ACTIONS OF FINITE GROUPS ON KNOT COMPLEMENTS

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We examine the symmetry of the complement of a non-trivial knot $K$ in $S^3$ and obtain a classification of the actions of finite groups on the complement of a non-trivial knot in the case where the actions extend to non-fixed point free actions on the three sphere. We prove the result by showing first an extension theorem which says that any action of finite group on a non-trivial knot complement extends to an action on the three sphere and then by applying the solution of the Smith conjecture.

Let $N(K)$ be a regular neighborhood of $K$; $m$, $l$ be a meridian and a preferred longitude of $K$ in $\partial N(K)$ respectively; $[m], [l]$ be the homology classes in $H_1(\partial N(K), Z)$ represented by $m, l$ respectively. A knot is called a hyperbolic knot if $S^3 - K$ has a hyperbolic structure. See [R], or [B, Z] for the standard terminology that we use. The main results of this note are the following. Theorem 1 also follows from the recent result of Gordon and Luecke [G, L]. Since the proof is simple, it is included here for completeness.

**Theorem 1.** If $K$ is a hyperbolic knot, then any self-diffeomorphism of the knot complement $S^3 - \text{int}(N(K))$ extends to a self-diffeomorphism of $S^3$.

Satellite knots have property P by Gabai's work, and torus knots are also known to have property P. One obtains the following theorem.

**Corollary 1.** Any self-diffeomorphism of a non-trivial knot complement $S^3 - N(K)$ extends to a self-diffeomorphism of $S^3$.

**Theorem 2.** If $G$ is a finite group acting smoothly on the complement $S^3 - \text{int}(N(K))$ of a non-trivial knot $K$, then the group $G$ is a cyclic or a dihedral group, and the $G$-action extends to a $G$-action on $S^3$. In particular, if $K$ is a hyperbolic knot, then $\text{Out}(\pi_1(S^3 - K))$ (or what is the same $\text{Isom}(S^3 - K)$) is a cyclic or a dihedral group.

With the help of a computer, Riley [Ri] has calculated the
Out$(\pi_1(S^3 - K))$ for the following hyperbolic knots, $5_2, 6_3, 7_7, 8_21, 9_{35}, 9_{43},$ and $9_{48}$, the corresponding groups are: $D_2, D_4, D_4, D_2, D_6, Z_2,$ and $D_6$. The theorem explains the general fact behind Riley’s work. Combining with the work of Culler, Gordon, Luecke, Shalen (see [CGLS]), Bleiler and Scharlemann [B, S] on the property P of non-trivial knots invariant under non-trivial periodic automorphisms of $S^3$, we have the following.

**Corollary 2.** If there exists a finite group acting smoothly non-trivially on a knot complement in $S^3$, then the knot has property P. In particular, if $K$ is a hyperbolic knot with non-trivial Out$(\pi_1(S^3 - K))$, then $K$ has property P.

If the group $G$ in Theorem 2 is cyclic, the $G$-action on the knot complement can be described more explicitly. Before stating the corollary, let us make the following conventions. A $2\pi/n$-rotation of $S^3$ is a $Z_n$-action which is conjugate to the orientation preserving $Z_n$-action generated by $A$ where $A$ sends a point $(x, z)$ in $S^3 = R^1 \times C \cup \{\text{infinity}\}$ to $(x, e^{2\pi i/n}z)$ and infinity to infinity. The circle $\{(x, z) | z = 0\} \cup \{\text{infinity}\}$ is called the axis of the rotation. A twisted $2\pi/n$-rotation of $S^3$ is an action conjugate to the non-orientation preserving $Z_n$-action generated by $\alpha$, where $\alpha$ is described as follows. Represent $S^3$ as $(R^1 \times C) \cup \{\text{infinity}\}$, $\alpha$ is the automorphism sending $(x, z)$ to $(-x, -e^{2\pi i/n}z)$, and infinity to infinity. The circle $\{(x, z) | z = 0\} \cup \{\text{infinity}\}$ is called the axis of the twisted rotation. A reflection of $S^3$ through two points is an action conjugate to the orientation reversing involution of $S^3$ generated by $\beta$, where $\beta$ is the automorphism of $S^3$ considered as $R^3 \cup \{\text{infinity}\}$ sending $x$ to $-x$, for $x$ in $R^3$, and infinity to infinity.

**Corollary 3.** The smooth action of a cyclic group $Z_n$ on a non-trivial knot complement $S^3 - \text{int}(N(K))$ are classified as follows.

(I) The action preserves the orientation. There are two cases.

