Pacific Journal of Mathematics

FACE NUMBER INEQUALITIES FOR MATROID COMPLEXES AND COHEN-MACAULAY TYPES OF STANLEY-REISNER RINGS OF DISTRIBUTIVE LATTICES

TAKAYUKI HIBI

Vol. 154, No. 2 June 1992

PACIFIC JOURNAL OF MATHEMATICS
Vol. 154, No. 2, 1992

FACE NUMBER INEQUALITIES FOR MATROID COMPLEXES AND COHEN-MACAULAY TYPES OF STANLEY-REISNER RINGS OF DISTRIBUTIVE LATTICES

TAKAYUKI HIBI

We discuss two topics related with combinatorial study of canonical modules of Stanley-Reisner rings, viz., (i) some linear inequalities on the number of faces of a matroid complex and (ii) a formula to compute the Cohen-Macaulay type of the Stanley-Reisner ring of a finite distributive lattice.

Introduction. We study the following two problems in the field of commutative algebra and combinatorics:

(i) What can be said about the number of faces of a matroid complex?
(ii) How can we calculate the Cohen-Macaulay type of the Stanley-Reisner ring of the order complex of a finite distributive lattice?

Recently, some topics on Hilbert functions of noetherian graded algebras have been studied by several authors, e.g., **[Sta3]**, **[Sta7]**, **[G-M-R]**, **[R-R]** and **[H5]** from viewpoints of commutative algebra, algebraic geometry and combinatorics. In the first half of the present paper, we are concerned with Hilbert functions of Stanley-Reisner rings of matroid complexes. Via well-known facts **[H-K]**, **[Sta3]** on canonical modules of Cohen-Macaulay graded integral domains, Stanley **[Sta7]** found certain linear inequalities for the Hilbert function of a Cohen-Macaulay graded integral domain. Based on an idea of J. Herzog (cf. Corollary (1.5)), we see that the same linear inequalities as in **[Sta7]** hold for the Hilbert function of the Stanley-Reisner ring of a matroid complex (cf. Theorem (1.8)).

On the other hand, it would be of interest to find a combinatorial formula to compute the Cohen-Macaulay type (i.e., the minimal number of generators of the canonical module) of the Stanley-Reisner ring of a Cohen-Macaulay complex, e.g., **[H7]**. In the latter half of this paper, we find a formula for the computation of the Cohen-Macaulay type of the Stanley-Reisner ring of the order complex of a finite distributive lattice. In fact, our main result (cf. Theorem (2.10)) guarantees that the Cohen-Macaulay type of the Stanley-Reisner ring of the order complex of a finite distributive lattice is equal to the number of distinct equivalence classes of a certain equivalence relation (cf. (2.8)) on the set of linear extensions of a finite partially ordered set associated with the distributive lattice.

1. Level rings and matroid complexes.

(1.1) Let k be a field and A a semi-standard k-algebra, that is, A is a commutative graded ring $\bigoplus_{n\geq 0} A_n$ satisfying (i) $A_0 = k$, (ii) A is finitely generated as a k-algebra, and (iii) A is integral over the subalgebra $k[A_1]$ of A generated by A_1. The *Hilbert function* of A is defined to be

$$H(A, n) := \dim_k A_n \quad \text{for } n = 0, 1, \dots,$$

while the *Hilbert series* of A is given by

$$F(A, \lambda) := \sum_{n=0}^{\infty} H(A, n)\lambda^n.$$

Since A is finitely generated as a $k[A_1]$-algebra and is integral over $k[A_1]$, it follows that A is finitely generated as a $k[A_1]$-module. Hence, well-known properties on Hilbert series, e.g., [**Mat**, pp. 94–95], guarantee that

$$F(A, \lambda) = (h_0 + h_1\lambda + \cdots + h_s\lambda^s)/(1-\lambda)^d$$

for some integers $h_0, h_1, \dots, h_s$ with $h_s \neq 0$. Here d is the Krull dimension of A. We say that the vector $h(A) := (h_0, h_1, \dots, h_s)$ is the *h-vector* of A.

(1.2) Suppose that a semi-standard k-algebra $A = \bigoplus_{n\geq 0} A_n$ is Cohen-Macaulay. Let K_A be the canonical module [**H-K**] of A. It is known [**H-K**, Corollary (6.7)] that there exists a graded ideal I of A with $I \cong K_A$ (as graded modules over A, up to shift in grading) if and only if A is generically Gorenstein, i.e., the localization $A_{\mathfrak{q}}$ is Gorenstein for every minimal prime ideal $\mathfrak{q}$ of A. Also, see [**H3**, Lemma (1.7)].

