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HENDRA GUNAWAN

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## A GENERALIZATION OF MAXIMAL FUNCTIONS ON COMPACT SEMISIMPLE LIE GROUPS

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Let  $G$  be a compact semisimple Lie group with finite centre. For each positive number  $s$ , let  $\mu_{sH}$  denote the  $\text{Ad}(G)$ -invariant probability measure carried on the conjugacy class of  $\exp(sH)$  in  $G$ . With this one-parameter family of measures, we define the maximal operator  $\mathcal{M}_H$  on  $\mathcal{E}(G)$ . We then estimate the Fourier transform of  $\mu_{sH}$  and of some derived distributions. Our result leads to the boundedness of  $\mathcal{M}_H$  on  $L^p(G)$ , for all  $p$  greater than some index  $p_0$  in  $(1, 2)$ . This generalizes a recent result of M. Cowling and C. Meaney [2].

**Introduction.** Let  $G$  be a compact semisimple Lie group of rank  $l$  with finite centre, and with its Haar measure normalized to have total mass 1. Let  $\mathfrak{g}$  denote its Lie algebra, and let  $\mathfrak{h}$  be a maximal toral subalgebra of  $\mathfrak{g}$ . We denote by  $\Phi$  the root system of  $(\mathfrak{g}^{\mathfrak{e}}, \mathfrak{h}^{\mathfrak{e}})$ , and fix  $\Delta = \{\alpha_j : j \in I\}$ , where  $I = \{1, \dots, l\}$ , to be a base of  $\Phi$  (as in [3, §10.1]). With respect to  $\Delta$ , we write  $\Phi^+$  for the set of positive roots, whose members are of the form

$$\alpha = \sum_{j \in I} n_j(\alpha) \alpha_j,$$

with  $n_j(\alpha) \in \mathbf{Z}^+ \cup \{0\}$  for all  $j \in I$ , and  $\Lambda^+$  for the set of dominant weights, which parametrizes the dual object of  $G$ .

We equip the Lie algebra  $\mathfrak{g}$  with the positive definite inner product  $(\cdot, \cdot)$  derived from the Killing form. For each  $\nu \in \mathfrak{h}^*$ , we define  $H_\nu \in \mathfrak{h}$  by

$$\nu(H) = (H_\nu, H) \quad \forall H \in \mathfrak{h}.$$

We also transfer the inner product to  $\mathfrak{h}^*$  via

$$(\nu, \nu') = (H_\nu, H_{\nu'}) \quad \forall \nu, \nu' \in \mathfrak{h}^*.$$

The norm on  $\mathfrak{h}^*$  and  $\mathfrak{h}$ , induced by these inner products, will then be denoted by  $|\cdot|$ .

We choose a regular element  $H \in \mathfrak{h}$ , for which  $\alpha(H) \neq 0$  for all  $\alpha \in \Phi^+$ , and fix  $R > 0$  such that  $\exp(sH)$  is regular in  $G$  for any

$s \in (0, R)$ . For a continuous function  $f$  on  $G$ , the maximal function  $\mathcal{M}_H f$  is defined by

$$\mathcal{M}_H f(x) = \sup_{s \in (0, R)} |\mu_{sH} * f(x)| \quad \forall x \in G,$$

where  $\mu_{sH}$  is the  $\text{Ad}(G)$ -invariant probability measure carried on the conjugacy class of  $\exp(sH)$  in  $G$ . This definition generalizes one in the paper of Cowling and Meaney [2], in which  $H$  was a particular regular element of  $\mathfrak{h}$ . Our main results are the following.

**THEOREM A.** *For all  $k = 0, 1, 2, \dots$ , there exist positive constants  $C_k = C_k(H)$  such that*

$$\left| \left( \frac{\partial}{\partial s} \right)^k \hat{\mu}_{sH}(\lambda) \right| \leq C_k \frac{(1 + |\lambda|)^k}{(1 + s|\lambda|)^\gamma} \quad \forall s \in (0, R), \lambda \in \Lambda^+,$$

where  $\gamma = \min_{j \in I} |\{\alpha \in \Phi^+ : n_j(\alpha) \geq 1\}|$ .

It is clear that Theorem A, together with the arguments of [2], imply the boundedness of  $\mathcal{M}_H$  on  $L^p(G)$  for all  $p > 1 + (2\gamma)^{-1}$ . So we state

**THEOREM B.** *For all  $p > 1 + (2\gamma)^{-1}$ , with  $\gamma$  as above, there exist positive constants  $C_p = C_p(H)$  such that*

$$\|\mathcal{M}_H f\|_p \leq C_p \|f\|_p \quad \forall f \in \mathcal{C}(G).$$

We prove Theorem A by handling first the case when  $G$  is simple, and then extend the result to the semisimple case. Our method is based on arguments of representation theory, involving formulae for characters and dimensions, a study of root systems, the theory of weights, and properties of the Weyl group, all developed in the first part of this note. The proof of Theorem A will be given in the second part. It is clear that Theorem A is sharp since the explicit expression used in [2] for the particular case in which  $H = H_p$  shows no improvement is possible. In the third part of this note, we give an example which shows that Theorem B too is sharp at least in the case where  $G = \text{SU}(2)$ .

Some related results can be found in M. Christ [1] and C. D. Sogge and E. M. Stein [5].

Throughout this note, the expressions  $C$ ,  $C_k$ , and  $C_{k_1, k_2, k_3}$  denote various positive constants which possibly vary from line to line. These constants may depend on  $G$ , and some may also depend on the choice

of  $H$ . When a constant,  $C$  say, depends on  $H$ , we write  $C(H)$  in place of  $C$ .

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**1. Representation theoretic arguments.** We shall assume throughout this part that the Lie algebra  $\mathfrak{g}$  is simple.