(a) The action on $S^3 - \text{int}(N(K))$ is free. Then the action is induced by a fixed point free $Z_n$-action on $S^3$. $K$ is invariant under the action.

(b) The action is not free. Then the $Z_n$-action is induced by a $2\pi/n$-rotation of $S^3$ about a trivial knot $L$. $K$ is invariant under the rotation. $K$ is disjoint from $L$, or $K$ intersects $L$ transversely in two points. If the latter happens, $n = 2$.

(II) The $Z_n$-action on $S^3 - \text{int}(N(K))$ does not preserve the orientation. Then the $Z_n$-action has fixed points in $S^3$, and is of even order.
There are four kinds:

(c) \( n = 2 \). Then the action is induced by a reflection \( R \) of \( S^3 \) through two points, or is induced by a reflection \( R' \) of \( S^3 \) with respect to a two-sphere, which is the same as a twisted \( \pi \)-rotation of \( S^3 \). \( K \) is invariant under the involution. There are three types of \( Z_2 \)-actions on \( S^3 - \text{int}(N(K)) \).

(c)\(_1 \) \( K \) is disjoint from the two fixed points of the reflection \( R \). In this case the \( Z_2 \)-action on \( S^3 - \text{int}(N(K)) \) has two fixed points.

(c)\(_2 \) \( K \) contains the two fixed points of \( R \). In this case, the \( Z_2 \)-action is a free action on \( S^3 - \text{int}(N(K)) \).

(c)\(_3 \) \( K \) intersects the 2-sphere fixed points of \( R' \) transversely in two points. In this case, \( K \) is of the form \( K = L\#(-L) \) for some knot \( L \).

(d) \( n \geq 4 \). Then the action is induced by a twisted \( 2\pi/n \)-rotation of \( S^3 \) about an axis \( L \). \( K \) is invariant, and is disjoint from \( L \).

We state the following as a corollary for convenience.

**Corollary 4.** If a cyclic group \( Z_n \) generated by \( g \) acts smoothly on a non-trivial knot complement \( S^3 - \text{int}(N(K)) \) such that \( g_*([l]) = -[l] \) in \( H_1(\partial N(K), \mathbb{Z}) \), then \( g \) is an involution.

Combining Corollaries 3 and 4, smooth action of dihedral groups on a knot complement can also be classified. We omit it here.

Recall that a knot \( K \) is invertible if \( K \) is oriented equivalent to \(-K\), the inverted knot of \( K \); \( K \) is amphicheiral if \( K \) is equivalent to its mirror-image \( K^* \).

**Corollary 5.** If \( K \) is a hyperbolic knot in \( S^3 \), then the following holds.

(a) \( K \) is invertible if and only if \( K \) is invariant under a \( \pi \)-rotation in \( S^3 \) about an axis \( L \) such that \( L \) intersects \( K \) transversely in two points.

(b) \( K \) is amphicheiral if and only if \( K \) is invariant under a twisted \( 2\pi/n \)-rotation of \( S^3 \) about an axis missing \( K \), for \( n \geq 4 \), or \( K \) is invariant under a reflection of \( S^3 \) through two points missing \( K \).

(c) If \( K \) is both invertible and amphicheiral, then \( K \) is invariant under a reflection of \( S^3 \) through two points contained in \( K \).

In §1, we prove Theorem 1. In §2, we prove Theorem 2, and its corollaries. In the appendix, we prove the following proposition concerning smooth non-orientation preserving cyclic group actions on \( S^3 \).
PROPOSITION. Any smooth non-orientation preserving cyclic group action on the 3-sphere is conjugate to a twisted rotation or a reflection of the sphere through two points.

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1. Proof of Theorem 1. Let $K$ be a hyperbolic knot in $S^3$ with $S^3 - K$ having a hyperbolic metric; $N(K)$ be a regular neighborhood of $K$ such that $\partial N(K)$ is a flat torus in $S^3 - K$ with respect to the hyperbolic metric; $m, l$ be a meridian and a preferred longitude of $K$ respectively, $m, l$ lie in $\partial N(K)$ and be realized as geodesics. $m, l$ will also be used to denote the elements in $\pi_1(S^3 - \text{int}(N(K)))$ represented by them. Let $[m], [l]$ be the homology classes in $H_1(\partial N(K), Z)$ represented by $m, l$ respectively. Let $h$ be a self-diffeomorphism of $S^3 - \text{int}(N(K))$. Our goal is to prove that $h^*[m]$ is $\pm[m]$ in $H_1(\partial N(K), Z)$. Since if this condition is satisfied,

