ACTIONS OF FINITE GROUPS ON KNOT COMPLEMENTS

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We examine the symmetry of the complement of a non-trivial knot $K$ in $S^3$ and obtain a classification of the actions of finite groups on the complement of a non-trivial knot in the case where the actions extend to non-fixed point free actions on the three sphere. We prove the result by showing first an extension theorem which says that any action of finite group on a non-trivial knot complement extends to an action on the three sphere and then by applying the solution of the Smith conjecture.

Let $N(K)$ be a regular neighborhood of $K$; $m$, $l$ be a meridian and a preferred longitude of $K$ in $\partial N(K)$ respectively; $[m]$, $[l]$ be the homology classes in $H_1(\partial N(K), \mathbb{Z})$ represented by $m$, $l$ respectively. A knot is called a hyperbolic knot if $S^3 - K$ has a hyperbolic structure. See [R], or [B, Z] for the standard terminology that we use. The main results of this note are the following. Theorem 1 also follows from the recent result of Gordon and Luecke [G, L]. Since the proof is simple, it is included here for completeness.

**Theorem 1.** If $K$ is a hyperbolic knot, then any self-diffeomorphism of the knot complement $S^3 - \text{int}(N(K))$ extends to a self-diffeomorphism of $S^3$.

Satellite knots have property P by Gabai’s work, and torus knots are also known to have property P. One obtains the following theorem.

**Corollary 1.** Any self-diffeomorphism of a non-trivial knot complement $S^3 - N(K)$ extends to a self-diffeomorphism of $S^3$.

**Theorem 2.** If $G$ is a finite group acting smoothly on the complement $S^3 - \text{int}(N(K))$ of a non-trivial knot $K$, then the group $G$ is a cyclic or a dihedral group, and the $G$-action extends to a $G$-action on $S^3$. In particular, if $K$ is a hyperbolic knot, then $\text{Out}(\pi_1(S^3 - K))$ (or what is the same $\text{Isom}(S^3 - K)$) is a cyclic or a dihedral group.

With the help of a computer, Riley [Ri] has calculated the
Out(\pi_1(S^3 - K)) for the following hyperbolic knots, 5_2, 6_3, 7_7, 8_{21}, 9_{35}, 9_{43}, and 9_{48}, the corresponding groups are: D_2, D_4, D_4, D_2, D_6, Z_2, and D_6. The theorem explains the general fact behind Riley's work. Combining with the work of Culler, Gordon, Luecke, Shalen (see [CGLS]), Bleiler and Scharlemann [B, S] on the property P of non-trivial knots invariant under non-trivial periodic automorphisms of S^3, we have the following.

**Corollary 2.** If there exists a finite group acting smoothly non-trivially on a knot complement in S^3, then the knot has property P. In particular, if K is a hyperbolic knot with non-trivial Out(\pi_1(S^3 - K)), then K has property P.

If the group G in Theorem 2 is cyclic, the G-action on the knot complement can be described more explicitly. Before stating the corollary, let us make the following conventions. A 2\pi/n-rotation of S^3 is a Z_n-action which is conjugate to the orientation preserving Z_n-action generated by A where A sends a point (x, z) in S^3 = R^1 \times C \cup \{\text{infinity}\} to (x, e^{2\pi i/n}z) and infinity to infinity. The circle \{(x, z)|z = 0\} \cup \{\text{infinity}\} is called the axis of the rotation. A twisted 2\pi/n-rotation of S^3 is an action conjugate to the non-orientation preserving Z_n-action generated by \alpha, where \alpha is described as follows. Represent S^3 as (R^1 \times C) \cup \{\text{infinity}\}, \alpha is the automorphism sending (x, z) to (−x, −e^{2\pi i/n}z), and infinity to infinity. The circle \{(x, z)|z = 0\} \cup \{\text{infinity}\} is called the axis of the twisted rotation. A reflection of S^3 through two points is an action conjugate to the orientation reversing involution of S^3 generated by \beta, where \beta is the automorphism of S^3 considered as R^3 \cup \{\text{infinity}\} sending x to −x, for x in R^3, and infinity to infinity.

**Corollary 3.** The smooth action of a cyclic group Z_n on a non-trivial knot complement S^3 − \text{int}(N(K)) are classified as follows.

(I) The action preserves the orientation. There are two cases.

(a) The action on S^3 − \text{int}(N(K)) is free. Then the action is induced by a fixed point free Z_n-action on S^3. K is invariant under the action.

(b) The action is not free. Then the Z_n-action is induced by a 2\pi/n-rotation of S^3 about a trivial knot L. K is invariant under the rotation. K is disjoint from L, or K intersects L transversely in two points. If the latter happens, n = 2.