(1.3) Proposition. *Let a Cohen-Macaulay semi-standard k-algebra* $A = \bigoplus_{n\geq 0} A_n$ *be generically Gorenstein, and let* $I = \bigoplus_{n\geq a}(I \cap A_n)$, $I \cap A_a \neq (0)$, *be a graded ideal of* A *with* $I \cong K_A$. *Suppose that there exists a non-zero divisor* $\vartheta \in I \cap A_a$ *on* A. *Then the h-vector* $h(A) = (h_0, h_1, \dots, h_s)$ *of* A *satisfies the linear inequality*

$$(*) \qquad h_0 + h_1 + \cdots + h_i \leq h_s + h_{s-1} + \cdots + h_{s-i}$$

for every $0 \leq i \leq s$.

Proof. Since $\vartheta \in I \cap A_a$ is a non-zero divisor on A, the dimension of $I/\vartheta A$ as an A-module is less than the Krull dimension of A if $\vartheta A \neq I$. Thus the proof of **[Sta7**, Theorem (2.1)] is valid in our situation without modification. □

(1.4) We say that a Cohen-Macaulay semi-standard k-algebra $A = \bigoplus_{n\geq 0} A_n$ is *level* **[Sta2]** if the canonical module $K_A = \bigoplus_{n\geq a}(K_A)_n$ with $(K_A)_a \neq (0)$, $a \in \mathbf{Z}$, of A is generated by $(K_A)_a$ as an A-module. In other words, A is level if and only if the Cohen-Macaulay type of A coincides with the last component of the h-vector of A. Consult, e.g., **[H2**, pp. 343–345].

(1.5) COROLLARY. *Suppose that a Cohen-Macaulay semi-standard k-algebra $A = \bigoplus_{n\geq 0} A_n$ is both generically Gorenstein and level. Then the h-vector $h(A) = (h_0, h_1, \ldots, h_s)$ of A satisfies the linear inequality* $(*)$ *for every* $0 \leq i \leq s$.

Proof. A routine technique enables us to assume that k is an infinite field. Let $I = \bigoplus_{n\geq a}(I \cap A_n)$, $I \cap A_a \neq (0)$, be a graded ideal of A with $I \cong K_A$. Thanks to Proposition (1.3), what we must show is the existence of a non-zero divisor $\vartheta \in I \cap A_a$ on A. Let $\mathcal{N}_A$ be the set of prime ideals of A which belong to the ideal (0). Since A is Cohen-Macaulay, we know that the Krull dimension of $A/\mathfrak{q}$ equals that of A for each $\mathfrak{q} \in \mathcal{N}_A$. We write $\mathcal{U}$ for the (set-theoretic) union of all prime ideals $\mathfrak{q} \in \mathcal{N}_A$. Recall (e.g., **[Mat**, p. 38]) that the set $\mathcal{U}$ coincides with the set of zero divisors on A. If $I \cap A_a \subset \mathcal{U}$, then $I \cap A_a \subset \mathfrak{q}$ for some $\mathfrak{q} \in \mathcal{N}_A$ since k is infinite (see, e.g., **[Her**, Problem 21, p. 136]). Now, A is level, thus I is generated by $I \cap A_a$ as an A-module. Hence, if $I \cap A_a \subset \mathfrak{q}$ then $I \subset \mathfrak{q}$, thus the Krull dimension of A/I is equal to that of A, which contradicts **[H-K**, Corollary (6.13)]. □

The author is grateful to Professor Jürgen Herzog for suggesting the above proof. We remark that Corollary (1.5) is false if we drop the assumption that A is generically Gorenstein.

(1.6) Let V be a finite set, called the *vertex set*, and Δ a *simplicial complex* on V. Thus Δ is a collection of subsets of V such that (i) $\{x\} \in \Delta$ for every $x \in V$ and (ii) $\sigma \in \Delta$, $\tau \subset \sigma$ imply $\tau \in \Delta$. Each element of Δ is called a *face* of Δ. Set $d := \max\{\#(\sigma)\,;\, \sigma \in \Delta\}$. Here $\#(\sigma)$ is the cardinality of σ as a set. Then the *dimension* of Δ is defined to be $\dim \Delta := d - 1$. We say that Δ is *pure* if every maximal face has the same cardinality. We write $f_i = f_i(\Delta)$, $0 \leq i < d$, for the number of faces σ of Δ with $\#(\sigma) = i+1$. Thus $f_0 = \#(V)$. We say that $f(\Delta) := (f_0, f_1, \ldots, f_{d-1})$ is the *f-vector* of Δ. Define the *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of Δ by the formula

$$\sum_{i=0}^{d} f_{i-1}(\lambda - 1)^{d-i} = \sum_{i=0}^{d} h_i \lambda^{d-i}$$

with $f_{-1} = 1$. Consult, e.g., **[Sta4]** and **[Hoc]** for further information.

