s_0 \) to the order \( m \) for every separable extension \( E \) of \( F \) of degree \( n \). Then for any \( \Phi \) and a function \( \phi_V \) such that \( \phi \) is cuspidal and discrete with any choice of \( \otimes \phi_v (v \notin V) \), we have that \( a(\pi, \phi_V, j, \Phi, \omega, s_0) \) is zero.

We shall now recall the relation between the summands in (2.5) and the \( L \)-function \( L(s, \pi \otimes \omega \times \tilde{\pi}) \). Let \( \psi \) be an additive non-trivial character of \( \mathbb{A} \) modulo \( F \) (into the unit circle in \( \mathbb{C} \)), and denote by \( \psi_v \) its component at \( v \). An irreducible admissible \( G_v \)-module \( \pi_v \) is called generic if \( \text{Hom}_{\mathbb{N}_v} (\pi_v, \psi_v) \neq \{0\} \). By [GK], or Corollary 5.17 of [BZ], such \( \pi_v \) embeds in the \( G_v \)-module \( \text{Ind}(\psi_v; G_v, N_v) \) from the character \( n = (n_{ij}) \mapsto \psi(n) = \psi(\sum_{1 \leq i < n} n_{i,i+1}) \) of the unipotent upper triangular subgroup \( N_v \) of \( G_v \). Moreover, this embedding is unique, equivalently the dimension of \( \text{Hom}_{\mathbb{N}_v} (\pi_v, \psi_v) \) is at most one. The embedding is given by \( \pi_v \ni \xi \mapsto W_\xi \), where \( W_\xi(g) = \lambda(\pi(g)\xi) (g \in G) \) and \( \lambda 
eq 0 \) is a fixed element in \( \text{Hom}_{\mathbb{N}_v} (\pi_v, \psi_v) \).

Since \( \pi_v \) is admissible, each of the functions \( W_\xi \) is smooth (under right action by \( G_v \)). If \( \pi_v \) is generic, denote by \( W(\pi_v) \) its realization in \( \text{Ind}(\psi_v); W(\pi_v) \) is called the Whittaker model of \( \pi_v \). It is well-known that any component of a cuspidal \( G \)-module is generic.

Given \( \pi \), consider \( W'_v \neq 0 \) in \( W(\pi_v) \) for all \( v \), such that \( W'_v \) is the normalized unramified vector \( W^0_v \) (it is \( K_v \)-invariant and \( W^0_v(1) = 1 \)) for all \( v \notin V \). The function \( \phi'(x) = \sum_{p \in \mathbb{N} \setminus \mathbb{Q}} W'(px) \), where \( W'(x) = \prod_v W'_v(x_v) \), is a cuspidal function in the space of \( \pi \subset L_0(G) \). Substituting the series definition of \( E(x, \Phi, \omega, s) = \sum_{Z \subset \mathbb{Q} \setminus G} \frac{f(\gamma \chi, s)}{\Phi(\epsilon \chi) \omega(\alpha \chi)} d\chi \) in

\[
\int_{Z \subset \mathbb{Q} \setminus G} \phi''(x) \phi'(x) E(x, \Phi, \omega, s) \, dx \quad (\phi'' \in \pi \subset L_0(G))
\]

one obtains

\[
\int_{Z \subset \mathbb{Q} \setminus G} \phi''(x) \phi'(x) f(x, s) \, dx = \int_{Z \subset \mathbb{N} \setminus G} \phi''(x) W'(x) f(x, s) \, dx.
\]

Since \( W'(nx) = \psi(n) W'(x) \), and \( \int_{\mathbb{N} \setminus \mathbb{N}} \phi''(nx) \psi(n) \, dn = W_{\phi''}(x) \) is the Whittaker function associated to the cusp form \( \phi'' \), the integral is equal to

\[
\int_{\mathbb{N} \setminus \mathbb{G}} W_{\phi''}(x) W'(x) f(x, s) \, dx
\]

\[
= \int_{\mathbb{N} \setminus \mathbb{G}} W_{\phi''}(x) W'(x) \Phi(\epsilon x) \omega(\det x) |\det x|^s \, dx.
\]
If \( \phi'' \) is also of the form \( \phi''(x) = \sum_{p \in \mathbb{N} \setminus \mathbb{Q}} W''(px) \), where \( W''(x) = \prod_v W''_v(x_v) \) is factorizable, then \( W''_\phi = W'' \) and the integral factorizes as a product over all \( v \) of the local integrals

\[
\int_{N_v \backslash G_v} W''_v(x) \overline{W_v}(x) \Phi_v(e x) \omega_v(\det x) |\det x|_v^\frac{1}{2} \, dx,
\]

provided that \( \Phi(x) = \prod_v \Phi_v(x_v) \).