$$h|_{\partial N(K)} : \partial N(K) \to \partial N(K)$$

extends to be a self-diffeomorphism of $N(K)$ which in turn gives an extension of $h$ to $S^3$ by gluing. By Mostow Rigidity, one can assume that $h$ is a hyperbolic isometry. $h_*([l]) = \varepsilon_1 [l]$ with $\varepsilon_1$ being $\pm 1$ in $H_1(\partial N(K), Z)$, because $\pm[l]$ are the only primitive homology classes in $H_1(\partial N(K), Z)$ which vanish in $H_1(S^3 - \text{int}(N(K)), Z)$ under the inclusion homomorphism. $h_*$ is an automorphism of $H_1(\partial N(K), Z)$; hence $h_* [m] = \varepsilon [m] + a[l]$, where $\varepsilon_2 = \pm 1$, and $a$ is in $Z$. Our goal is to show $a = 0$. If $\varepsilon_1 = \varepsilon_2$, i.e., $h$ is orientation preserving, the result is trivial because on one hand $h$, being an isometry of a hyperbolic manifold of finite volume, is of finite order (i.e., composition of $h$ finite times is the identity map; see [M, B], or [Th]), on the other hand the matrix $\begin{bmatrix} \varepsilon_1 & a \\ 0 & \varepsilon_2 \end{bmatrix}$ has infinite order if $a$ is non-zero. Therefore, we need only to consider the case where $\varepsilon_1 = -\varepsilon_2$. Suppose conversely $a \neq 0$. Then by Culler, Gordon, Luecke, Shalen [CGLS], one has that $a = \pm 1$, and that $K$ does not have property P. Since the matrix $\begin{bmatrix} \varepsilon_1 & a \\ 0 & \varepsilon_2 \end{bmatrix}$ is of order two, $h_* h_* = \text{id}$ in $H_1(\partial N(K), Z)$. Consider the orientation preserving isometry $g = h \circ h$. $g$ is of finite order; hence it generates a finite cyclic group $G$ acting isometrically on the flat torus $\partial N(K)$. Because $g_*([m]) = [m]$ and $g_*[l] = [l]$ in $H_1(\partial N(K), Z)$, $G$ preserves the foliations $\partial N(K)$ by geodesic
meridians and by geodesic longitudes. The following lemma shows that the $G$-action on $\partial N(K)$ can be extended to a $G$-action on $N(K)$.

**Lemma 1.** If $G$ acts isometrically on a flat boundary $\partial N$ of a solid torus $N$ and $g_\ast[m] = \pm[m]$, $g_\ast[l] = \pm[l]$ in $H_1(\partial N, Z)$ where $g$ is a generator of $G$, $m$, $l$ are a meridian and a longitude of $\partial N$ respectively, then the $G$-action can be extended to an action on $N$. Moreover the extended $G$-action on the core of $N$ preserves a flat Riemannian metric on it.

**Proof.** Parametrize $\partial N$ by $(u, v)$, where $u, v$ are the unit complex numbers such that $S^1 \times \{v\}$ and $\{u\} \times S^1$ correspond to the geodesic meridian $m$ and the geodesic longitude $l$ in $\partial N$. Since the action on the homology group $H_1(\partial N, Z)$ satisfies the conditions above, the $G$-action on $\partial N$ corresponds now to a $G$-action on $S^1 \times S^1$ preserving the standard product metric and the product structure. Extending the $G$-action on $\partial N$ to $N$ is the same as extending the $G$-action on $S^1 \times S^1$ to $D^2 \times S^1$. The extension of the latter is trivial. To see this, for $g \in G$, we have,

$$g(u, v) = (\phi(u, g), \psi(v, g))$$

where $u, v \in S^1$, $\phi(u, g) = \alpha u$, or $\alpha \bar{u}$, and $\psi(v, g) = \beta v$ or $\beta \bar{v}$, for some roots of unity $\alpha$ and $\beta$. The extension of the $G$-action to $D^2 \times S^1$ is given by the same formula with $u$ in $D^2 = \{z \in \mathbb{C} | |z| \leq 1\}$. The extended $G$-action still preserves the product metric and acts on the core $\{0\} \times S^1$ isometrically with respect to the flat metric induced from $D^2 \times S^1$.