1.1. We start with some formulae for characters and dimensions of representations of  $G$ . To each  $\lambda \in \Lambda^+$ , we associate the representation  $\pi_\lambda$ , the set of weights  $\varpi_\lambda$ , the character  $\chi_\lambda$ , and the dimension  $d_\lambda = \chi_\lambda(\mathbf{1})$ . For all  $\lambda \in \Lambda^+$ , we have (see [3, §22])

$$\chi_\lambda(\exp(H)) = \sum_{\lambda' \in \varpi_\lambda} m_\lambda(\lambda') \exp(i\lambda'(H)),$$

where  $m_\lambda(\lambda') \in \mathbf{Z}^+$  is the multiplicity of  $\lambda'$  in  $\pi_\lambda$ . Accordingly,

$$d_\lambda = \sum_{\lambda' \in \varpi_\lambda} m_\lambda(\lambda').$$

Let  $\mathscr{W}$  be the Weyl group of  $(\mathfrak{g}^c, \mathfrak{h}^c)$ , generated by the reflections  $\sigma_\alpha$  corresponding to  $\alpha \in \Delta$ . Introduce the special element  $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ . For all  $\lambda \in \Lambda^+$ , the character and dimension formulae of Weyl read (see [3, §24.3])

$$\chi_\lambda(\exp(H)) = \frac{\sum_{\sigma \in \mathscr{W}} \det(\sigma) \exp(i\sigma(\lambda + \rho)(H))}{\prod_{\alpha \in \Phi^+} 2i \sin \frac{1}{2}\alpha(H)}$$

and

$$d_\lambda = \prod_{\alpha \in \Phi^+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)}.$$

1.2. It is well known that  $\mathfrak{g}^c$  has the root space decomposition (see [7, p. 273])

$$\mathfrak{g}^c = \mathfrak{g}^c \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha^c,$$

where  $\mathfrak{g}_\alpha^c$  denotes the root subspace of  $\mathfrak{g}^c$  corresponding to  $\alpha \in \Phi$ .

Assuming  $l \geq 2$ , we choose  $j_0 \in I$ , and then remove  $\alpha_{j_0}$  from  $\Delta$  to obtain

$$\Delta_0 = \{\alpha_j : j \in I_0\}, \quad \text{where } I_0 = I \setminus \{j_0\}.$$

Set  $\Phi_0^+ = \{\alpha \in \Phi^+ : n_{j_0}(\alpha) = 0\}$ , and put  $\Phi_0 = \Phi_0^+ \cup -\Phi_0^+$ . Clearly  $\Phi_0 = -\Phi_0$  and  $\sigma_\alpha \Phi_0 = \Phi_0$  for all  $\sigma_\alpha$  ( $\alpha \in \Delta_0$ ). This shows that

$\Phi_0$  is a root system (see [7, p. 370]). Let  $\mathfrak{h}_0$  be the subspace of  $\mathfrak{h}$  spanned by  $H_\alpha$  ( $\alpha \in \Phi_0$ ). Then one may verify that

$$\mathfrak{g}_0^c = \mathfrak{h}_0^c \oplus \bigoplus_{\alpha \in \Phi_0} \mathfrak{g}_\alpha^c$$

is a semisimple subalgebra of  $\mathfrak{g}^c$ , with maximal toral subalgebra  $\mathfrak{h}_0^c$  (see [7, Ex. 30 of Ch. 4]). Evidently  $\Phi_0$  is the root system of  $(\mathfrak{g}_0^c, \mathfrak{h}_0^c)$ ,  $\Delta_0$  is a base of  $\Phi_0$ , and  $\Phi_0^+$  is the set of positive roots with respect to  $\Delta_0$ .

Write  $\Phi_0$  as a disjoint union of irreducible root systems, say

$$\Phi_0 = \Phi_{01} \cup \cdots \cup \Phi_{0r}.$$

Let  $q \in \{1, \dots, r\}$ . Denote by  $\mathfrak{h}_{0q}$  the subspace of  $\mathfrak{h}_0$  spanned by  $H_\alpha$  ( $\alpha \in \Phi_{0q}$ ). Then we find that

$$\mathfrak{g}_{0q}^c = \mathfrak{h}_{0q}^c \oplus \bigoplus_{\alpha \in \Phi_{0q}} \mathfrak{g}_\alpha^c$$

is a simple ideal of  $\mathfrak{g}_0^c$ , with maximal toral subalgebra  $\mathfrak{h}_{0q}^c$ . We also note that

$$\mathfrak{h}_0^c = \mathfrak{h}_{01}^c \oplus \cdots \oplus \mathfrak{h}_{0r}^c$$

and

$$\mathfrak{g}_0^c = \mathfrak{g}_{01}^c \oplus \cdots \oplus \mathfrak{g}_{0r}^c.$$

Now denote by  $(\cdot, \cdot)_0$  and  $(\cdot, \cdot)_{0q}$  the inner products of  $\mathfrak{g}_0$  and  $\mathfrak{g}_{0q}$  respectively. Then we have (see [3, Lemma 5.1])

$$(\cdot, \cdot)_0|_{\mathfrak{g}_{0q} \times \mathfrak{g}_{0q}} = (\cdot, \cdot)_{0q},$$

and so

$$(X, Y)_0 = (X_1, Y_1)_{01} + \cdots + (X_r, Y_r)_{0r}$$

for all  $X = X_1 + \cdots + X_r$ ,  $Y = Y_1 + \cdots + Y_r \in \mathfrak{g}_0$ , with  $X_q, Y_q \in \mathfrak{g}_{0q}$ . Further, since  $\mathfrak{g}$  and  $\mathfrak{g}_{0q}$  are simple, there exists a positive constant  $C_q$  satisfying (see [4, p. 242])

$$(X, Y)_{0q} = C_q(X, Y) \quad \forall X, Y \in \mathfrak{g}_{0q}.$$

We transfer these inner products to the corresponding dual spaces in the usual way.

Let  $\Lambda_0^+$  denote the set of dominant weights with respect to  $\Delta_0$ . We need to determine the set of fundamental dominant weights in  $\Lambda_0^+$ . Suppose  $\{\omega_j: j \in I\}$  is the set of fundamental dominant weights in  $\Lambda^+$ , for which (see [3, §13.1] for definition)

$$2 \frac{(\omega_j, \alpha_k)}{(\alpha_k, \alpha_k)} = \delta_{jk} \quad \forall j, k \in I.$$

If we now set

$$\tilde{\omega}_j = \omega_j - \text{proj}_{\omega_{j_0}}(\omega_j) \quad \forall j \in I,$$

then we have the following facts.

*Fact 1.* For each  $j \in I_0$ ,  $\tilde{\omega}_j \in \mathfrak{h}_{0Q}^*$  whenever  $\alpha_j \in \mathfrak{h}_{0Q}^*$ .