When \( W'_v = W'_0 = W''_0 \), and \( \Phi_v \) is the characteristic function \( \Phi^0_v \) of \( R^0_v \) (and \( v \not\in V \)), the integral (3.1) is easily seen (on using Schur function computations; see [F3], p. 305) to be equal to \( L(s, \pi_v \otimes \omega_v \times \kappa_v) \). For a non-archimedean \( v \in V \) the \( L \)-factor is defined in [JPS], Theorem 2.7, as a "g.c.d" of the integrals (3.1) for all \( W'_1, W'_2 \in W(\pi_v) \) and \( \Phi_v \). In the archimedean case the \( L \)-factor is defined in [JS1], Theorem 5.1. It is shown in [JPS] and [JS1] that the \( L \)-factor lies in the span of the integrals (3.1). The product of the \( L \)-factors, as well as the various manipulations above, converges absolutely for \( s \) in some right half plane.

4. Lemma. The functions \( W'_v \in W(\pi_v) \) (and so \( \phi' \in \pi \)) can be chosen to have the property that \( \phi' \) factorizes as \( \bigotimes_v \phi'_v \).

Proof. Since \( W'_v \) is \( K_v \)-invariant for \( v \not\in V \), so is \( \phi' \), and we have

\[
\phi'(x) = \phi'_V(x_v) \prod_{v \not\in V} \phi'_v(x_v),
\]

where \( \phi'_v \) is the \( K_v \)-invariant function on \( G_v \) which takes the value 1 at 1 and is the eigenfunction of the operators \( \pi_v(\phi_v) \), \( \phi_v \in \mathbb{H}_v \), with the eigenvalue \( t(\pi_v) \).

The space \( \pi \subset L_0(G) \) is spanned by factorizable functions, namely \( \phi' \) is a finite sum over \( j (1 \leq j \leq J) \) of products \( \bigotimes_v \phi'_{j_v} \) of functions \( \phi'_{j_v} \) on \( G_v \) (which are smooth, compactly supported modulo \( Z_v \), transform under \( Z_v \) via \( \epsilon_v \), with \( \phi'_{j_v} = \phi'_v \) for all \( v \not\in V \). Each of the functions \( \phi'_{j_v} \) is (right) invariant under a congruence subgroup \( K'_v \) of the standard compact subgroup \( K_v \) of \( G_v \). Namely \( \phi'_{j_v} \) is a non-zero vector in the finite dimensional space \( \pi_v K'_v \) of \( K'_v \)-fixed vectors in \( \pi_v \). The Hecke algebra \( \mathbb{H}(K'_v) \) of \( K'_v \)-biinvariant compactly supported functions on \( G_v \), which transform under \( Z_v \) via \( \epsilon_v^{-1} \) generate the algebra of endomorphisms of the finite dimensional space \( \pi_v K'_v \). Consider \( \phi_v \in \mathbb{H}(K'_v) \) such that \( \pi_v(\phi_v) \) acts
as an orthogonal projection on $\phi'_v$. Then $(\bigotimes_v \phi'_v)(\phi')$ lies in $\pi$, is of the form $\bigotimes_v \phi'_v$, and is defined by the Whittaker functions $\pi_v(\phi_v) W'_v$, as required.

**Proof of Theorem 1.** For $\pi$ as in the theorem, and $s_0$ as in (2.3) $j$, we shall choose $W'_v \in W(\pi_v)$ with factorizable $\phi'_v(x) = \bigotimes_v \phi'_v(x_v) = \sum_{p \in \mathcal{Q}} W'(px)$ and proceed to show the vanishing of the corresponding summand in (2.5) $j$. Recall that by the assumption of Theorem 1 there is an $F$-place $v_2$ such that $\pi_{v_2}$ is supercuspidal. Let $v_1$ be another $F$-place in $V$, say where $\pi$ and $\omega$ are unramified. Put $V'' = V - \{v_2\}$ and $V'$ for $V'' - \{v_1\}$.