(II) The Z_n-action on S^3 − \text{int}(N(K)) does not preserve the orientation. Then the Z_n-action has fixed points in S^3, and is of even order.
There are four kinds:

(c) \( n = 2 \). Then the action is induced by a reflection \( R \) of \( S^3 \) through two points, or is induced by a reflection \( R' \) of \( S^3 \) with respect to a two-sphere, which is the same as a twisted \( \pi \)-rotation of \( S^3 \). \( K \) is invariant under the involution. There are three types of \( \mathbb{Z}_2 \)-actions on \( S^3 - \text{int}(N(K)) \).

(c)\(_1\) \( K \) is disjoint from the two fixed points of the reflection \( R \). In this case the \( \mathbb{Z}_2 \)-action on \( S^3 - \text{int}(N(K)) \) has two fixed points.

(c)\(_2\) \( K \) contains the two fixed points of \( R \). In this case, the \( \mathbb{Z}_2 \)-action is a free action on \( S^3 - \text{int}(N(K)) \).

(c)\(_3\) \( K \) intersects the 2-sphere fixed points of \( R' \) transversely in two points. In this case, \( K \) is of the form \( K = L#(-L) \) for some knot \( L \).

(d) \( n \geq 4 \). Then the action is induced by a twisted \( 2\pi/n \)-rotation of \( S^3 \) about an axis \( L \). \( K \) is invariant, and is disjoint from \( L \).

We state the following as a corollary for convenience.

**Corollary 4.** If a cyclic group \( \mathbb{Z}_n \) generated by \( g \) acts smoothly on a non-trivial knot complement \( S^3 - \text{int}(N(K)) \) such that \( g_*([L]) = -[L] \) in \( H_1(\partial N(K), \mathbb{Z}) \), then \( g \) is an involution.

Combining Corollaries 3 and 4, smooth action of dihedral groups on a knot complement can also be classified. We omit it here.

Recall that a knot \( K \) is invertible if \( K \) is oriented equivalent to \(-K\), the inverted knot of \( K \); \( K \) is amphicheiral if \( K \) is equivalent to its mirror-image \( K^* \).

**Corollary 5.** If \( K \) is a hyperbolic knot in \( S^3 \), then the following holds.

(a) \( K \) is invertible if and only if \( K \) is invariant under a \( \pi \)-rotation in \( S^3 \) about an axis \( L \) such that \( L \) intersects \( K \) transversely in two points.

(b) \( K \) is amphicheiral if and only if \( K \) is invariant under a twisted \( 2\pi/n \)-rotation of \( S^3 \) about an axis missing \( K \), for \( n \geq 4 \), or \( K \) is invariant under a reflection of \( S^3 \) through two points missing \( K \).

(c) If \( K \) is both invertible and amphicheiral, then \( K \) is invariant under a reflection of \( S^3 \) through two points contained in \( K \).

In §1, we prove Theorem 1. In §2, we prove Theorem 2, and its corollaries. In the appendix, we prove the following proposition concerning smooth non-orientation preserving cyclic group actions on \( S^3 \).
**Proposition.** Any smooth non-orientation preserving cyclic group action on the 3-sphere is conjugate to a twisted rotation or a reflection of the sphere through two points.

**Acknowledgment.** The author would like to thank his thesis advisor M. Freedman, and X.-S. Lin for many discussions on knot theory. He also thanks the referee for the comments and for pointing out that Corollary 5(a) was a result of Kawauchi [Ka].

1. **Proof of Theorem 1.** Let \( K \) be a hyperbolic knot in \( S^3 \) with \( S^3-K \) having a hyperbolic metric; \( N(K) \) be a regular neighborhood of \( K \) such that \( \partial N(K) \) is a flat torus in \( S^3-K \) with respect to the hyperbolic metric; \( m, l \) be a meridian and a preferred longitude of \( K \) respectively, \( m, l \) lie in \( \partial N(K) \) and be realized as geodesics. \( m, l \) will also be used to denote the elements in \( \pi_1(S^3-\text{int}(N(K))) \) represented by them. Let \([m], [l]\) be the homology classes in \( H_1(\partial N(K), \mathbb{Z}) \) represented by \( m, l \) respectively. Let \( h \) be a self-diffeomorphism of \( S^3-\text{int}(N(K)) \). Our goal is to prove that \( h^*[m] \) is \( \pm[m] \) in \( H_1(\partial N(K), \mathbb{Z}) \). Since if this condition is satisfied,

\[ h|_{\partial N(K)}: \partial N(K) \to \partial N(K) \]