We have now a cyclic group $G$ which acts on $S^3$ preserving $K$. If $G$ is non-trivial, then $K$ has property P by Corollary 7 of Culler, Gordon, Luecke, Shalen [CGLS] which contradicts $a \neq 0$. Therefore $h \circ h = \text{id}$ in $S^3 - \text{int}(N(K))$. It is easy to check, using $a = \pm 1$, $h_\ast([m]) = -\epsilon_1[m] + a[l]$ and $h_\ast([l]) = \epsilon_1[l]$, that

$$h_\ast(-2\epsilon_1 a[m] + [l]) = -\epsilon_1(-2\epsilon_1 a[m] + [l]).$$

Note that $[l]$, and $-2\epsilon_1 a[m] + [l]$ are primitive elements, and are the $(\pm 1)$-eigenvectors of $h_\ast$ in $H_1(\partial N(K), Z)$. The algebraic intersection number of $[l]$ and $-2\epsilon_1 a[m] + [l]$ is $\pm 2$. The following lemma shows that $h$ has fixed points in $\partial N(K)$.
**Lemma 2.** Suppose $h$ is an orientation reversing fixed point free involution of a torus $T^2$, then the $(\pm 1)$-eigenspaces of $h_*$ are generated by two primitive classes with $\pm 1$ as their algebraic intersection number.

**Proof.** Since any orientation reversing fixed point free involution of $T^2$ has the quotient space homeomorphic to the Klein bottle, and since the Klein bottle has only one orientable two-fold cover up to covering equivalence, any two orientation reversing fixed point free involutions on $T^2$ are conjugate. Because the hypothesis and the conclusion of the lemma are invariant under conjugation, the lemma follows by checking a concrete example. Take $T^2$ to be $S^1 \times S^1$ parametrized by $(u, v)$, where $u, v \in S^1$, the unit circle in the complex plane. Let $h: T^2 \to T^2$ be the automorphism sending $(u, v)$ to $(\bar{u}, -v)$. $h$ generates a fixed point free orientation reversing involution of $T^2$. The 1-eigenspace of $h_*$ is generated by the homology class of the curve $\{1\} \times S^1$, and the $(-1)$-eigenspace of $h_*$ is generated by the homology class of the curve $S^1 \times \{1\}$. Hence the algebraic intersection number of the primitive generators of $(\pm 1)$-eigenspaces is $\pm 1$.

By the lemma, $h$ has fixed points in $\partial N(K)$. However, $h$ is an orientation reversing involution, $\text{Fix}(h|_{\partial N(K)})$ is a 1-dimensional submanifold. This implies that $\text{Fix}(h)$ contains a 2-manifold, say $F$. We claim that this is impossible. By Smith theory (see [B], Theorem 5.1), for the $Z_2$-action generated by $h$ on the 1-dimensional $Z_2$-homology sphere $S^3 - \text{int}(N(K))$, the fixed point set $\text{Fix}(h)$ is a $Z_2$-homology sphere of dimension at most one. Hence $\text{Fix}(h) (= F)$ is an annulus or a Möbius band.

**Case 1.** $F$ is an annulus. Since $S^3 - K$ has a hyperbolic structure, $S^3 = \text{int}(N(K))$ is annulus free. Hence $F$ is parallel to an annulus in $\partial N(K)$. In particular, $F$ is separating. The two components of the complement of $F$ in $S^3 - \text{int}(N(K))$ are interchanged by $h$ and hence are homeomorphic. Therefore both of them are solid tori. This implies that $S^3 - \text{int}(N(K))$ is the union of two solid tori along an annulus in their boundaries which contradicts the existence of the hyperbolic structure of finite volume in $S^3 - K$.

**Case 2.** $F$ is a Möbius band. $\partial F$ is now a simple closed curve in $\partial N(K)$ fixed by $h$, and hence $[\partial F]$ is in the 1-eigenspace of $h_*$ which is generated by $[l]$, or by $2a[m] + [l]$ according to $\varepsilon_1 = 1$, or $-1$. Thus $\partial F$ and $K$ bound an annulus $A$ in $N(K)$. The Möbius band $F \cup_{\partial} A$ in $S^3$ has $K$ as its boundary. Let $L$ be the core of
the Möbius band. If \( L \) is non-trivial, \( K \) is the cable knot of \( L \). This contradicts that \( K \) is a hyperbolic knot. If \( L \) is the trivial knot, then \( K \) is the \((2,n)\)-torus which is again absurd.

This completes the proof of Theorem 1.

Since any non-trivial knot with property \( P \) has the property that any self-diffeomorphism of the knot complement preserves the meridian, and since the only non-trivial knots which are not known to have property \( P \) are some hyperbolic knots by the work of Gabai and others, Corollary 2 follows from Theorem 1.