Consider the matrix coefficient $\phi'_v(x) = (\pi_v(x^{-1}) \phi'_v, \phi'_v)$ of the supercuspidal $G_{v_2}$-module $\pi_{v_2}$. Note that $\phi'_v$ is a $C^\infty_c$-function on $G_{v_2}$ modulo $Z_{v_2}$, and $(\cdot, \cdot)$ denotes the natural inner product. The function $\phi'_v$ is smooth and compactly supported on $G_{v_2}$ modulo $Z_{v_2}$, and it is a supercuspidal form $(\int \phi'_v(xny) dn = 0, n \in N_{v_2} = \text{unipotent radical of any parabolic subgroup of } G_{v_2})$. It is well-known that a function $\phi = \bigotimes \phi_v$ whose component at $v_2$ is $\phi'_v$ is cuspidal. By the Schur orthogonality relations, the convolution operator $\pi_{v_2}(\phi'_v)$ acts as an orthogonal projection on the subspace generated by $\phi'_v$. Working with $\phi = \bigotimes \phi_v$ whose component at $v_2$ is $\phi'_v$, we then have that $\phi$ is cuspidal and that the sum in (2.5) $j$ ranges only over the $\phi (= \phi')$ whose component at $v_2$ is $\phi'_v$ (up to a scalar multiple).

As in the proof of Lemma 4, for each $v \in V'$ we may choose $\phi'_v$ in $H(K'_v)$ such that $\pi_v(\phi'_v)$ acts as an orthogonal projection to the subspace of $\pi_v$ spanned by $\phi'_v$. Choosing the components $\phi_v$ of $\phi$ at $v \in V'$ to be of the form $\phi''_v * \phi'_v$, with any $\phi''_v$, the sum in (2.5) $j$ for our $\pi$ extends only over those $\phi$ in the orthonormal basis of the chosen $\pi \subset L_0(G)$ whose component at $v \neq v_1$ is $\phi'_v$. But $\phi$ is left $G$-invariant, being a cusp form, and $G = G \prod_{v \neq v_1} G_v$. Hence the only $\phi$ which contributes to the sum in (2.5) $j$ is $\phi'$, whatever $\phi_v$ is.

We still need to choose $\phi_{v_1}$ such that $\phi = \bigotimes \phi_v$ be discrete. It suffices to choose $\phi_{v_1}$ to be supported on the regular elliptic set in $G_{v_1}$. Moreover, since $\phi'_v$ is right invariant under a compact open subgroup $K'_v$ of $K_{v_1} \subset G_{v_1}$, we can choose the support of $\phi_{v_1}$ to be contained in $Z_{v_1}K'_v$. Then $\pi_{v_1}(\phi_{v_1})$ acts as a scalar on $\phi'_v$, and we normalize $\phi_{v_1}$ so that this scalar be one.

In conclusion, for any choice of $W'_v \in W(\pi_v)$ for all $v$, with $W'_v = \bigotimes_v \phi'_v$.
$W^0_v$ for $v \not\in V$, and any choice of $\varphi_v$ ($v \in V'$), we have that
\[
\int_{ZG \backslash G} (\pi_{V'}(\varphi_{V'})\phi')(x)\overline{\phi'}(x)E(x, \Phi, \omega, s) \, dx \\
= \prod_{v \in V} \int_{N_v \backslash G_v} (\pi_v(\varphi_v)W^0_v)(x)\overline{W^0_v}(x)\Phi_v(x)\omega_v(\det x)\left| \det x \right|_v^s \, dx \\
\cdot \prod_{v \not\in V} L(s, \pi_v \otimes \omega_v \times \tilde{\pi}_v)
\]
vanishes at $s_0$ to the order $m$. Here $\pi_{v_1}(\varphi_{v_1})W^0_{v_1} = W^0_{v_1}$. In fact we may choose $W^0_{v_1}$ to be $W^0_v \in W(\pi_{v_1})$, and $\Phi_{v_1}$ to be $\Phi^0_{v_1}$. Since $\pi_{v_1}$ and $\omega_{v_1}$ are unramified, the corresponding integral is then equal to the $L$-factor, so $v_1$ can be deleted from the set $V$.

To complete the proof of Theorem 1, note that the $L$-function $L(s, \pi_v \otimes \omega_v \times \tilde{\pi}_v)$ lies in the span of the integrals (3.1). Hence the assumption for every separable extension $E$ of $F$ of degree $n$ that $L(s, \omega|E)$ vanishes at $s = s_0$ to the order $m$, implies the vanishing of $\prod L(s, \pi_v \otimes \omega_v \times \tilde{\pi}_v)$ to the order $m$. This completes the proof of Theorem 1.

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