*Proof.* For all  $j, k \in I_0$ , we have

$$\begin{aligned} 2 \frac{(\tilde{\omega}_j, \alpha_k)}{(\alpha_k, \alpha_k)} &= 2 \frac{(\omega_j, \alpha_k)}{(\alpha_k, \alpha_k)} - 2 \frac{(\text{proj}_{\omega_{j_0}}(\omega_j), \alpha_k)}{(\alpha_k, \alpha_k)} \\ &= 2 \frac{(\omega_j, \alpha_k)}{(\alpha_k, \alpha_k)} - 2 \frac{(\omega_j, \omega_{j_0}) (\omega_{j_0}, \alpha_k)}{(\omega_{j_0}, \omega_{j_0}) (\alpha_k, \alpha_k)} \\ &= 2 \frac{(\omega_j, \alpha_k)}{(\alpha_k, \alpha_k)} - 0 = \delta_{jk}. \end{aligned}$$

Now take  $j \in I_0$ , and let  $Q \in \{1, \dots, r\}$  such that  $\alpha_j \in \mathfrak{h}_{0Q}^*$ . Clearly

$$\tilde{\omega}_j \perp \mathfrak{h}_{0q}^* \quad \forall q \neq Q.$$

Writing  $\tilde{\omega}_j = \tilde{\omega}_{j1} + \dots + \tilde{\omega}_{jr}$ , with  $\tilde{\omega}_{jq} \in \mathfrak{h}_{0q}^*$  for all  $q \in \{1, \dots, r\}$ , we find that

$$\tilde{\omega}_{jq} = 0 \quad \forall q \neq Q.$$

We therefore have

$$\tilde{\omega}_j = \tilde{\omega}_{jQ} \in \mathfrak{h}_{0Q}^*,$$

as stated. □

*Fact 2.*  $\{\tilde{\omega}_j : j \in I_0\}$  is the set of fundamental dominant weights in  $\Lambda_0^+$ .

*Proof.* Take  $j, k \in I_0$ . Suppose  $\tilde{\omega}_j \in \mathfrak{h}_{0q}^*$  and  $\alpha_k \in \mathfrak{h}_{0q'}^*$  for some  $q, q' \in \{1, \dots, r\}$ . If  $q \neq q'$ , then clearly  $(\tilde{\omega}_j, \alpha_k)_0 = 0$ ; otherwise we have

$$2 \frac{(\tilde{\omega}_j, \alpha_k)_0}{(\alpha_k, \alpha_k)_0} = 2 \frac{(\tilde{\omega}_j, \alpha_k)_{0q}}{(\alpha_k, \alpha_k)_{0q}} = 2 \frac{(\tilde{\omega}_j, \alpha_k)}{(\alpha_k, \alpha_k)} = \delta_{jk}.$$

Using Fact 1, the assertion follows. □

*Fact 3.* Suppose  $\lambda = \sum_{j \in I} n_j \omega_j \in \Lambda^+$ . Then  $\lambda$  can be rewritten as

$$\lambda = \lambda_0 + \lambda_1$$

where  $\lambda_0 = \sum_{j \in I_0} n_j \tilde{\omega}_j \in \Lambda_0^+$  (with the same  $n_j$ 's) and  $\lambda_1 = \text{proj}_{\omega_{j_0}}(\lambda)$ .

*Proof.* Noting that  $\tilde{\omega}_{j_0} = 0$ , we have

$$\begin{aligned}
\lambda &= \sum_{j \in I} n_j \omega_j = \sum_{j \in I} n_j \tilde{\omega}_j + \sum_{j \in I} n_j \operatorname{proj}_{\omega_{j_0}}(\omega_j) \\
&= \sum_{j \in I_0} n_j \tilde{\omega}_j + \sum_{j \in I} n_j \frac{(\omega_j, \omega_{j_0})}{(\omega_{j_0}, \omega_{j_0})} \omega_{j_0} \\
&= \sum_{j \in I_0} n_j \tilde{\omega}_j + \frac{(\sum_{j \in I} n_j \omega_j, \omega_{j_0})}{(\omega_{j_0}, \omega_{j_0})} \omega_{j_0} \\
&= \sum_{j \in I_0} n_j \tilde{\omega}_j + \frac{(\lambda, \omega_{j_0})}{(\omega_{j_0}, \omega_{j_0})} \omega_{j_0} \\
&= \sum_{j \in I_0} n_j \tilde{\omega}_j + \operatorname{proj}_{\omega_{j_0}}(\lambda) = \lambda_0 + \lambda_1,
\end{aligned}$$

as claimed.  $\square$

**REMARK.** It is well known that the special element  $\rho$  is a dominant weight in  $\Lambda^+$ . Indeed,  $\rho = \sum_{j \in I} \omega_j$  (see [3, Lemma 13.3A]). By Fact 3, we may rewrite  $\rho = \rho_0 + \rho_1$  where  $\rho_0 = \sum_{j \in I_0} \tilde{\omega}_j \in \Lambda_0^+$  and  $\rho_1 = \operatorname{proj}_{\omega_{j_0}}(\rho)$ . But then  $\rho_0 = \frac{1}{2} \sum_{\alpha \in \Phi_0^+} \alpha$ , giving  $\rho_1 = \frac{1}{2} \sum_{\alpha \in \Phi_1^+} \alpha$  where  $\Phi_1^+ = \Phi^+ \setminus \Phi_0^+$ . As another consequence, we also have  $\rho_1 = c \omega_{j_0}$  for some  $c > 0$ . But we know that  $2(\omega_{j_0}, \alpha_{j_0})/(\alpha_{j_0}, \alpha_{j_0}) = 1$ , and so we find  $c = 2(\rho_1, \alpha_{j_0})/(\alpha_{j_0}, \alpha_{j_0})$ . Hence we determine  $\omega_{j_0} = \frac{1}{2}((\alpha_{j_0}, \alpha_{j_0})/(\rho_1, \alpha_{j_0}))\rho_1$ , with  $\rho_1 = \frac{1}{2} \sum_{\alpha \in \Phi_1^+} \alpha$ . This offers a method of finding the fundamental dominant weight  $\omega_{j_0}$  for any given  $j_0 \in I$ .

Introduce  $\mathfrak{h}_1 = \{H \in \mathfrak{h} : \alpha(H) = 0 \ \forall \alpha \in \Delta_0\}$ . Obviously  $\mathfrak{h}_1$  is a subalgebra of  $\mathfrak{h}$ , which is spanned by  $H_{\rho_1}$  (by the above remark). Moreover, we have (like Fact 3)

*Fact 4.* Every  $H \in \mathfrak{h}$  can be written as

$$H = H_0 + H_1$$

where  $H_0 \in \mathfrak{h}_0$  and  $H_1 \in \mathfrak{h}_1$ .