2. Proof of Theorem 2. We shall still use the same notations introduced in §1. Hence \( K \) is a non-trivial knot in \( S^3 \); \( N(K) \) is a regular neighborhood of \( K \); \( m, l \) are a meridian and a preferred longitude of \( K \) respectively. \( m, l \) lie in \( \partial N(K) \). Our first observation is that there exists a flat metric on \( \partial N(K) \) such that \( G \) acts on \( \partial N(K) \) isometrically. This follows from the Geometrization Theorem that any action of a finite group \( G \) on a 2-manifold is equivalent to a geometric group action (see [E]). Fix the metric on \( \partial N(K) \), and realize \( m, l \) by geodesics in \( \partial N(K) \). Theorem 1 shows that the \( G \)-action on \( \partial N(K) \) preserves the geodesic meridians and geodesic longitudes in \( \partial N(K) \). By Lemma 1, the \( G \)-action on \( \partial N(K) \) extends to a \( G \)-action on \( N(K) \) such that the extended \( G \)-action preserves a flat metric on \( K \). Hence the \( G \)-action on \( S^3 \setminus \text{int}(N(K)) \) extends to a \( G \)-action on \( S^3 \) which preserves \( K \) and acts on \( K \) preserving a flat metric \( d \). The restriction of the \( G \)-action to \( K \) gives a representation:

\[
\sigma: G \rightarrow \text{Isom}(K, d).
\]

The solution of the Smith Conjecture shows that \( \sigma \) is a monomorphism. To see this, let \( h \in \ker(\sigma) \), and \( H \) be the cyclic group by \( h \). Then \( H \) acts on \( S^3 \) with fixed point set containing \( K \), and \( H \) preserves each geodesic meridian in \( \partial N(K) \). Moreover, \( h_*(\langle l \rangle) = \langle l \rangle \) in \( H_1(\partial N(K), \mathbb{Z}) \). There are now two cases that might happen.

Case 1. \( h_*([m]) = [m] \). \( h \) is now an orientation preserving homeomorphism because \( h_*([l]) = [l] \) and \( h_*([m]) = [m] \) imply that \( h \) is an orientation preserving homeomorphism in \( H_1(\partial N(K), \mathbb{Z}) \). Therefore the \( H \)-action on a geodesic meridian \( m \) is a rotation. Suppose \( h \neq \text{id} \); then \( H \) acts non-trivially on \( m \). Therefore \( K \) is the only fixed point set of \( h \) in \( N(K) \). By Smith theory, \( \text{Fix}(h) = K \), which then contradicts the solution of the Smith Conjecture.
Case 2. $h_\ast([m]) = -[m]$. $h$ is now an orientation reversing homeomorphism. Since $h \circ h \in \ker(\sigma)$, and $h_\ast h_\ast([m]) = [m]$, one has $h \circ h = \text{id}$ by the solution of Case 1. Hence $h$ is an orientation reversing involution of $S^3$ with fixed point set containing $K$. Because the Fix($h$) is a submanifold of odd codimension and contains $K$, Fix($h$) contains a 2-manifold. By Smith Theory, the Fix($h$) is a $Z_2$-homology sphere. Hence Fix($h$) is a 2-sphere and contains $K$. This implies that $K$ is a trivial knot which is absurd.

Therefore $G$ is a subgroup of Isom$(K, d)$. It is well known that a finite subgroup of Isom$(K, d)$ is a cyclic or a dihedral group. In case $K$ is a hyperbolic knot, Out$(\pi_1(S^3 - K))$ acts isometrically on $S^3 - \text{int}(N(K))$ where $\partial N(K)$ is a flat torus in $S^3 = K$ (see [M, B], or [Th]). Hence Out$(\pi_1(S^3 - K))$ (or the same Isom$(S^3 - K)$) is a cyclic or a dihedral group.

Proof of Corollary 3. By Theorem 2 and its proof, the $Z_n$-action extends to a $Z_n$-action on $S^3$ such that $K$ is invariant and $K$ intersects the fixed point set of a nontrivial element $f$ in $Z_n$ if and only if Fix($\sigma(f)$) $\cap K \neq \emptyset$. But Fix($\sigma(f)$) $\cap K \neq \emptyset$ if and only if $\sigma(f)$ is a reflection on $K$ which in turn is the same as $f_\ast([l]) = -[l]$ in $H_1(\partial N(K), Z)$. Moreover, in this case, $K$ intersects Fix($f$) transversely in two points. The classification is now reduced to the classification of smooth cyclic group actions on $S^3$.

(I) The $Z_n$-action preserves the orientation.