(1.7) A simplicial complex Δ on the vertex set V is called a *matroid complex* (or G-complex **[Sta2]**) if the following conditions are satisfied:

(i) If $\sigma, \tau \in \Delta$ and $\#(\sigma) < \#(\tau)$, then there exists $x \in \tau$ such that $x \notin \sigma$ and $\sigma \cup \{x\} \in \Delta$.
(ii) $\dim(\Delta - x) = \dim \Delta$ for every $x \in V$. Here $\Delta - x$ is the subcomplex $\{\sigma \in \Delta; x \notin \sigma\}$ of Δ on $V - \{x\}$.

We remark that the above condition (ii) is required only to avoid the inessential case; if $\dim(\Delta - x) < \dim \Delta$ then Δ is a cone over $\Delta - x$ with apex x, thus we should study $\Delta - x$ rather than Δ.

For example, let V be a finite set of non-zero vectors of a vector space over a field and suppose that the dimension of the subspace spanned by V is equal to the dimension of the subspace spanned by $V - \{x\}$ for every $x \in V$. Then the set Δ of linearly independent subsets of V is a matroid complex.

Now, what can be said about the h-vector of an arbitrary matroid complex?

(1.8) THEOREM. *Suppose that $h(\Delta) = (h_0, h_1, \ldots, h_d)$ is the h-vector of a matroid complex Δ of dimension $d-1$. Then we have the linear inequality*

$$h_0 + h_1 + \cdots + h_i \le h_d + h_{d-1} + \cdots + h_{d-i}$$

for every $0 \le i \le d$.

Proof. Let $V = \{X_1, X_2, \ldots, X_t\}$ be the vertex set of Δ and $k[\Delta] = k[X_1, X_2, \ldots, X_t]/I_\Delta$ the *Stanley-Reisner ring* (**[Sta1]**, **[Rei]**) of Δ over a field k with the standard grading, i.e., each $\deg(X_i) = 1$. Then the Krull dimension of $k[\Delta]$ is d, and the Hilbert series of $k[\Delta]$ is just

$$F(k[\Delta], \lambda) = (h_0 + h_1\lambda + \cdots + h_d\lambda^d)/(1-\lambda)^d,$$

see, e.g., [**Sta4**, pp. 62–68]. It is known (and, in fact, not difficult to prove) that a matroid complex is "doubly" Cohen-Macaulay in the sense of [**Bac**]. In other words, $k[\Delta]$ is a level ring with $h_d \neq 0$. See also [**Sta2**]. Moreover, $k[\Delta]$ is generically Gorenstein [**Sta4**, p. 80]. Hence Corollary (1.5) enables us to obtain the required inequality. □

(1.9) *Conjecture.* Work in the same notation as in Theorem (1.8). Then we have the following linear inequalities:

(i) $h_i \leq h_{d-i}$ for every $0 \leq i \leq [d/2]$, and
(ii) $h_0 \leq h_1 \leq \cdots \leq h_{[d/2]}$.

Consult [**H4**] for further information on the inequalities in the above Conjecture (1.9). We easily see the inequality $h_1 \leq h_2$ when $d \geq 3$. Also, note that, thanks to [**H4**], the above conjecture is weaker than that of [**Sta2**, p. 59].

On the other hand, a log-concavity conjecture on f-vectors of matroid complexes is presented by Mason [**Mas**]. Some partial results on this conjecture are obtained by Dowling [**Dow**] and by Mahoney [**Mah**].

It would, of course, be of great interest to find a combinatorial characterization of the h-vectors of matroid complexes.

The f-vectors (or h-vectors) of various classes of simplicial complexes have been studied by several combinatorialists. We refer the reader to, e.g., [**B-K**] for a survey of the topic.

2. Cohen-Macaulay types of distributive lattices.

(2.1) Given a finite partially ordered set (*poset* for short) P we write $\mathscr{J}(P)$ for the poset which consists of all *poset ideals* (or *order ideals* [**Sta6**, p. 100]) of P, ordered by inclusion. Then $\mathscr{J}(P)$ is a *distributive lattice* [**Sta6**, p. 105]. On the other hand, the fundamental theorem for finite distributive lattices, e.g., [**Sta6**, Theorem (3.4.1)] guarantees that, for every finite distributive lattice L, there exists a unique poset P for which $L \cong \mathscr{J}(P)$.

(2.2) Let $\rho(P; \ell)$ be the number of *chains* [**Sta6**, p. 99]

$$\mathscr{M} : \varnothing = I_0 \subsetneqq I_1 \subsetneqq \cdots \subsetneqq I_{\ell+1} = P$$

of *length* $\ell + 1$ (cf. [**Sta6**, p. 99]) in the distributive lattice $\mathscr{J}(P)$ such that

(i) $I_{i+1} - I_i$ is a *clutter* [**Sta6**, p. 100] in P for each $0 \leq i \leq \ell$, and

(ii) for every $1 \le i \le \ell$, there exist $y \in I_{i+1} - I_i$ and $x \in I_i - I_{i-1}$ with $x < y$ in P.