**REMARK.**  $H_0 \in \mathfrak{h}_0$  means that  $H_0 = H_{\nu_0}$ , where  $\nu_0 \in \operatorname{span}(\Delta_0)$ , while  $H_1 \in \mathfrak{h}_1$  means that  $H_1 = H_{\nu_1}$ , where  $\nu_1 = r\rho_1$  for some  $r \in \mathbf{R}$ . Thus clearly  $\mathfrak{h}_0 \perp \mathfrak{h}_1$ , and so Fact 4 actually states that  $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ .

Suppose we are in  $(\mathfrak{g}_0, \mathfrak{h}_0)$ . To each  $\lambda_0 \in \Lambda_0^+$ , we associate the representation  $\tilde{\pi}_{\lambda_0}$ , the set of weights  $\tilde{\omega}_{\lambda_0}$ , the character  $\tilde{\chi}_{\lambda_0}$ , and the

dimension  $\tilde{d}_{\lambda_0}$ . For all  $\lambda_0 \in \Lambda_0^+$  and  $H_0 \in \mathfrak{h}_0$ , we have

$$\tilde{\chi}_{\lambda_0}(\exp(H_0)) = \sum_{\lambda' \in \tilde{\omega}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda') \exp(i\lambda'(H_0))$$

and

$$\tilde{d}_{\lambda_0} = \sum_{\lambda' \in \tilde{\omega}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda'),$$

with  $\tilde{m}_{\lambda_0}(\lambda') \in \mathbf{Z}^+$  being the multiplicity of  $\lambda'$  in  $\tilde{\pi}_{\lambda_0}$ .

Let  $\mathscr{W}_0$  (or  $\mathscr{W}[\Delta_0]$  if necessary) denote the subgroup of  $\mathscr{W}$  generated by  $\sigma_\alpha$  ( $\alpha \in \Delta_0$ ). The Weyl formulae then read

$$\tilde{\chi}_{\lambda_0}(\exp(H_0)) = \frac{\sum_{\tau \in \mathscr{W}_0} \det(\tau) \exp(i\tau(\lambda_0 + \rho_0)(H_0))}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\alpha(H_0)}$$

and

$$\tilde{d}_{\lambda_0} = \prod_{\alpha \in \Phi_0^+} \frac{(\lambda_0 + \rho_0, \alpha)}{(\rho_0, \alpha)}.$$

We should note that the inner product in the expression above is really the inner product of  $\mathfrak{g}$ . Indeed, we may calculate

$$\begin{aligned} \tilde{d}_{\lambda_0} &= \lim_{s \rightarrow 0} \tilde{\chi}_{\lambda_0}(\exp(sH_{\rho_0})) \\ &= \lim_{s \rightarrow 0} \frac{\sum_{\tau \in \mathscr{W}_0} \det(\tau) \exp(i\tau(\lambda_0 + \rho_0)(sH_{\rho_0}))}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\alpha(sH_{\rho_0})} \\ &= \lim_{s \rightarrow 0} \frac{\sum_{\tau \in \mathscr{W}_0} \det(\tau) \exp(i\tau \rho_0(sH_{\lambda_0 + \rho_0}))}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\alpha(sH_{\rho_0})} \\ &= \lim_{s \rightarrow 0} \frac{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\alpha(sH_{\lambda_0 + \rho_0})}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\alpha(sH_{\rho_0})} \\ &= \prod_{\alpha \in \Phi_0^+} \frac{\alpha(H_{\lambda_0 + \rho_0})}{\alpha(H_{\rho_0})} = \prod_{\alpha \in \Phi_0^+} \frac{(\lambda_0 + \rho_0, \alpha)}{(\rho_0, \alpha)} \end{aligned}$$

(see [8, p. 106] for clarification).

Allowing  $\mathscr{W}$  to act, one may observe that all the above facts still hold for the system constituted by  $\sigma\Phi_0$  ( $\sigma \in \mathscr{W}$ ), as well as for that by  $\Phi_0$ . Moreover, the two facts below explain the connection between one system and another.

*Fact 5.*  $\sigma\mathscr{W}[\Delta_0]\sigma^{-1} = \mathscr{W}[\sigma\Delta_0]$  for any  $\sigma \in \mathscr{W}$ .

*Proof.* Obvious (see [3, Lemma 9.2] for justification).  $\square$

*Fact 6.*  $\tilde{\chi}_{\sigma\lambda_0}(\exp(H_{\sigma\nu_0})) = \tilde{\chi}_{\lambda_0}(\exp(H_{\nu_0}))$  for any  $\sigma \in \mathscr{W}$ .

*Proof.* For any  $\sigma \in \mathscr{W}$ , we have (by Fact 5)

$$\begin{aligned} \tilde{\chi}_{\sigma\lambda_0}(\exp(H_{\sigma\nu_0})) &= \frac{\sum_{\tau \in \mathscr{W}[\sigma\Delta_0]} \det(\tau) \exp(i\tau(\sigma\lambda_0 + \sigma\rho_0)(H_{\sigma\nu_0}))}{\prod_{\alpha \in \sigma\Phi_0^+} 2i \sin \frac{1}{2}\alpha(H_{\sigma\nu_0})} \\ &= \frac{\sum_{\tau \in \sigma\mathscr{W}[\Delta_0]\sigma^{-1}} \det(\tau) \exp(i\tau\sigma(\lambda_0 + \rho_0)(H_{\sigma\nu_0}))}{\prod_{\alpha \in \sigma\Phi_0^+} 2i \sin \frac{1}{2}\alpha(H_{\sigma\nu_0})} \\ &= \frac{\sum_{\tau \in \mathscr{W}[\Delta_0]} \det(\sigma\tau\sigma^{-1}) \exp(i\sigma\tau(\lambda_0 + \rho_0)(H_{\sigma\nu_0}))}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\sigma\alpha(H_{\sigma\nu_0})} \\ &= \frac{\sum_{\tau \in \mathscr{W}[\Delta_0]} \det(\tau) \exp(i\tau(\lambda_0 + \rho_0)(H_{\nu_0}))}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\alpha(H_{\nu_0})} \\ &= \tilde{\chi}_{\lambda_0}(\exp(H_{\nu_0})), \end{aligned}$$

as stated.  $\square$

**2. The proof of the theorem.** The outline of the proof is as follows. We first look for an estimate for all  $s \in (0, R)$ , then examine the decay for large  $s$ , and finally combine the results. The result obtained is valid under the assumption that  $G$  is simple, but then it extends to every semisimple Lie group  $G$ .