If the $Z_n$-action on $S^3$ is fixed point free, we have (a). Otherwise, by Smith theory, the fixed point set is a knot, say $L$. The solution of the Smith Conjecture shows that $L$ is a trivial knot, and the $Z_n$-action is a $2\pi/n$-rotation about $L$. Let $g$ be a generator of the $Z_n$-action. If $L$ intersects $K$, then by the remark above, we have $g_\ast([l]) = [l]$, and $\sigma(g)$ is a reflection in $K$. Hence Fix($g \circ g$) contains $K$. However $gg$ is orientation preserving. Therefore the solution of the Smith Conjecture implies that $g \circ g$ is the identity, i.e., $n = 2$. This proves (b).

(II) The $Z_n$-action does not preserve the orientation.

Let $g$ still be the generator of the $Z_n$-action on $S^3$. Since $g$ reverses the orientation, $g$ has fixed points in $S^3$, $n$ is even, and Fix($g$) is a submanifold of odd codimension in $S^3$.

(c) $n = 2$.

By Smith theory, Fix($g$) is a $Z_2$-homology sphere. Hence Fix($g$) is the two points set or the 2-sphere. If Fix($g$) is the two points set,
by Livesay's theorem [L], the \( Z_2 \)-action is a reflection of \( S^3 \) through two points; if \( \text{Fix}(g) \) is a 2-sphere, then the action is a reflection of \( S^3 \) with respect to a 2-sphere by Schönflies theorem. Now the \( Z_2 \)-action is classified as follows. If \( g_*(|[l]|) = [l] \), then \( \text{Fix}(g) \cap K = \emptyset \). In this case \( \text{Fix}(G) \) cannot be a 2-sphere. To see this, \( \text{Fix}(g) \cap K = \emptyset \). In this case \( \text{Fix}(G) \) cannot be a 2-sphere. To see this, \( \text{Fix}(g) \cap K = \emptyset \) implies the fixed point set of \( g \) in \( S^3 \) is actually in \( S^3 - \text{int}(N(K)) \). By Smith theory, for the \( g \)-involution on the one-dimensional homology sphere \( S^3 - \text{int}(N(K)) \), \( \text{Fix}(g|_{S^3 - \text{int}(N(K))}) \) is a \( Z_2 \)-homology sphere of dimension at most one. Hence \( \text{Fix}(g) \) are two points. This gives (c). If \( g_*(|[l]|) = -[l] \), then \( \sigma(g) \) is a reflection in \( K \), and \( K \) intersects \( \text{Fix}(g) \) transversely in two points. (c)_2, (c)_3 follow from the above mentioned classification of the orientation reversing involutions of \( S^3 \).

(d) \( n \geq 4 \).

The result is a consequence of the following proposition which will be proven in the appendix.

**Proposition.** Any smooth cyclic group action on \( S^3 \) which does not preserve the orientation is conjugate to a twisted rotation of \( S^3 \), or to a reflection of \( S^3 \) through two points.

Applying the proposition, we need only to check that \( K \) is disjoint from the axis of the twisted rotation \( g \). However the axis of \( g \) is \( \text{Fix}(g \circ g) \). \( \text{Fix}(g \circ g) \) does not intersect \( K \) follows now from \( g_*(|[l]|) = [l] \), and \( g \circ g \neq \text{id} \). This completes the proof of (d).

Corollary 4 is actually proven in the proof of Corollary 3.

**Proof of Corollary 5.** (a) By Proposition 3.19 of [B, Z], \( K \) is invertible if and only if there is an automorphism

\[
\phi: \pi_1(S^3 - \text{int}(N(K))) \to \pi_1(S^3 - \text{int}(N(K)))
\]

such that \( \phi(m) = m^{-1} \) and \( \phi(l) = l^{-1} \). Since \( K \) is a hyperbolic knot, Mostow Rigidity Theorem shows that \( \phi \) can be realized by a hyperbolic isometry \( h: S^3 - \text{int}(N(K)) \to S^3 - \text{int}(N(K)) \) such that \( h_*(|[m]|) = -[m] \), and \( h_*(|[l]|) = -[l] \) in \( H_1(\partial N(K), \mathbb{Z}) \). Here we have assumed that \( \partial N(K) \) is a flat torus in \( S^3 - K \). The condition \( h_*(|[l]|) = -[l] \) implies that \( h \) is an involution by Corollary 4. Because \( h_*(|[m]|) = -[m] \), \( h \) is orientation preserving. Hence by Corollary 3, the \( Z_2 \)-action generated by the extension of \( h \) on \( S^3 \) is induced by a \( \pi \)-rotation of \( S^3 \) about an axis \( L \). \( H_*(|[l]|) = -[l] \) implies that
\( L \) intersects \( K \) transversely in two points. Therefore \( K \) is invariant under a \( \pi \)-rotation about an axis intersecting \( K \) at two points. The inverse implication is trivial.