Then $\rho(P; \ell) = 0$ if $\ell < \mathrm{rank}(P)$ and $\rho(P; \mathrm{rank}(P)) \neq 0$. Here $\mathrm{rank}(P)$ is the *rank* **[Sta6**, p. 99**]** of P.

(2.3) We now study the Stanley-Reisner ring

$$k[\Delta(L)] = k[X_\alpha; \alpha \in L]/I_{\Delta(L)},$$

with each $\deg(X_\alpha) = 1$, of the *order complex* $\Delta(L)$ (cf. **[Sta6**, p. 120**]**) of a finite distributive lattice L over a field k. It is well known, e.g., **[B-G-S]** that $k[\Delta(L)]$ is Cohen-Macaulay. We are interested in the Cohen-Macaulay type $\mathrm{type}(k[\Delta(L)])$ of $k[\Delta(L)]$, i.e., the minimal number of generators of the canonical module $K_{k[\Delta(L)]}$ of $k[\Delta(L)]$ as a $k[\Delta(L)]$-module. We refer the reader to, e.g., **[B-G-S]** and **[Sta6**, Chap. 4, §5**]** for the information on the h-vector of the order complex of a finite distributive lattice. Also, consult **[H1]**, **[H3]** and **[H6]** for some topics on commutative algebra related with distributive lattices.

(2.4) PROPOSITION. *The Cohen-Macaulay type* $\mathrm{type}(k[\Delta(L)])$ *of the Stanley-Reisner ring* $k[\Delta(L)]$ *of the order complex* $\Delta(L)$ *of a finite distributive lattice* $L = \mathcal{J}(P)$ *is*

$$(**) \qquad \mathrm{type}(k[\Delta(L)]) = \rho(P; \mathrm{rank}(P)) + \rho(P; \mathrm{rank}(P) + 1) + \cdots.$$

Proof. Suppose that $\#(P) = n$, say $P = \{p_1, p_2, \dots, p_n\}$, and we write $e(I) = (e_1, e_2, \dots, e_n) \in \mathbf{R}^n$ for the incident vector of a poset ideal I of P, i.e., $e_i = 1$ if $p_i \in I$ and $e_i = 0$ otherwise. Thus in particular $e(\varnothing) = (0, 0, \dots, 0)$ and $e(P) = (1, 1, \dots, 1)$. If $\mathscr{M}$ is a chain in L of the form $(\bigstar)$ $(\varnothing \subset) I_0 \subsetneqq I_1 \subsetneqq \cdots \subsetneqq I_\ell$ $(\subset P)$ with each $I_i \in \mathcal{J}(P)$, then we write $[\mathscr{M}]$ for the convex hull of $\{e(I_0), e(I_1), \dots, e(I_\ell)\}$ in $\mathbf{R}^n$. Thus $[\mathscr{M}]$ is an ℓ-simplex in $\mathbf{R}^n$. Let $\mathscr{C} = \mathscr{C}(L)$ be the set of chains in $L = \mathcal{J}(P)$ and $\mathscr{P} = \mathscr{P}(L)$ the convex hull of $\{e(I); I \in \mathcal{J}(P)\}$ in $\mathbf{R}^n$. Hence $\mathscr{P} \subset \mathbf{R}^n$ is a convex polytope of dimension n. We identify $\{[\mathscr{M}]; \mathscr{M} \in \mathscr{C}\}$ with the order complex $\Delta(L)$ of L. It is known, e.g., **[Sta5**, p. 17**]** that $\{[\mathscr{M}]; \mathscr{M} \in \mathscr{C}\}$ is a triangulation of $\mathscr{P}$; hence $\mathscr{P}$ is a geometric realization of $\Delta(L)$.

Now, let $\mathscr{I}$ be the ideal of the Stanley-Reisner ring $k[\Delta(L)] = k[X_\alpha; \alpha \in L]/I_{\Delta(L)}$ which is generated by those square-free monomials $\prod_{\alpha \in \mathscr{M}} X_\alpha$ with $[\mathscr{M}] \in \Delta(L) - \partial\Delta(L)$. Here $\partial\Delta(L)$ is the boundary of $\Delta(L)$. Then, by virtue of **[Sta4**, Theorem (7.3), p. 81**]**, $\mathscr{I}$ is isomorphic to the canonical module $K_{k[\Delta(L)]}$ of $k[\Delta(L)]$. On the other hand, thanks to [**Sta5**, p. 10], if $\mathscr{M} \in \mathscr{C}$ is of the form $(\bigstar)$, then $[\mathscr{M}] \in \Delta(L) - \partial\Delta(L)$ if and only if the following conditions are satisfied: (i) $I_0 = \varnothing$, (ii) $I_\ell = P$, and (iii) each $I_{i+1} - I_i$ is a clutter. Hence, it follows immediately that the minimal number of generators of $\mathscr{I}$ as a $k[\Delta(L)]$-module is just $(**)$ as required. □