2.1. For all  $s \in (0, R)$ ,  $\lambda \in \Lambda^+$ , we have (see [2, p. 813])

$$\hat{\mu}_{sH}(\lambda) = \frac{\chi_\lambda(\exp(sH))}{d\lambda}.$$

Using the multiplicity formulae, we write

$$\hat{\mu}_{sH}(\lambda) = \frac{\sum_{\lambda' \in \varpi_\lambda} m_\lambda(\lambda') \exp(i\lambda'(sH))}{\sum_{\lambda' \in \varpi_\lambda} m_\lambda(\lambda')}.$$

Hence, we have

$$\begin{aligned} \left| \left( \frac{\partial}{\partial s} \right)^k \hat{\mu}_{sH}(\lambda) \right| &\leq \frac{\sum_{\lambda' \in \varpi_\lambda} m_\lambda(\lambda') \left| \left( \frac{\partial}{\partial s} \right)^k \exp(i\lambda'(sH)) \right|}{\sum_{\lambda' \in \varpi_\lambda} m_\lambda(\lambda')} \\ &\leq |H|^k |\lambda|^k = C_k(H) |\lambda|^k, \end{aligned}$$

for all  $k = 0, 1, 2, \dots$

2.2. By the Weyl formulae, for all  $s \in (0, R)$ ,  $\lambda \in \Lambda^+$ , we have

$$\hat{\mu}_{sH}(\lambda) = \frac{\sum_{\sigma \in \mathscr{W}} \det(\sigma) \exp(i(\lambda + \rho)(sH))}{\prod_{\alpha \in \Phi^+} 2i \sin \frac{1}{2}\alpha(sH)} \prod_{\alpha \in \Phi^+} \frac{(\rho, \alpha)}{(\lambda + \rho, \alpha)}.$$

In the case  $l = 1$ , one can easily obtain

$$\left| \left( \frac{\partial}{\partial s} \right)^k \hat{\mu}_{sH}(\lambda) \right| \leq C_k(H) \frac{|\lambda + \rho|^k}{s^{|\lambda + \rho|}},$$

for all  $k = 0, 1, 2, \dots$ . So assume, hereafter, that  $l \geq 2$ .

For each  $\lambda \in \Lambda^+$ , choose  $j_0 \in I$  for which  $(\lambda + \rho, \alpha_{j_0})$  is maximal. As before, we write  $\Delta_0 = \Delta \setminus \{\alpha_{j_0}\}$ ,  $\Phi_0^+ = \{\alpha \in \Phi^+ : n_{j_0}(\alpha) = 0\}$ , and  $\Phi_1^+ = \{\alpha \in \Phi^+ : n_{j_0}(\alpha) \geq 1\}$ . (Note that  $\Phi_1^+ = \Phi^+ \setminus \Phi_0^+$ , and that  $\Phi_1^+$  depends on the choice of  $j_0$ , and so depends on  $\lambda$ .) Clearly, if  $\alpha \in \Phi_0^+$ , then

$$(\lambda + \rho, \alpha) \geq (\rho, \alpha) \geq C,$$

and if  $\alpha \in \Phi_1^+$ , then (by the choice of  $j_0$ )

$$(\lambda + \rho, \alpha) \geq n_{j_0}(\alpha)(\lambda + \rho, \alpha_{j_0}) \geq C|\lambda + \rho|.$$

Moreover,

$$\gamma = \min_{j \in I} |\{\alpha \in \Phi^+ : n_j(\alpha) \geq 1\}| \leq |\Phi_1^+|.$$

Recall that  $\mathscr{W}_0$  is the subgroup of  $\mathscr{W}$  generated by  $\sigma_\alpha$  ( $\alpha \in \Delta_0$ ). For an appropriate  $\mathscr{S} \subset \mathscr{W}$ , we write  $\mathscr{W} = \bigcup_{\sigma \in \mathscr{S}} \sigma \mathscr{W}_0$  (disjoint union). We then obtain

$$\hat{\mu}_{sH}(\lambda) = \sum_{\sigma \in \mathscr{S}} \left( \frac{\sum_{\tau \in \mathscr{W}_0} \det(\sigma\tau) \exp(i\sigma\tau(\lambda + \rho)(sH))}{\prod_{\alpha \in \Phi^+} 2i \sin \frac{1}{2}\alpha(sH)} \prod_{\alpha \in \Phi^+} \frac{(\rho, \alpha)}{(\lambda + \rho, \alpha)} \right).$$

For each reflection  $\sigma_\alpha \in \mathscr{W}$ , we know that  $\det(\sigma_\alpha) = -1$ ,  $\sigma_\alpha \alpha = -\alpha$ , and  $\sigma_\alpha(\Phi^+ \setminus \{\alpha\}) = \Phi^+ \setminus \{\alpha\}$  (see [3, Lemma 10.2B]). Thus, for any  $\sigma \in \mathscr{W}$ , we have

$$\prod_{\alpha \in \Phi^+} 2i \sin \frac{1}{2}\alpha(sH) = \det(\sigma) \prod_{\alpha \in \Phi^+} 2i \sin \frac{1}{2}\sigma\alpha(sH).$$

It follows that

$$\begin{aligned} \hat{\mu}_{sH}(\lambda) &= \sum_{\sigma \in \mathscr{S}} \left( \frac{\sum_{\tau \in \mathscr{W}_0} \det(\tau) \exp(i\sigma\tau(\lambda + \rho)(sH))}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\sigma\alpha(sH)} \prod_{\alpha \in \Phi_0^+} \frac{(\rho, \alpha)}{(\lambda + \rho, \alpha)} \right) \\ &\quad \times \left( \prod_{\alpha \in \Phi_1^+} \frac{1}{2i \sin \frac{1}{2}\sigma\alpha(sH)} \frac{(\rho, \alpha)}{(\lambda + \rho, \alpha)} \right). \end{aligned}$$