extends to be a self-diffeomorphism of \( N(K) \) which in turn gives an extension of \( h \) to \( S^3 \) by gluing. By Mostow Rigidity, one can assume that \( h \) is a hyperbolic isometry. \( h_*([l]) = \varepsilon_1[l] \) with \( \varepsilon_1 \) being \( \pm1 \) in \( H_1(\partial N(K), \mathbb{Z}) \), because \( \pm[l] \) are the only primitive homology classes in \( H_1(\partial N(K), \mathbb{Z}) \) which vanish in \( H_1(S^3-\text{int}(N(K)), \mathbb{Z}) \) under the inclusion homomorphism. \( h_* \) is an automorphism of \( H_1(\partial N(K), \mathbb{Z}) \); hence \( h_*[m] = \varepsilon_2[m] + a[l] \), where \( \varepsilon_2 = \pm1 \), and \( a \) is in \( \mathbb{Z} \). Our goal is to show \( a = 0 \). If \( \varepsilon_1 = \varepsilon_2 \), i.e., \( h \) is orientation preserving, the result is trivial because on one hand \( h \), being an isometry of a hyperbolic manifold of finite volume, is of finite order (i.e., composition of \( h \) finite times is the identity map; see [M, B], or [Th]), on the other hand the matrix \( \begin{pmatrix} \varepsilon_2 & a \\ 0 & \varepsilon_1 \end{pmatrix} \) has infinite order if \( a \) is non-zero. Therefore, we need only to consider the case where \( \varepsilon_1 = -\varepsilon_2 \). Suppose conversely \( a \neq 0 \). Then by Culler, Gordon, Luecke, Shalen [CGLS], one has that \( a = \pm1 \), and that \( K \) does not have property P. Since the matrix \( \begin{pmatrix} \varepsilon_2 & a \\ 0 & \varepsilon_1 \end{pmatrix} \) is of order two, \( h_*h_* = \text{id} \) in \( H_1(\partial N(K), \mathbb{Z}) \). Consider the orientation preserving isometry \( g = h \circ h \). \( g \) is of finite order; hence it generates a finite cyclic group \( G \) acting isometrically on the flat torus \( \partial N(K) \). Because \( g_*([m]) = [m] \) and \( g_*[l] = [l] \) in \( H_1(\partial N(K), \mathbb{Z}) \), \( G \) preserves the foliations \( \partial N(K) \) by geodesic
meridians and by geodesic longitudes. The following lemma shows that the $G$-action on $\partial N(K)$ can be extended to a $G$-action on $N(K)$.

**Lemma 1.** If $G$ acts isometrically on a flat boundary $\partial N$ of a solid torus $N$ and $g_\ast[m] = \pm[m], g_\ast[l] = \pm[l]$ in $H_1(\partial N, \mathbb{Z})$ where $g$ is a generator of $G$, $m, l$ are a meridian and a longitude of $\partial N$ respectively, then the $G$-action can be extended to an action on $N$. Moreover the extended $G$-action on the core of $N$ preserves a flat Riemannian metric on it.

**Proof.** Parametrize $\partial N$ by $(u, v)$, where $u, v$ are the unit complex numbers such that $S^1 \times \{v\}$ and $\{u\} \times S^1$ correspond to the geodesic meridian $m$ and the geodesic longitude $l$ in $\partial N$. Since the action on the homology group $H_1(\partial N, \mathbb{Z})$ satisfies the conditions above, the $G$-action on $\partial N$ corresponds now to a $G$-action on $S^1 \times S^1$ preserving the standard product metric and the product structure. Extending the $G$-action on $\partial N$ to $N$ is the same as extending the $G$-action on $S^1 \times S^1$ to $D^2 \times S^1$. The extension of the latter is trivial. To see this, for $g \in G$, we have,

$$g(u, v) = (\phi(u, g), \psi(v, g))$$

where $u, v \in S^1$, $\phi(u, g) = \alpha u$, or $\alpha \bar{u}$, and $\psi(v, g) = \beta v$ or $\beta \bar{v}$, for some roots of unity $\alpha$ and $\beta$. The extension of the $G$-action to $D^2 \times S^1$ is given by the same formula with $u$ in $D^2 = \{z \in \mathbb{C} | |z| \leq 1\}$. The extended $G$-action still preserves the product metric and acts on the core $\{0\} \times S^1$ isometrically with respect to the flat metric induced from $D^2 \times S^1$.