We should remark that the ideal $\mathscr{I}$ in the above proof is generated by $\{\Pi_{\alpha\in\mathscr{M}} X_\alpha ; [\mathscr{M}] \in \Delta(L) - \partial\Delta(L), \#(\mathscr{M}) = \operatorname{rank}(P) + 2\}$ as a $k[\Delta(\mathscr{L})]$-module if and only if $\rho(P; \ell) = 0$ for every $\ell \neq \operatorname{rank}(P)$. In other words,

(2.5) COROLLARY. *The Stanley-Reisner ring* $k[\Delta(L)]$ *of the order complex* $\Delta(L)$ *of a finite distributive lattice* $L = \mathscr{J}(P)$ *is level if and only if* $\rho(P; \ell) = 0$ *for every* $\ell \neq \operatorname{rank}(P)$.

(2.6) Let $\mathbf{N}$ be the set of non-negative integers and P a finite poset. We say that a map $\sigma: P \to \mathbf{N}$ is *strictly order-preserving* if $x < y$ in P implies $\sigma(x) < \sigma(y)$ in $\mathbf{N}$. We write $\mathscr{B}(P; \ell)$ for the set of strictly order-preserving maps $\sigma: P \to \mathbf{N}$ such that (i) $\sigma(P) = \{0, 1, \dots, \ell\}$ and (ii) $\sigma^{-1}(\{i-1, i\})$ is *not* a clutter in P for every $1 \leq i \leq \ell$.

(2.7) LEMMA. $\rho(P; \ell) = \#(\mathscr{B}(P; \ell))$.

Proof. Given a chain $\mathscr{M} : \varnothing = I_0 \subsetneqq I_1 \subsetneqq \cdots \subsetneqq I_{\ell+1} = P$ in the distributive lattice $\mathscr{J}(P)$ which satisfies the conditions (i) and (ii) in (2.2), we can define a map $\sigma: P \to \mathbf{N}$ in $\mathscr{B}(P; \ell)$ by $\sigma(x) = i$ if $x \in I_{i+1} - I_i$. On the other hand, if $\sigma \in \mathscr{B}(P; \ell)$, then $\varnothing \subsetneqq \sigma^{-1}(\{0\}) \subsetneqq \sigma^{-1}(\{0, 1\}) \subsetneqq \cdots \subsetneqq \sigma^{-1}(\{0, 1, \dots, \ell - 1\}) \subsetneqq P$ is a chain in $\mathscr{J}(P)$ with the properties (i) and (ii) in (2.2). □

(2.8) We recall that a *linear extension* [**Sta6**, p. 110] of a finite poset P is a strictly order-preserving map $\sigma: P \to \mathbf{N}$ such that $\sigma(P) = \{1, 2, \dots, \#(P)\}$. If σ is a linear extension of P, then there exists a unique sequence $\mathscr{D}(\sigma) = (d_1, d_2, \dots, d_\ell) \in \mathbf{Z}^\ell$, $0 \leq \ell = \ell(\sigma) \in \mathbf{Z}$, with $1 \leq d_1 < d_2 < \cdots < d_\ell < \#(P)$ such that

(i) $\sigma^{-1}(\{d_i + 1, d_i + 2, \dots, d_{i+1}\})$ is a clutter in P for each $0 \leq i \leq \ell$, where we set $d_0 = 0$ and $d_{\ell+1} = \#(P)$, and
(ii) for every $1 \leq i \leq \ell$, there exists $x \in \sigma^{-1}(\{d_{i-1} + 1, \dots, d_i\})$ with $x < \sigma^{-1}(d_i + 1)$ in P.

We say that two linear extensions σ and τ of P are *equivalent* (written as $\sigma \sim \tau$) if $\mathscr{D}(\sigma) = \mathscr{D}(\tau)$ $(= (d_1, d_2, \dots, d_\ell))$ and $\sigma^{-1}(\{1, 2, \dots, d_{i+1}\}) = \tau^{-1}(\{1, 2, \dots, d_{i+1}\})$ for every $0 \leq i \leq \ell$.