Now fix  $\sigma \in \mathcal{S}$ . We write  $H = H_{\sigma\nu}$ , with  $\nu = \nu_0 + \nu_1$ , where  $\nu_0 \in \text{span}(\Delta_0)$  and  $\nu_1 = r\rho_1$  for some  $r \in \mathbf{R}$ . Then put  $H_0 = H_{\nu_0}$  and  $H_1 = H_{\nu_1}$ . Next recall that  $\Lambda_0^+$  is the set of dominant weights corresponding to  $\Phi_0^+$ . For each  $\lambda \in \Lambda^+$ , we write  $\lambda = \lambda_0 + \lambda_1$ , where  $\lambda_0 \in \Lambda_0^+$  and  $\lambda_1 = c\rho_1$  for some  $c \in \mathbf{R}^+$ . Hence, for all  $\alpha \in \Phi_0^+$ , we have  $(\rho, \alpha) = (\rho_0, \alpha)$  and  $(\lambda + \rho, \alpha) = (\lambda_0 + \rho_0, \alpha)$ . Further, for all  $\alpha \in \Phi_0^+$ ,

$$\begin{aligned} \sigma\alpha(H) &= (\sigma\alpha, \sigma\nu) = (\alpha, \nu) \\ &= (\alpha, \nu_0 + \nu_1) = (\alpha, \nu_0) \quad (\text{as } \nu_1 \perp \alpha) \\ &= \alpha(H_{\nu_0}) = \alpha(H_0), \end{aligned}$$

and whenever  $\tau \in \mathcal{W}_0$ ,

$$\begin{aligned} \sigma\tau(\lambda + \rho)(H) &= (\sigma\tau(\lambda + \rho), \sigma\nu) = (\tau(\lambda + \rho), \nu) \\ &= (\tau(\lambda_0 + \rho_0) + \tau(\lambda_1 + \rho_1), \nu_0 + \nu_1) \\ &= (\tau(\lambda_0 + \rho_0) + (\lambda_1 + \rho_1), \nu_0 + \nu_1) \quad (\text{as } \tau \in \mathcal{W}_0) \\ &= (\tau(\lambda_0 + \rho_0), \nu_0) + (\lambda_1 + \rho_1, \nu_1) \quad (\text{by orthogonality}) \\ &= \tau(\lambda_0 + \rho_0)(H_{\nu_0}) + (\lambda_1 + \rho_1)(H_{\nu_1}) \\ &= \tau(\lambda_0 + \rho_0)(H_0) + (\lambda_1 + \rho_1)(H_1). \end{aligned}$$

It turns out that

$$\begin{aligned} &\frac{\sum_{\tau \in \mathcal{W}_0} \det(\tau) \exp(i\sigma\tau(\lambda + \rho)(sH))}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2} \sigma\alpha(sH)} \prod_{\alpha \in \Phi_0^+} \frac{(\rho, \alpha)}{(\lambda + \rho, \alpha)} \\ &= \exp(i(\lambda_1 + \rho_1)(sH_1)) \frac{\sum_{\tau \in \mathcal{W}_0} \det(\tau) \exp(i\tau(\lambda_0 + \rho_0)(sH_0))}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2} \alpha(sH_0)} \\ &\quad \times \prod_{\alpha \in \Phi_0^+} \frac{(\rho_0, \alpha)}{(\lambda_0 + \rho_0, \alpha)} \\ &= \exp(i(\lambda_1 + \rho_1)(sH_1)) \frac{\tilde{\chi}_{\lambda_0}(\exp(sH_0))}{\tilde{d}_{\lambda_0}} \\ &= \exp(i(\lambda_1 + \rho_1)(sH_1)) \frac{\sum_{\lambda' \in \tilde{\alpha}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda') \exp(i\lambda'(sH_0))}{\sum_{\lambda' \in \tilde{\alpha}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda')} \\ &= \frac{\sum_{\lambda' \in \tilde{\alpha}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda') \exp(i(\lambda' + \lambda_1 + \rho_1)(sH))}{\sum_{\lambda' \in \tilde{\alpha}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda')} \quad (\text{by orthogonality}). \end{aligned}$$

So we have

$$\hat{\mu}_{sH}(\lambda) = \sum_{\sigma \in \mathcal{S}} \left( \frac{\sum_{\lambda' \in \tilde{\omega}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda') \exp(i(\lambda' + \lambda_1 + \rho_1)(sH))}{\sum_{\lambda' \in \tilde{\omega}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda')} \right) \times \left( \prod_{\alpha \in \Phi_1^+} \frac{1}{\sigma \alpha(sH)} \frac{\sigma \alpha(sH)}{2i \sin \frac{1}{2} \sigma \alpha(sH)} \frac{(\rho, \alpha)}{(\lambda + \rho, \alpha)} \right).$$

For all  $k = 0, 1, 2, \dots$ , we have the estimates

$$(1) \quad \left| \left( \frac{\partial}{\partial s} \right)^k \frac{\sum_{\lambda' \in \tilde{\omega}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda') \exp(i(\lambda' + \lambda_1 + \rho_1)(sH))}{\sum_{\lambda' \in \tilde{\omega}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda')} \right| \leq |H|^k |\lambda + \rho|^k,$$

$$(2) \quad \left| \left( \frac{\partial}{\partial s} \right)^k \prod_{\alpha \in \Phi_1^+} \sigma \alpha(sH)^{-1} \right| \leq C_k(H) s^{-k - |\Phi_1^+|},$$

$$(3) \quad \left| \left( \frac{\partial}{\partial s} \right)^k \prod_{\alpha \in \Phi_1^+} \frac{\sigma \alpha(sH)}{2i \sin \frac{1}{2} \sigma \alpha(sH)} \right| \leq C_k \quad (\text{by Leibniz' rule}),$$

$$(4) \quad \left| \prod_{\alpha \in \Phi_1^+} \frac{(\rho, \alpha)}{(\lambda + \rho, \alpha)} \right| \leq C |\lambda + \rho|^{-|\Phi_1^+|}$$

(as  $(\lambda + \rho, \alpha) \geq C |\lambda + \rho|$  for all  $\alpha \in \Phi_1^+$ ).

Therefore, by Leibniz' rule for the derivatives of products, we obtain

$$\begin{aligned} & \left| \left( \frac{\partial}{\partial s} \right)^k \hat{\mu}_{sH}(\lambda) \right| \\ & \leq \sum_{\sigma \in \mathcal{S}} \sum_{k_1 + k_2 + k_3 = k} C_{k_1, k_2, k_3} \left| \left( \frac{\partial}{\partial s} \right)^{k_1} (\text{1st term}) \right| \\ & \quad \times \left| \left( \frac{\partial}{\partial s} \right)^{k_2} (\text{2nd term}) \right| \left| \left( \frac{\partial}{\partial s} \right)^{k_3} (\text{3rd term}) \right| |\text{4th term}| \\ & \leq \sum_{\sigma \in \mathcal{S}} \sum_{k_1 + k_2 + k_3 = k} C_{k_1, k_2, k_3}(H) |H|^{k_1} |\lambda + \rho|^{k_1} s^{-k_2} (s|\lambda + \rho|)^{-|\Phi_1^+|} \\ & \leq C_k(H) (1 + |H|)^k \frac{|\lambda + \rho|^k}{(s|\lambda + \rho|)^{|\Phi_1^+|}} \quad (\text{provided } s|\lambda + \rho| > 1) \\ & \leq C_k(H) \frac{|\lambda + \rho|^k}{(s|\lambda + \rho|)^\gamma} \quad (\text{as } \gamma \leq |\Phi_1^+|), \end{aligned}$$

for all  $k = 0, 1, 2, \dots$ , as desired.