We have now a cyclic group $G$ which acts on $S^3$ preserving $K$. If $G$ is non-trivial, then $K$ has property P by Corollary 7 of Culler, Gordon, Luecke, Shalen [CGLS] which contradicts $a \neq 0$. Therefore $h \circ h = \text{id}$ in $S^3 - \text{int}(N(K))$. It is easy to check, using $a = \pm 1$, $h_\ast([m]) = -\varepsilon_1[m] + a[l]$ and $h_\ast([l]) = \varepsilon_1[l]$, that

$$h_\ast(-2\varepsilon_1 a[m] + [l]) = -\varepsilon_1(-2\varepsilon_1 a[m] + [l]).$$

Note that $[l]$, and $-2\varepsilon_1 a[m] + [l]$ are primitive elements, and are the $(\pm 1)$-eigenvectors of $h_\ast$ in $H_1(\partial N(K), \mathbb{Z})$. The algebraic intersection number of $[l]$, and $-2\varepsilon_1 a[m] + [l]$ is $\pm 2$. The following lemma shows that $h$ has fixed points in $\partial N(K)$. 


**Lemma 2.** Suppose $h$ is an orientation reversing fixed point free involution of a torus $T^2$, then the $(\pm 1)$-eigenspaces of $h_*$ are generated by two primitive classes with $\pm 1$ as their algebraic intersection number.

**Proof.** Since any orientation reversing fixed point free involution of $T^2$ has the quotient space homeomorphic to the Klein bottle, and since the Klein bottle has only one orientable two-fold cover up to covering equivalence, any two orientation reversing fixed point free involutions on $T^2$ are conjugate. Because the hypothesis and the conclusion of the lemma are invariant under conjugation, the lemma follows by checking a concrete example. Take $T^2$ to be $S^1 \times S^1$ parametrized by $(u, v)$, where $u, v \in S^1$, the unit circle in the complex plane. Let $h: T^2 \to T^2$ be the automorphism sending $(u, v)$ to $(\bar{u}, -v)$. $h$ generates a fixed point free orientation reversing involution of $T^2$. The $1$-eigenspace of $h_*$ is generated by the homology class of the curve $\{1\} \times S^1$, and the $(-1)$-eigenspace of $h_*$ is generated by the homology class of the curve $S^1 \times \{1\}$. Hence the algebraic intersection number of the primitive generators of $(\pm 1)$-eigenspaces is $\pm 1$.

By the lemma, $h$ has fixed points in $\partial N(K)$. However, $h$ is an orientation reversing involution, $\text{Fix}(h|_{\partial N(K)})$ is a 1-dimensional submanifold. This implies that $\text{Fix}(h)$ contains a 2-manifold, say $F$. We claim that this is impossible. By Smith theory (see [B], Theorem 5.1), for the $\mathbb{Z}_2$-action generated by $h$ on the 1-dimensional $\mathbb{Z}_2$-homology sphere $S^3 - \text{int}(N(K))$, the fixed point set $\text{Fix}(h)$ is a $\mathbb{Z}_2$-homology sphere of dimension at most one. Hence $\text{Fix}(h)$ (= $F$) is an annulus or a Möbius band.

**Case 1.** $F$ is an annulus. Since $S^3 - K$ has a hyperbolic structure, $S^3 = \text{int}(N(K))$ is annulus free. Hence $F$ is parallel to an annulus in $\partial N(K)$. In particular, $F$ is separating. The two components of the complement of $F$ in $S^3 - \text{int}(N(K))$ are interchanged by $h$ and hence are homeomorphic. Therefore both of them are solid tori. This implies that $S^3 - \text{int}(N(K))$ is the union of two solid tori along an annulus in their boundaries which contradicts the existence of the hyperbolic structure of finite volume in $S^3 - K$.

**Case 2.** $F$ is a Möbius band. $\partial F$ is now a simple closed curve in $\partial N(K)$ fixed by $h$, and hence $[\partial F]$ is in the 1-eigenspace of $h_*$ which is generated by $[l]$, or by $2a[m] + [l]$ according to $\varepsilon_1 = 1$, or $-1$. Thus $\partial F$ and $K$ bound an annulus $A$ in $N(K)$. The Möbius band $F \cup_{\partial} A$ in $S^3$ has $K$ as its boundary. Let $L$ be the core of
the Möbius band. If $L$ is non-trivial, $K$ is the cable knot of $L$. This contradicts that $K$ is a hyperbolic knot. If $L$ is the trivial knot, then $K$ is the $(2, n)$-torus which is again absurd.

This completes the proof of Theorem 1.

Since any non-trivial knot with property $P$ has the property that any self-diffeomorphism of the knot complement preserves the meridian, and since the only non-trivial knots which are not known to have property $P$ are some hyperbolic knots by the work of Gabai and others, Corollary 2 follows from Theorem 1.