(2.9) Given a linear extension σ of a finite poset P with $\mathscr{D}(\sigma)=(d_1, d_2, \ldots, d_\ell)$, we write $I_i(\sigma)$ for the poset ideal $\sigma^{-1}(\{1, 2, \ldots, d_i\})$ of P for each $1 \leq i \leq \ell+1$, where $d_{\ell+1} = \#(P)$. Also, set $I_0(\sigma) = \varnothing$. Then the chain

$$\mathscr{M}(\sigma) : \varnothing = I_0(\sigma) \subsetneqq I_1(\sigma) \subsetneqq \cdots \subsetneqq I_{\ell+1}(\sigma) = P$$

in the distributive lattice $\mathscr{J}(P)$ possesses the properties (i) and (ii) in (2.2).

On the other hand, for each chain $\mathscr{M}$ in (2.2), there exists a linear extension σ of P with $\mathscr{M} = \mathscr{M}(\sigma)$. Moreover, $\mathscr{M}(\sigma) = \mathscr{M}(\tau)$ if and only if σ and τ are equivalent.

We now come to the main result of this section in consequence of Proposition (2.4) with Lemma (2.7) and (2.9).

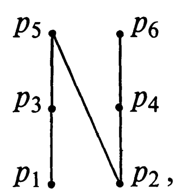
(2.10) **Theorem.** *The following quantities on a finite poset P are equal:*

(a) *the Cohen-Macaulay type* $\mathrm{type}(k[\Delta(L)])$ *of the Stanley-Reisner ring $k[\Delta(L)]$ of the order complex $\Delta(L)$ of the finite distributive lattice $L = \mathscr{J}(P)$,*

(b) *the number of strictly order preserving maps $\sigma: P \to \mathbf{N}$ such that $\sigma^{-1}(\{i-1, i\})$ is not a clutter in P for every $i \in \sigma(P)$ with $i \geq 1$,*

(c) *the number of distinct equivalence classes of the equivalence relation "$\sim$" in (2.8) on the set of linear extensions of the poset P.*

(2.11) Example. Let $P = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ be the following finite poset:



and we employ the notation, e.g., 214635 for denoting the linear extension σ of P with $\sigma(p_2) = 1$, $\sigma(p_1) = 2$, $\sigma(p_4) = 3$, $\sigma(p_6) = 4$, $\sigma(p_3) = 5$ and $\sigma(p_5) = 6$. Then the equivalence classes of the equivalence relation "$\sim$" in (2.8) on the set of linear extensions of the poset P are

$$\{123456, 123465, 124356, 124365, \\ 213456, 213465, 214356, 214365\},$$
$$\{123546, 213546\},$$
$$\{124635, 214635\},$$
$$\{132546, 132456\},$$
$$\{132465\},$$
$$\{241635, 241365\},$$
$$\{246135\}, \quad \text{and}$$
$$\{241356\}.$$

Hence the Cohen-Macaulay type $\text{type}(k[\Delta(L)])$ of the Stanley-Reisner ring $k[\Delta(L)]$ of the order complex $\Delta(L)$ of the distributive lattice $L = \mathscr{J}(P)$ is equal to eight. Note that the h-vector of $k[\Delta(L)]$ is $h(k[\Delta(L)]) = (1, 8, 9, 1)$.

It might be of interest to find a "nice" formula to compute the number of distinct equivalence classes of the equivalence relation " $\sim$ " in (2.8) on the set of linear extensions of P when P is, e.g., a *rooted tree* [**Sta6**, p. 294]).

We here turn to the problem of finding a chain condition of P for the Stanley-Reisner ring $k[\Delta(L)]$ to be level.

(2.12) The *altitude* of a finite poset P, written as $\text{alt}(P)$, is defined to be the maximal number $\ell \geq 0$ for which there exists a finite sequence $C_0, C_1, \ldots, C_r$ of chains in P such that

(i) every $y \in C_j$ is neither less than nor equal to each $x \in C_i$ if $0 \leq i < j \leq r$, and

(ii) the sum of the cardinalities of C_i's is $\ell + r + 1$.

Obviously, we have $\text{rank}(P) \leq \text{alt}(P)$.

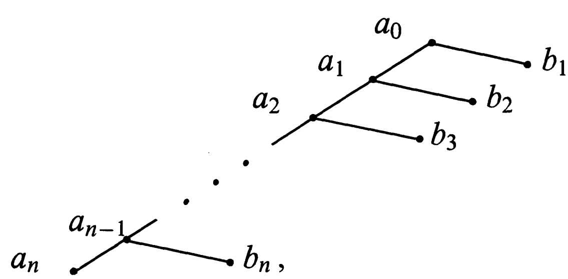
(2.13) Lemma. $\rho(P; \text{alt}(P)) \neq 0$.