Combining this with the previous estimate, we obtain the result.

2.3. We shall now extend our result to every semisimple Lie group  $G$ . The key is to prove that Fact 2 in §1.2 is still valid.

Let us write  $\Phi$  as a disjoint union of irreducible root systems

$$\Phi = \Phi^{(1)} \cup \dots \cup \Phi^{(n)},$$

and split  $\Delta$  into

$$\Delta = \Delta^{(1)} \cup \dots \cup \Delta^{(n)},$$

with  $\Delta^{(m)} = \Delta \cap \Phi^{(m)}$  being a base of  $\Phi^{(m)}$  for each  $m \in \{1, \dots, n\}$ .

The Lie algebra  $\mathfrak{g}^{\mathbb{C}}$  is now a direct sum of simple ideals

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{(1)\mathbb{C}} \oplus \dots \oplus \mathfrak{g}^{(n)\mathbb{C}}.$$

As before, we choose  $j_0 \in I$  and remove  $\alpha_{j_0}$  from  $\Delta$  to obtain

$$\Delta_0 = \Delta \setminus \{\alpha_{j_0}\}.$$

But  $\alpha_{j_0} \in \Delta^{(M)}$  for some  $M \in \{1, \dots, n\}$ , and so

$$\Delta_0 = \Delta^{(1)} \cup \dots \cup \Delta_0^{(M)} \cup \dots \cup \Delta^{(n)},$$

with  $\Delta_0^{(M)} = \Delta^{(M)} \setminus \{\alpha_{j_0}\}$ . The Lie algebra  $\mathfrak{g}_0$  (as in §1.2) then decomposes into

$$\mathfrak{g}_0^{\mathbb{C}} = \mathfrak{g}^{(1)\mathbb{C}} \oplus \dots \oplus \mathfrak{g}_0^{(M)\mathbb{C}} \oplus \dots \oplus \mathfrak{g}^{(n)\mathbb{C}},$$

where  $\mathfrak{g}_0^{(M)\mathbb{C}}$  is the Lie subalgebra corresponding to  $\Delta_0^{(M)}$ . Now let  $K$ ,  $K_0$ ,  $K^{(m)}$ , and  $K_0^{(M)}$  denote the Killing forms of  $\mathfrak{g}$ ,  $\mathfrak{g}_0$ ,  $\mathfrak{g}^{(m)}$ , and  $\mathfrak{g}_0^{(M)}$  respectively. Then, for each  $m \in \{1, \dots, n\}$ ,  $m \neq M$ , we have

$$K_0|_{\mathfrak{g}^{(m)} \times \mathfrak{g}^{(m)}} = K^{(m)} = K|_{\mathfrak{g}^{(m)} \times \mathfrak{g}^{(m)}};$$

while for  $m = M$ , the connection between  $K^{(M)}$  and  $K_0^{(M)}$  is explained in §1.2. We therefore find that Fact 2 still holds, and thus the extension is clear.

**3. An example: The sharpness of the estimate.** We shall here consider an example concerning the sharpness of the  $L^p$ -estimate.

Let  $G = \mathbf{SU}(2)$ , the Lie group consisting of  $2 \times 2$  complex matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

with  $|\alpha|^2 + |\beta|^2 = 1$ . Its Lie algebra  $\mathfrak{g}$  then contains all matrices of the form

$$\begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix}$$

with  $a \in \mathbf{R}$ ,  $b \in \mathbf{C}$ . Here  $\gamma = 1$  and the special element is

$$H_\rho = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

In  $\mathfrak{g}$ , one may define the norm  $|\cdot|$  by

$$\left| \begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix} \right| = (a^2 + |b|^2)^{1/2} \quad \forall a \in \mathbf{R}, b \in \mathbf{C}.$$

For any  $y \in G$ ,  $X \in \mathfrak{g}$ , one may observe that

$$X' = yXy^{-1} \in \mathfrak{g},$$

with  $|X'| = |X|$ . Conversely, for any  $X, X' \in \mathfrak{g}$  with  $|X| = |X'|$ , one can find  $y \in G$  such that  $X' = yXy^{-1}$ .

Denote by  $B_0(\pi)$  the ball in  $\mathfrak{g}$  which has centre 0 and radius  $\pi$ . It is then evident that the map  $\exp: B_0(\pi) \rightarrow G$  is injective. Indeed, for each  $x \in G$ , there exists a unique  $X \in B_0(\pi)$  for which  $x = \exp(X)$ . Diagonalizing such an  $X$ , one has

$$x = y \exp(\omega H_\rho) y^{-1}, \quad \text{where } \omega = |X|,$$

for some  $y \in G$ . It is seen here that  $\text{trace}(x) = 2 \cos \omega$ .

As suggested in [6], let us consider the function  $f: G \rightarrow \mathbf{R}^+$  given by

$$f(\exp(X)) = \begin{cases} \frac{|X|^{-2}}{\log |X|^{-1}}, & \text{if } 0 < |X| < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

One may observe that  $f \in L^p(G)$ , whenever  $1 \leq p \leq \frac{3}{2}$ . On the other hand, regarding the maximal function  $\mathcal{M}f = \mathcal{M}_{H_\rho} f$ , we claim that  $\mathcal{M}f(x) = \infty$  for all  $x \in G$ .