2. Proof of Theorem 2. We shall still use the same notations introduced in §1. Hence $K$ is a non-trivial knot in $S^3$; $N(K)$ is a regular neighborhood of $K$; $m, l$ are a meridian and a preferred longitude of $K$ respectively. $m, l$ lie in $\partial N(K)$. Our first observation is that there exists a flat metric on $\partial N(K)$ such that $G$ acts on $\partial N(K)$ isometrically. This follows from the Geometrization Theorem that any action of a finite group $G$ on a 2-manifold is equivalent to a geometric group action (see [E]). Fix the metric on $\partial N(K)$, and realize $m, l$ by geodesics in $\partial N(K)$. Theorem 1 shows that the $G$-action on $\partial N(K)$ preserves the geodesic meridians and geodesic longitudes in $\partial N(K)$. By Lemma 1, the $G$-action on $\partial N(K)$ extends to a $G$-action on $N(K)$ such that the extended $G$-action preserves a flat metric on $K$. Hence the $G$-action on $S^3 - \text{int}(N(K))$ extends to a $G$-action on $S^3$ which preserves $K$ and acts on $K$ preserving a flat metric $d$. The restriction of the $G$-action to $K$ gives a representation:

$$\sigma: G \to \text{Isom}(K, d).$$

The solution of the Smith Conjecture shows that $\sigma$ is a monomorphism. To see this, let $h \in \ker(\sigma)$, and $H$ be the cyclic group by $h$. Then $H$ acts on $S^3$ with fixed point set containing $K$, and $H$ preserves each geodesic meridian in $\partial N(K)$. Moreover, $h_*([l]) = [l]$ in $H_1(\partial N(K), \mathbb{Z})$. There are now two cases that might happen.

**Case 1.** $h_*([m]) = [m]$. $h$ is now an orientation preserving homeomorphism because $h_*([l]) = [l]$ and $h_*([m]) = [m]$ imply that $h$ is an orientation preserving homeomorphism in $H_1(\partial N(K), \mathbb{Z})$. Therefore the $H$-action on a geodesic meridian $m$ is a rotation. Suppose $h \neq \text{id}$; then $H$ acts non-trivially on $m$. Therefore $K$ is the only fixed point set of $h$ in $N(K)$. By Smith theory, $\text{Fix}(h) = K$, which then contradicts the solution of the Smith Conjecture.
Case 2. \( h_*([m]) = -[m] \). \( h \) is now an orientation reversing homeomorphism. Since \( h \circ h \in \ker(\sigma) \), and \( h_*h_*([m]) = [m] \), one has \( h \circ h = \text{id} \) by the solution of Case 1. Hence \( h \) is an orientation reversing involution of \( S^3 \) with fixed point set containing \( K \). Because the \( \text{Fix}(h) \) is a submanifold of odd codimension and contains \( K \), \( \text{Fix}(h) \) contains a 2-manifold. By Smith Theory, the \( \text{Fix}(h) \) is a \( \mathbb{Z}_2 \)-homology sphere. Hence \( \text{Fix}(h) \) is a 2-sphere and contains \( K \). This implies that \( K \) is a trivial knot which is absurd.

Therefore \( G \) is a subgroup of \( \text{Isom}(K, d) \). It is well known that a finite subgroup of \( \text{Isom}(K, d) \) is a cyclic or a dihedral group. In case \( K \) is a hyperbolic knot, \( \text{Out}(\pi_1(S^3 - K)) \) acts isometrically on \( S^3 - \text{int}(N(K)) \) where \( \partial N(K) \) is a flat torus in \( S^3 = K \) (see [M, B], or [Th]). Hence \( \text{Out}(\pi_1(S^3 - K)) \) (or the same \( \text{Isom}(S^3 - K) \)) is a cyclic or a dihedral group.

Proof of Corollary 3. By Theorem 2 and its proof, the \( \mathbb{Z}_n \)-action extends to a \( \mathbb{Z}_n \)-action on \( S^3 \) such that \( K \) is invariant and \( K \) intersects the fixed point set of a nontrivial element \( f \) in \( \mathbb{Z}_n \) if and only if \( \text{Fix}(\sigma(f)) \cap K \neq \emptyset \). But \( \text{Fix}(\sigma(f)) \cap K \neq \emptyset \) if and only if \( \sigma(f) \) is a reflection on \( K \) which in turn is the same as \( f_*([l]) = [-l] \) in \( H_1(\partial N(K), Z) \). Moreover, in this case, \( K \) intersects \( \text{Fix}(f) \) transversely in two points. The classification is now reduced to the classification of smooth cyclic group actions on \( S^3 \).

(I) The \( \mathbb{Z}_n \)-action preserves the orientation.