Proof. Work in the same notation as in (2.12) with $\ell = \text{alt}(P)$. We write Q for the subposet $C_0 \cup C_1 \cup \cdots \cup C_r$ of P. Then we have $\text{alt}(P) = \text{alt}(Q)$. On the other hand, there exists a unique $\tau \in \mathscr{B}(Q; \text{alt}(Q))$ such that $\tau(\alpha) \leq \tau(\beta)$ if $\alpha \in C_i$ and $\beta \in C_j$ with $0 \leq i < j \leq r$. Let I_i, $0 \leq i \leq \text{alt}(P)$, be the poset ideal of P which consists of those elements $x \in P$ such that $x < \alpha$ for some $\alpha \in Q$ with $\tau(\alpha) \leq i$. In particular $I_0 = \varnothing$. Also, we set $I_{\text{alt}(P)+1} = P$. Then, the chain $\varnothing = I_0 \subsetneqq I_1 \subsetneqq \cdots \subsetneqq I_{\mathrm{alt}(P)+1} = P$ in the distributive lattice $\mathscr{J}(P)$ satisfies the conditions (i) and (ii) in (2.2). Thus $\rho(P; \mathrm{alt}(P)) \neq 0$ as desired. □

Hence, we have $\rho(P; \ell) = 0$ if either $\ell < \mathrm{rank}(P)$ or $\ell > \mathrm{alt}(P)$ and $\rho(P; \mathrm{rank}(P)) \neq 0$, $\rho(P; \mathrm{alt}(P)) \neq 0$. Thus, thanks to Corollary (2.5), we immediately obtain

(2.14) COROLLARY. *The Stanley-Reisner ring* $k[\Delta(L)]$ *of the order complex* $\Delta(L)$ *of a finite distributive lattice* $L = \mathscr{J}(P)$ *is level if and only if* $\mathrm{rank}(P) = \mathrm{alt}(P)$.

(2.15) EXAMPLE. If C_n is the following finite poset



then the Stanley-Reisner ring $k[\Delta(L)]$ of the order complex $\Delta(L)$ of the finite distributive lattice $L = \mathscr{J}(C_n)$ is level with the Cohen-Macaulay type $\mathrm{type}(k[\Delta(L)]) = n!$.

(2.16) Recall that the *height* (resp. *depth*) $\mathrm{height}_P(\alpha)$ (resp. $\mathrm{depth}_P(\alpha)$) of an element α of a finite poset P is the maximal number $\ell \geq 0$ for which there exists a chain in P of the form $\alpha_\ell < \alpha_{\ell-1} < \cdots < \alpha_0 = \alpha$ (resp. $\alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_\ell$). Thus we have $\mathrm{height}_P(\alpha) + \mathrm{depth}_P(\alpha) \leq \mathrm{rank}(P)$ for every element $\alpha \in P$. On the other hand, if α and β are incomparable elements of P, then $\mathrm{height}_P(\alpha) + \mathrm{depth}_P(\beta) \leq \mathrm{alt}(P)$. We write $P^{(+)}$ for the subposet of P which consists of all elements $\alpha \in P$ with $\mathrm{height}_P(\alpha) + \mathrm{depth}_P(\alpha) = \mathrm{rank}(P)$.

(2.17) COROLLARY. *Suppose that the Stanley-Reisner ring* $k[\Delta(L)]$ *of the order complex* $\Delta(L)$ *of a finite distributive lattice* $L = \mathscr{J}(P)$ *is level. If* α *and* β *are incomparable elements of the poset* P, *then we have the inequality* $\mathrm{height}_P(\alpha) + \mathrm{depth}_P(\beta) \leq \mathrm{rank}(P)$. *Thus, in particular, the subposet* $P^{(+)}$ *of* P *is the* ordinal sum **[Sta6**, p. 100] *of clutters.*

(2.18) We say that a finite poset P satisfies the *λ-chain condition* [**Sta6**, p. 219] if $P = P^{(+)}$. It is known, e.g., [**Sta6**, Corollary (4.5.17)] that a poset P satisfies the λ-chain condition if and only if the last non-zero component of the h-vector of the order complex $\Delta(L)$ of the distributive lattice $L = \mathscr{J}(P)$ is equal to one.

(2.19) COROLLARY. *The Stanley-Reisner ring $k[\Delta(L)]$ of the order complex $\Delta(L)$ of a finite distributive lattice $L = \mathscr{J}(P)$ is Gorenstein, i.e.,* $\mathrm{type}(k[\Delta(L)]) = 1$, *if and only if the poset P is the ordinal sum of clutters.*

REFERENCES

[Bac] K. Baclawski, *Cohen-Macaulay connectivity and geometric lattices*, European J. Combin., **3** (1982), 293–305.

[B-G-S] A. Björner, A. Garsia and R. Stanley, *An introduction to Cohen-Macaulay partially ordered sets*, in *Ordered Sets* (I. Rival, ed.), Reidel, Dordrecht/Boston, (1982), 583–615.

[B-K] A. Björner and G. Kalai, *On f-vectors and homology*, in *Combinatorial Mathematics*, Annals of the New York Academy of Sciences, **555** (1989), 63–80.