Before verifying our claim, we remark that

$$f(-\exp(X)) = f(\exp(X'))$$

where  $|X'| = \pi - |X|$ . Moreover,  $f(yxy^{-1}) = f(x)$  for all  $x, y \in G$ . In fact, for all  $x, y \in G$ , we have

$$\begin{aligned} f(yxy^{-1}) &= f(y \exp(X) y^{-1}) \quad (\text{for some } X \in \mathfrak{g}) \\ &= f(\exp(yXy^{-1})) \\ &= f(\exp(X')) \quad (\text{where } |X'| = |X|) \\ &= f(\exp(X)) = f(x). \end{aligned}$$

Similarly, we observe that  $\mathcal{M} f(yxy^{-1}) = \mathcal{M} f(x)$  for all  $x, y \in G$ . To be precise, for all  $x, y \in G$ , we have

$$\begin{aligned} \mathcal{M} f(yxy^{-1}) &= \sup_{s \in (0, \pi)} \int_G f(y \times y^{-1} g \exp(sH_\rho) g^{-1}) dg \\ &= \sup_{s \in (0, \pi)} \int_G f(xy^{-1} g \exp(sH_\rho) g^{-1} y) dg \\ &= \sup_{s \in (0, \pi)} \int_G f(xg' \exp(sH_\rho) g'^{-1}) dg' = \mathcal{M} f(x). \end{aligned}$$

We shall now verify our claim. First, for  $x = \pm \mathbf{1}$ , we have

$$\begin{aligned} \mathcal{M} f(\pm \mathbf{1}) &= \sup_{s \in (0, \pi)} \int_G f(\pm g \exp(sH_\rho) g^{-1}) dg \\ &= \sup_{s \in (0, \pi)} \int_G f(\pm \exp(sH_\rho)) dg = \sup_{s \in (0, \pi)} f(\pm \exp(sH_\rho)) \\ &= \sup_{s \in (0, \frac{1}{2})} \frac{s^{-2}}{\log s^{-1}} = \infty. \end{aligned}$$

Next, for  $x \neq \pm \mathbf{1}$ , we may assume that  $x = \exp(\frac{t}{2}H_\rho)$  for some  $0 < t < 2\pi$ , and hence

$$\begin{aligned} \mathcal{M} f(x) &= \mathcal{M} f(\exp(\frac{t}{2}H_\rho)) \\ &= \sup_{s \in (0, \pi)} \int_G f(\exp(\frac{t}{2}H_\rho) g \exp(sH_\rho) g^{-1}) dg \\ &\geq \int_G f(\exp(\frac{t}{2}H_\rho) g \exp(\frac{t}{2}H_\rho) g^{-1}) dg. \end{aligned}$$

Writing each  $g \in G$  as  $g = h_\theta k_\phi h_{\theta'}$ , where  $h_\theta = \exp(\frac{\theta}{2}H_\rho)$  and  $k_\phi$  is the matrix of rotation with angle  $\frac{\phi}{2}$ , we have (see [9, pp. 99–100])

$$\begin{aligned} \mathcal{M} f(x) &\geq \frac{1}{16\pi^2} \int_{-2\pi}^{2\pi} \int_0^{2\pi} \int_0^\pi f(h_t h_\theta k_\phi h_{\theta'} h_t h_{-\theta'} k_{-\phi} h_{-\theta}) \sin \phi d\phi d\theta d\theta' \\ &= \frac{1}{16\pi^2} \int_{-2\pi}^{2\pi} \int_0^{2\pi} \int_0^\pi f(h_t h_\theta k_\phi h_t k_{-\phi} h_{-\theta}) \sin \phi d\phi d\theta d\theta' \\ &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi f(h_t h_\theta k_\phi h_t k_{-\phi} h_{-\theta}) \sin \phi d\phi d\theta \\ &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi f(h_{-\theta} h_t h_\theta k_\phi h_t k_{-\phi}) \sin \phi d\phi d\theta \\ &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi f(h_t k_\phi h_t k_{-\phi}) \sin \phi d\phi d\theta \\ &= \frac{1}{2} \int_0^\pi f(h_t k_\phi h_t k_{-\phi}) \sin \phi d\phi. \end{aligned}$$

Let us now investigate the integrand. Multiplying out, we get

$$h_t k_\phi h_t k_{-\phi} = \begin{pmatrix} e^{it} \cos^2 \frac{\phi}{2} + \sin^2 \frac{\phi}{2} & \cos \frac{\phi}{2} \sin \frac{\phi}{2} (1 - e^{it}) \\ -\cos \frac{\phi}{2} \sin \frac{\phi}{2} (1 - e^{-it}) & e^{-it} \cos^2 \frac{\phi}{2} + \sin^2 \frac{\phi}{2} \end{pmatrix}.$$

As seen before, this matrix is similar to  $\exp(\omega H_\rho)$ , where

$$\omega = \cos^{-1} \left( \sin^2 \frac{\phi}{2} + \cos^2 \frac{\phi}{2} \cos t \right).$$

By observation (thanks to John Cornwall for making it easier), there exists a constant  $C = C_t \in (0, 1)$  such that

$$\cos(\pi - \phi) \leq \sin^2 \frac{\phi}{2} + \cos^2 \frac{\phi}{2} \cos t \leq \cos C(\pi - \phi) \quad \forall \phi \in (\pi - \frac{1}{2}, \pi),$$

and accordingly

$$0 < C(\pi - \phi) \leq \omega \leq \pi - \phi < \frac{1}{2} \quad \forall \phi \in (\pi - \frac{1}{2}, \pi).$$

Hence we find that

$$f(h_t k_\phi h_t k_{-\phi}) = f(\exp(\omega H_\rho)) = \frac{\omega^{-2}}{\log \omega^{-1}} \geq \frac{(\pi - \phi)^{-2}}{\log\{C(\pi - \phi)\}^{-1}},$$

for all  $\phi \in (\pi - \frac{1}{2}, \pi)$ . It therefore follows that

$$\begin{aligned} \mathcal{M} f(x) &\geq \frac{1}{2} \int_{\pi-1/2}^{\pi} \frac{(\pi - \phi)^{-2}}{\log\{C(\pi - \phi)\}^{-1}} \sin \phi \, d\phi \\ &\geq \frac{1}{4} \int_{\pi-1/2}^{\pi} \frac{(\pi - \phi)^{-1}}{\log\{C(\pi - \phi)\}^{-1}} \, d\phi \\ &= \frac{1}{4} \int_0^{C/2} \frac{\varphi^{-1}}{\log \varphi^{-1}} \, d\varphi = \frac{1}{4} \int_{\log(2/C)}^{\infty} \frac{d\psi}{\psi} = \infty, \end{aligned}$$

as claimed.

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THE UNIVERSITY OF NEW SOUTH WALES  
KENSINGTON, N.S.W. 2033  
AUSTRALIA

*Correspondence address:* Jurusan Matematika  
Institut Teknologi Bandung  
Jalan Ganesha 10 Bandung  
Indonesia

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clemens@math.utah.edu

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Los Angeles, CA 90024-1555  
christ@math.ucla.edu

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University of California, San Diego  
La Jolla, CA 92093  
tenright@ucsd.edu

NICHOLAS ERCOLANI  
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ercolani@math.arizona.edu

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Stanford University  
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finn@gauss.stanford.edu

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vfr@math.berkeley.edu

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spk@gauss.stanford.edu

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