If the \( \mathbb{Z}_n \)-action on \( S^3 \) is fixed point free, we have (a). Otherwise, by Smith theory, the fixed point set is a knot, say \( L \). The solution of the Smith Conjecture shows that \( L \) is a trivial knot, and the \( \mathbb{Z}_n \)-action is a \( 2\pi/n \)-rotation about \( L \). Let \( g \) be a generator of the \( \mathbb{Z}_n \)-action. If \( L \) intersects \( K \), then by the remark above, we have \( g_*([l]) = [l] \), and \( \sigma(g) \) is a reflection in \( K \). Hence \( \text{Fix}(g \circ g) \) contains \( K \). However \( gg \) is orientation preserving. Therefore the solution of the Smith Conjecture implies that \( g \circ g \) is the identity, i.e., \( n = 2 \). This proves (b).

(II) The \( \mathbb{Z}_n \)-action does not preserve the orientation.

Let \( g \) still be the generator of the \( \mathbb{Z}_n \)-action on \( S^3 \). Since \( g \) reverses the orientation, \( g \) has fixed points in \( S^3 \), \( n \) is even, and \( \text{Fix}(g) \) is a submanifold of odd codimension in \( S^3 \).

(c) \( n = 2 \).

By Smith theory, \( \text{Fix}(g) \) is a \( \mathbb{Z}_2 \)-homology sphere. Hence \( \text{Fix}(g) \) is the two points set or the 2-sphere. If \( \text{Fix}(g) \) is the two points set,
by Livesay’s theorem [L], the $Z_2$-action is a reflection of $S^3$ through two points; if $\text{Fix}(g)$ is a 2-sphere, then the action is a reflection of $S^3$ with respect to a 2-sphere by Schonflies theorem. Now the $Z_2$-action is classified as follows. If $g_*([l]) = [l]$, then $\text{Fix}(g) \cap K = \emptyset$. In this case $\text{Fix}(G)$ cannot be a 2-sphere. To see this, $\text{Fix}(g) \cap K = \emptyset$. In this case $\text{Fix}(G)$ cannot be a 2-sphere. To see this, $\text{Fix}(g) \cap K = \emptyset$ implies the fixed point set of $g$ in $S^3$ is actually in $S^3 - \text{int}(N(K))$. By Smith theory, for the $g$ involution on the one-dimensional homology sphere $S^3 - \text{int}(N(K))$, $\text{Fix}(g|_{S^3 - \text{int}(N(K))})$ is a $Z_2$-homology sphere of dimension at most one. Hence $\text{Fix}(g)$ are two points. This gives (c). If $g_*([l]) = -[l]$, then $\sigma(g)$ is a reflection in $K$, and $K$ intersects $\text{Fix}(g)$ transversely in two points. (c)$_2$, (c)$_3$ follow from the above mentioned classification of the orientation reversing involutions of $S^3$.

(d) $n \geq 4$.

The result is a consequence of the following proposition which will be proven in the appendix.

**Proposition.** Any smooth cyclic group action on $S^3$ which does not preserve the orientation is conjugate to a twisted rotation of $S^3$, or to a reflection of $S^3$ through two points.

Applying the proposition, we need only to check that $K$ is disjoint from the axis of the twisted rotation $g$. However the axis of $g$ is $\text{Fix}(g \circ g)$. $\text{Fix}(g \circ g)$ does not intersect $K$ follows now from $g_*g_*([l]) = [l]$, and $g \circ g \neq \text{id}$. This completes the proof of (d).

Corollary 4 is actually proven in the proof of Corollary 3.

**Proof of Corollary 5.** (a) By Proposition 3.19 of [B, Z], $K$ is invertible if and only if there is an automorphism

$$\phi: \pi_1(S^3 - \text{int}(N(K))) \to \pi_1(S^3 - \text{int}(N(K)))$$

such that $\phi(m) = m^{-1}$ and $\phi(l) = l^{-1}$. Since $K$ is a hyperbolic knot, Mostow Rigidity Theorem shows that $\phi$ can be realized by a hyperbolic isometry $h: S^3 - \text{int}(N(K)) \to S^3 - \text{int}(N(K))$ such that $h_*([m]) = -[m]$, and $h_*([l]) = -[l]$ in $H_1(\partial N(K), \mathbb{Z})$. Here we have assumed that $\partial N(K)$ is a flat torus in $S^3 - K$. The condition $h_*([l]) = -[l]$ implies that $h$ is an involution by Corollary 4. Because $h_*([m]) = -[m]$, $h$ is orientation preserving. Hence by Corollary 3, the $Z_2$-action generated by the extension of $h$ on $S^3$ is induced by a $\pi$-rotation of $S^3$ about an axis $L$. $H_*([l]) = -[l]$ implies that
L intersects K transversely in two points. Therefore K is invariant under a \( \pi \)-rotation about an axis intersecting K at two points. The inverse implication is trivial.