(b) By Proposition 3.19 of [B, Z], \( K \) is amphicheiral if and only if there is an automorphism
\[
\phi: \pi_1(S^3 - \text{int}(N(K))) \rightarrow \pi_1(S^3 - \text{int}(N(K)))
\]
such that \( \phi(m) = m^{-1} \) and \( \phi(l) = l \). Realize \( \phi \) by an isometry \( h: S^3 - \text{int}(N(K)) \rightarrow S^3 - \text{int}(N(K)) \). \( h \) is orientation reversing since \( h_*([m]) = -[m] \), and \( h_*([l]) = [l] \) in \( H_1(\partial N(K), Z) \). \( h \) generates a smooth cyclic group action on \( S^3 - \text{int}(N(K)) \) which does not preserve the orientation. Hence by Corollary 3, \( h \) is induced by a twisted rotation of \( S^3 \) about an axis \( L \) missing \( K \) if the order of \( h \) is at least four. If the order of \( h \) is two, the \( h \) involution is the case \( (c)_1 \) in Corollary 3 because \( h_*([l]) = [l] \). Therefore, in this case \( K \) is invariant under a reflection of \( S^3 \) through two points missing \( K \). Then the condition is clearly sufficient.

(c) If the knot is both invertible and amphicheiral, then there exists an automorphism
\[
\phi: \pi_1(S^3 - \text{int}(N(K))) \rightarrow \pi_1(S^3 - \text{int}(N(K)))
\]
such that \( \phi(m) = m \), and \( \phi(l) = l^{-1} \). \( \phi \) is the composition of the two automorphisms coming from (a) and (b). Realize \( \phi \) by an orientation reversing hyperbolic isometry \( h \) such that \( h_*([m]) = [m] \), and \( h_*([l]) = -[l] \) in \( H_1(\partial N(K), Z) \). By Corollary 4, \( h_*([l]) = -[l] \) and \( h_*([m]) = [m] \) imply \( h \) is an orientation reversing involution of \( S^3 - \text{int}(N(K)) \rightarrow S^3 - N(K) \). By Corollary 3, \( h \) is the case \( (c)_2 \) or the case \( (c)_3 \). Case \( (c)_3 \) cannot happen since \( K \) is a prime knot. Hence \( K \) is invariant under the reflection of \( S^3 \) through two points contained in \( K \).

Appendix. We prove the following proposition concerning smooth cyclic group action on the 3-sphere which does not preserve the orientation.

**Proposition.** Any smooth non-orientation preserving cyclic group action on \( S^3 \) is conjugate to a twisted rotation of \( S^3 \), or to a reflection of \( S^3 \) through two points.

**Proof.** Let \( g \) be a generator of the \( Z_n \)-action. \( n \) has to be even. \( g \) is orientation reversing, and hence has fixed points in \( S^3 \). If \( n = 2 \),
we have shown in the proof of Corollary 3 (c) that the result holds. Assume $n \geq 4$ from now on. Let $h = g \circ g$. $h$ is an orientation preserving automorphism of order $m$, and has fixed points. The solution of the Smith Conjecture shows that the Fix($h$) is a trivial knot, say $L$. Now $L$ is invariant under $g$. $g$ acts on $L$ with fixed point and is of order two in $L$. Hence the action of $g$ on $L$ is a reflection by the classification of $\mathbb{Z}_2$-action on the circle. Take a $\mathbb{Z}_n$-equivariant regular neighborhood $N(L)$ of $L$ in $S^3$ (see [B]). By the choice of the regular neighborhood, one knows that the action of $\mathbb{Z}_n$ on $N(L)$ is standard. Therefore by choosing the generator $g$ of the $\mathbb{Z}_n$-action appropriately, we can assume that the restriction of $g$ on $N(L) = D^2 \times S^1$ is conjugate to $\alpha$, where