[Dow] T. A. Dowling, *On the independent set numbers of a finite matroid*, Ann. Discrete Math., **8** (1980), 21–28.

[G-M-R] A. Geramita, P. Maroscia and L. Roberts, *The Hilbert function of a reduced k-algebra*, J. London Math. Soc., (2) **28** (1983), 443–452.

[Her] I. N. Herstein, *Topics in Algebra*, Blaisdell, New York/Toronto/London, 1964.

[H-K] J. Herzog and E. Kunz, *Der Kanonische Modul eines Cohen-Macaulay-Rings*, Lect. Notes in Math., No. 238, Springer-Verlag, Berlin/Heidelberg/New York, 1971.

[H1] T. Hibi, *Distributive lattices, affine semigroup rings and algebras with straightening laws*, in *Commutative Algebra and Combinatorics* (M. Nagata and H. Matsumura, eds.), Advanced Studies in Pure Math., Vol. 11, North-Holland, Amsterdam, (1987), 93–109.

[H2] ——, *Level rings and algebras with straightening laws*, J. Algebra, **117** (1988), 343–362.

[H3] ——, *Canonical ideals of Cohen-Macaulay partially ordered sets*, Nagoya Math. J., **112** (1988), 1–24.

[H4] ——, *What can be said about pure 0-sequences?*, J. Combinatorial Theory Ser. A, **50** (1989), 319–322.

[H5] ——, *Flawless 0-sequences and Hilbert functions of Cohen-Macaulay integral domains*, J. Pure and Appl. Algebra, **60** (1989), 245–251.

[H6] ——, *Hilbert functions of Cohen-Macaulay integral domains and chain conditions of finite partially ordered sets*, J. Pure and Appl. Algebra, **72** (1991), 265–273.

[H7] ——, *Cohen-Macaulay type of the face poset of a plane graph*, to appear.

[Hoc] M. Hochster, *Cohen-Macaulay rings, combinatorics, and simplicial complexes*, in *Ring Theory* II (B. R. McDonald and R. Morris, eds.), Lect. Notes in Pure and Appl. Math., No. 26, Dekker, New York, (1977), 171–223.

[Mah] C. Mahoney, *On the unimodality of the independent set numbers of a class of matroids*, J. Combinatorial Theory, Ser. B, **39** (1985), 77–85.

[Mas] J. H. Mason, *Matroids: Unimodal conjectures and Motzkin's theorem*, in *Combinatorics* (D. J. A. Welsh and D. R. Woodall, eds.), Inst. Math. Appl., Southend-on-Sea, England, (1972), 207–220.

[Mat] H. Matsumura, *Commutative Ring Theory* (translated by M. Reid), Cambridge Univ. Press, Cambridge, 1986.

[Rei] G. Reisner, *Cohen-Macaulay quotients of polynomial rings*, Advances in Math., **21** (1976), 30–49.

[R-R] L. Roberts and M. Roitman, *On Hilbert functions of reduced and of integral algebras*, J. Pure and Appl. Algebra, **56** (1989), 85–104.

[Sta1] R. Stanley, *The upper bound conjecture and Cohen-Macaulay rings*, Stud. Appl. Math., **54** (1975), 135–142.

[Sta2] ——, *Cohen-Macaulay complexes*, in *Higher Combinatorics* (M. Aigner, ed.), NATO Advanced Study Institute Series, Reidel, Dordrecht/Boston, (1977), 51–62.

[Sta3] ——, *Hilbert functions of graded algebras*, Advances in Math., **28** (1978), 57–83.

[Sta4] ——, *Combinatorics and Commutative Algebra*, Birkhäuser, Boston/Basel/Stuttgart, 1983.

[Sta5] ——, *Two poset polytopes*, Discrete Comput. Geom., **1** (1986), 9–23.

[Sta6] ——, *Enumerative Combinatorics, Volume* I, Wadsworth & Brooks/Cole, Monterey, Calif., 1986.

[Sta7] ——, *On the Hilbert function of a graded Cohen-Macaulay domain*, J. Pure Appl. Algebra, **73** (1991), 307–314.

Received April 8, 1991 and in revised form July 12, 1991.

HOKKAIDO UNIVERSITY
KITA-KU, SAPPORO 060, JAPAN

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the 1991 *Mathematics Subject Classification* scheme which can be found in the December index volumes of *Mathematical Reviews*. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024-1555-05.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* (ISSN 0030-8730) is published monthly except for July and August. Regular subscription rate: $190.00 a year (10 issues). Special rate: $95.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) is published monthly except for July and August. Second-class postage paid at Carmel Valley, California 93924, and additional mailing offices. Postmaster: send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Pacific Journal of Mathematics

Vol. 154, No. 2 June, 1992