(b) By Proposition 3.19 of [B, Z], K is amphicheiral if and only if there is an automorphism

\[
\phi: \pi_1(S^3 - \text{int}(N(K))) \rightarrow \pi_1(S^3 - \text{int}(N(K)))
\]

such that \( \phi(m) = m^{-1} \) and \( \phi(l) = l \). Realize \( \phi \) by an isometry \( h: S^3 - \text{int}(N(K)) \rightarrow S^3 - \text{int}(N(K)) \). \( h \) is orientation reversing since \( h_*([m]) = -[m] \), and \( h_*([l]) = [l] \) in \( H_1(\partial N(K), Z) \). \( h \) generates a smooth cyclic group action on \( S^3 - \text{int}(N(K)) \) which does not preserve the orientation. Hence by Corollary 3, \( h \) is induced by a twisted rotation of \( S^3 \) about an axis L missing K if the order of \( h \) is at least four. If the order of \( h \) is two, the \( h \) involution is the case (c) in Corollary 3 because \( h_*([l]) = [l] \). Therefore, in this case K is invariant under a reflection of \( S^3 \) through two points missing K. Then the condition is clearly sufficient.

(c) If the knot is both invertible and amphicheiral, then there exists an automorphism

\[
\phi: \pi_1(S^3 - \text{int}(N(K))) \rightarrow \pi_1(S^3 - \text{int}(N(K)))
\]

such that \( \phi(m) = m \), and \( \phi(l) = l^{-1} \). \( \phi \) is the composition of the two automorphisms coming from (a) and (b). Realize \( \phi \) by an orientation reversing hyperbolic isometry \( h \) such that \( h_*([m]) = [m] \), and \( h_*([l]) = -[l] \) in \( H_1(\partial N(K), Z) \). By Corollary 4, \( h_*([l]) = -[l] \) and \( h_*([m]) = [m] \) imply \( h \) is an orientation reversing involution of \( S^3 - \text{int}(N(K)) \rightarrow S^3 - N(K) \). By Corollary 3, \( h \) is the case (c) or the case (c). Case (c) cannot happen since K is a prime knot. Hence K is invariant under the reflection of \( S^3 \) through two points contained in K.

**Appendix.** We prove the following proposition concerning smooth cyclic group action on the 3-sphere which does not preserve the orientation.

**Proposition.** Any smooth non-orientation preserving cyclic group action on \( S^3 \) is conjugate to a twisted rotation of \( S^3 \), or to a reflection of \( S^3 \) through two points.

**Proof.** Let \( g \) be a generator of the \( Z_n \)-action. \( n \) has to be even. \( g \) is orientation reversing, and hence has fixed points in \( S^3 \). If \( n = 2 \),
we have shown in the proof of Corollary 3 (c) that the result holds. Assume \( n \geq 4 \) from now on. Let \( h = g \circ g \). \( h \) is an orientation preserving automorphism of order \( m \), and has fixed points. The solution of the Smith Conjecture shows that the \( \text{Fix}(h) \) is a trivial knot, say \( L \). Now \( L \) is invariant under \( g \). \( g \) acts on \( L \) with fixed point and is of order two in \( L \). Hence the action of \( g \) on \( L \) is a reflection by the classification of \( \mathbb{Z}_2 \)-action on the circle. Take a \( \mathbb{Z}_n \)-equivariant regular neighborhood \( N(L) \) of \( L \) in \( S^3 \) (see [B]). By the choice of the regular neighborhood, one knows that the action of \( \mathbb{Z}_n \) on \( N(L) \) is standard. Therefore by choosing the generator \( g \) of the \( \mathbb{Z}_n \)-action appropriately, we can assume that the restriction of \( g \) on \( N(L) = D^2 \times S^1 \) is conjugate to \( \alpha \), where

\[
\alpha : D^2 \times S^1 \to D^2 \times S^1
\]