$$\alpha : D^2 \times S^1 \rightarrow D^2 \times S^1$$

sends $(z, w)$ to $(e^{2\pi i/n}z, \overline{w})$, with $z$ in $D^2 = \{z \in C | |z| \leq 1\}$ and $w$ in $S^1 = \{z \in C | |z| = 1\}$. Note that $\alpha$ generates an orientation reversing $\mathbb{Z}_n$-action on $D^2 \times S^1$ with two fixed points in $\{0\} \times S^1$. Since $L$ is the trivial knot, $S^3 - \text{int}(N(L))$ is a solid torus. Let $\phi : S^3 = (S^3 - \text{int}(N(L))) \cup N(L) \rightarrow \overline{S}^3 = (S^1 \times D^2) \cup \text{id} (D^2 \times S^1)$ be a diffeomorphism taking $N(L)$ to $D^2 \times S^1$ such that $\phi g|_{N(L)} \phi^{-1} = \alpha$. Now extend $\alpha$ to be a self-diffeomorphism $\overline{\alpha}$ of $\overline{S}^3$ by sending $(z, w)S^1 \times D^2$ to $(e^{2\pi i/n}z, \overline{w})$ with $z \in S^1$ and $w \in D^2$. Then $\overline{\alpha}$ generates a twisted $2\pi/n$-rotation of $\overline{S}^3$. Our goal is to show that $\phi g \phi^{-1}$ is conjugate to $\overline{\alpha}$ in $\overline{S}^3$. This is consequence of the following claim.

**Claim.** $g' = \phi g \phi^{-1}|_{S^1 \times D^2}$ is conjugate to $\beta = \overline{\alpha}|_{S^1 \times D^2}$ by a piecewise smooth diffeomorphism $\psi$ such that $\psi$ is the identity map on $\partial(S^1 \times D^2)$.

Let us assume the claim and finish the proof. By gluing $\psi$ with $\text{Id}|_{D^2 \times S^1}$ along the boundaries, we obtain a piecewise smooth self-diffeomorphism of $\overline{S}^3$ which conjugates $\phi g \phi^{-1}$ to $\overline{\alpha}$. Therefore $\phi g \phi^{-1}$ is smoothly conjugate to $\overline{\alpha}$ by the work of Moise.

**Proof of the Claim.** By the choice of $\phi$, $g'$ is the same as $\beta$ on $\partial(S^1 \times D^2)$. Using the equivariant Dehn's lemma, we can find $n$ copies of disjoint properly embedded disks $D_1, D_2, \ldots, D_n$ with $\partial D_j = e^{2\pi j/n} \times \partial D^2$ in $S^1 \times D^2$, such that $g'(D_j) = D_{j+1}$ for $j =$
1, 2, ..., n, where $D_1 = D_{n+1}$. $g^j: D_j \to D_{j+1}$ is a diffeomorphism for each $j$. These disks cut $S^1 \times D^2$ into $n$ components, say $B_1, B_2, \ldots, B_n$ with $D_j \cup D_{j+1} \subset \partial B_j$, and each of $B_j$ is a 3-ball by Schoenflies' theorem. Let $D'_j = e^{2\pi ij/n} \times D^2 \ (\text{where } D'^{n+1}_{n+1} = D'_{n+1})$; $B'_j = \{e^{2\pi it/n} | j \leq t \leq j + 1 \} \times D^2$; and $E_j = \partial B'_j - (D_j \cup D_{j+1})$, the annulus, for each $i = 1, 2, \ldots, n$. The construction of $\psi$ is now as follows. Let $A_1: D_1 \to D'_1$ be a diffeomorphism which is the identity on $\partial D_1$. Define $A_2: D_2 \to D'_2$ to be $\beta|_{D'_1} A_1 g'^{-1}|_{D'_2}$. It is still a diffeomorphism which fixes $\partial D_2$ pointwise. Since $\partial B_1 = D_1 \cup E_1 \cup D_2$ and $\partial B'_1 = D'_1 \cup E_1 \cup D'_2$, glue $A_1, A_2$ and $\text{id}|_{E_1}$ along the boundaries, one obtains a piecewise smooth diffeomorphism from $\partial B_1 \to \partial B'_1$ which is the identity on $E_1$. Extend it to be a piecewise smooth diffeomorphism from $B_1$ to $B'_1$ by Alexander's lemma, and call it $\psi_1$. Now $\psi_j: B_j \to B'_j$ is defined to be

$$\beta|_{B'_j} \psi_1 g'^{j-1}|_{B_j}$$

for $j = 2, 3, \ldots, n$. All these piecewise smooth diffeomorphisms match on the $D_j$'s. Gluing them together along the $D_j$'s, we obtain a piecewise diffeomorphism $\psi: S^1 \times D^2 \to S^1 \times D^2$. Then $\psi|_{\partial (S^1 \times D^2)} = \text{id}$ and $\beta = \psi^{-1} \beta \psi$.

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