sends \((z, w)\) to \((e^{2\pi i/n}z, \bar{w})\), with \( z \) in \( D^2 = \{z \in C | |z| \leq 1\} \) and \( w \) in \( S^1 = \{z \in C | |z| = 1\} \). Note that \( \alpha \) generates an orientation reversing \( \mathbb{Z}_n \)-action on \( D^2 \times S^1 \) with two fixed points in \( \{0\} \times S^1 \). Since \( L \) is the trivial knot, \( S^3 - \text{int}(N(L)) \) is a solid torus. Let \( \phi : S^3 = (S^3 - \text{int}(N(L))) \cup N(L) \to \tilde{S}^3 = (S^1 \times D^2) \cup \text{id}(D^2 \times S^1) \) be a diffeomorphism taking \( N(L) \) to \( D^2 \times S^1 \) such that \( \phi g|_{N(L)} \phi^{-1} = \alpha \). Now extend \( \alpha \) to be a self-diffeomorphism \( \overline{\alpha} \) of \( \tilde{S}^3 \) by sending \((z, w)S^1 \times D^2 \) to \((e^{2\pi i/n}z, \bar{w})\) with \( z \in S^1 \) and \( w \in D^2 \). Then \( \overline{\alpha} \) generates a twisted \( 2\pi/n \)-rotation of \( \tilde{S}^3 \). Our goal is to show that \( \phi g \phi^{-1} \) is conjugate to \( \overline{\alpha} \) in \( \tilde{S}^3 \). This is consequence of the following claim.

**Claim.** \( g' = \phi g \phi^{-1}|_{S^1 \times D^2} \) is conjugate to \( \beta = \overline{\alpha}|_{S^1 \times D^2} \) by a piecewise smooth diffeomorphism \( \psi \) such that \( \psi \) is the identity map on \( \partial(S^1 \times D^2) \).

Let us assume the claim and finish the proof. By gluing \( \psi \) with \( \text{Id}|_{D^2 \times S^1} \) along the boundaries, we obtain a piecewise smooth self-diffeomorphism of \( \tilde{S}^3 \) which conjugates \( \phi g \phi^{-1} \) to \( \overline{\alpha} \). Therefore \( \phi g \phi^{-1} \) is smoothly conjugate to \( \overline{\alpha} \) by the work of Moise.

**Proof of the Claim.** By the choice of \( \phi \), \( g' \) is the same as \( \beta \) on \( \partial(S^1 \times D^2) \). Using the equivariant Dehn's lemma, we can find \( n \) copies of disjoint properly embedded disks \( D_1, D_2, \ldots, D_n \) with \( \partial D_j = e^{\pi ji/n} \times \partial D^2 \) in \( S^1 \times D^2 \), such that \( g'(D_j) = D_{j+1} \) for \( j = -1 \).
$D_1 = D_{n+1}$. \(g': D_j \to D_{j+1}\) is a diffeomorphism for each \(j\). These disks cut \(S^1 \times D^2\) into \(n\) components, say \(B_1, B_2, \ldots, B_n\) with \(D_j \cup D_{j+1} \subset \partial B_j\), and each of \(B_j\) is a 3-ball by Schonflies' theorem. Let \(D'_j = e^{2\pi ij/n} \times D^2\) (where \(D'_{n+1} = D'_1\)); \(B'_j = \{e^{2\pi it/n} | j \leq t \leq j + 1\} \times D^2\); and \(E_j = \partial B'_j - (D_j \cup D_{j+1})\), the annulus, for each \(i = 1, 2, \ldots, n\). The construction of \(\psi\) is now as follows. Let \(A_1: D_1 \to D'_1\) be a diffeomorphism which is the identity on \(\partial D_1\). Define \(A_2: D_2 \to D'_2\) to be \(\beta|_{D_j A_1 g^{-1}|D_2}\). It is still a diffeomorphism which fixes \(\partial D_2\) pointwise. Since \(\partial B_1 = D_1 \cup E_1 \cup D_2\) and \(\partial B'_1 = D'_1 \cup E_1 \cup D'_2\), glue \(A_1, A_2\) and \(\text{id}|_{E_1}\) along the boundaries, one obtains a piecewise smooth diffeomorphism from \(\partial B_1 \to \partial B'_1\) which is the identity on \(E_1\). Extend it to be a piecewise smooth diffeomorphism from \(B_1\) to \(B'_1\) by Alexander's lemma, and call it \(\psi_1\). Now \(\psi_j: B_j \to B'_j\) is defined to be

\[\beta_j|_{B'_j} \psi_1 g^{-1}|_{B_j}\]

for \(j = 2, 3, \ldots, n\). All these piecewise smooth diffeomorphisms match on the \(D_j\)'s. Gluing them together along the \(D_j\)'s, we obtain a piecewise diffeomorphism \(\psi: S^1 \times D^2 \to S^1 \times D^2\). Then \(\psi|_{\partial(S^1 \times D^2)} = \text{id}\) and \(\beta = \psi^{-1} \beta \psi\).

**References**


Received February 23, 1991. Supported in part by NSF DMS 86-3126.

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