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ON SIX-CONNECTED FINITE H -SPACES

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In this note we shall prove the following theorem.

MAIN THEOREM. *Let X be a 6-connected finite H -space with associative mod 2 homology. Further, suppose that $\text{Sq}^4 H^7(X; \mathbb{Z}_2) = 0$ and $\text{Sq}^{15} H^{15}(X; \mathbb{Z}_2) = 0$. Then X is either contractible or has the homotopy type of a product of seven-spheres.*

0. Introduction. It should be noted that there are several results related to this theorem. Lin showed that any finite H -space with associative mod 2 homology has its first nonvanishing homotopy in degrees 1, 3, 7, or 15 (or is contractible). A seven-sphere is an H -space, but not a mod 2 homotopy-associative one [4, 10]. Further work of Hubbuck [5], Sigrist and Suter [12], and others has shown that spaces whose mod 2 cohomology has the form

$$\Lambda(x_7, x_{11}) \quad \text{or} \quad \Lambda(x_7, x_{11}, x_{13})$$

are not realizable as H -spaces. (Here x_i denotes an element of degree i .) One is led to conjecture that

Conjecture 1. Every two-torsion-free 6-connected finite H -space is homotopy equivalent to a product of seven-spheres (or is acyclic).

Conjecture 2. Every two-torsion-free homotopy-associative 6-connected finite H -space is acyclic.

Conjecture 1 implies Conjecture 2 by [4, 11].

Henceforth, X will denote an H -space that satisfies the hypotheses of the Main Theorem, and $H^*(X)$ will denote $H^*(X; \mathbb{Z}_2)$. The proof of the Main Theorem will be accomplished in a series of steps, which we record here. Our goal is to show that under the hypotheses, X has mod 2 cohomology an exterior algebra on 7-dimensional generators. This relies heavily on the following theorem.

Steenrod Connections [8]. Let X be a finite simply-connected H -space with associative mod 2 homology. Then for $r \geq 0$, $k > 0$,

$$QH^{2'+2^{r+1}k-1}(X; Z_2) = \text{Sq}^{2^r k} QH^{2'+2^r k-1}(X; Z_2), \text{ and}$$

$$\text{Sq}^{2^r} QH^{2'+2^{r+1}k-1}(X; Z_2) = 0.$$

(Here QH^* denotes the indecomposable quotient.)

In §1 we shall use a relation in the Steenrod algebra and the methods of [1] to produce a new factorization of Sq^{16} . We then apply this factorization to show that $H^{23}(X) = 0$. This implies that $H^*(X)$ is an exterior algebra on generators in degrees of the form $2^d - 1$, $d \geq 3$, with trivial action of the Steenrod algebra. In §2 we use the Cartan formula for secondary operations, [7], and a particular factorization of the cube of a certain 8-dimensional cohomology class, [10], to show that $H^{15}(X) = 0$. In §3 we turn to the c -invariant, [14], to complete our calculations by showing that no algebra generators for $H^*(X)$ exist in degrees greater than seven. Once it is shown that the mod 2 cohomology is exterior on 7-dimensional generators, it follows by the Bockstein spectral sequence that the rational cohomology has the same form. But since the rational cohomology is isomorphic to the E_∞ term of the mod p Bockstein spectral for any prime p , it follows by [2] that $H^*(X; Z)$ has no odd torsion. Thus $H^*(X; Z)$ is torsion-free, and we may use the Hurewicz map together with the multiplication in X to obtain a homotopy equivalence

$$S^7 \times \cdots \times S^7 \rightarrow X.$$

1. $H^{23}(X)$. In this section we prove that there are no 23-dimensional generators in $H^*(X)$. We will also show that $H^*(X)$ is an exterior algebra with trivial action of the Steenrod algebra. We shall use the notation $\text{Sq}^{i,j}$ to denote $\text{Sq}^i \text{Sq}^j$.

THEOREM 1.1. *Let Y be a space and $x \in H^k(Y)$ be the reduction of an integral class. If x is in the intersection of the kernels of Sq^2 , Sq^7 , Sq^8 , and $\text{Sq}^{8,4}$, then there exist classes $v_i \in H^{k+i}(Y)$, $i = 3, 7, 8, 10, 12, 13, 14, 15$, such that*

$$(1.1) \quad \begin{aligned} \text{Sq}^{16}x &= \text{Sq}^{11,2}v_3 + (\text{Sq}^{7,2} + \text{Sq}^{6,3})v_7 \\ &\quad + (\text{Sq}^8 + \text{Sq}^{6,2})v_8 + \text{Sq}^{4,2}v_{10} + \text{Sq}^4v_{12} \\ &\quad + \text{Sq}^3v_{13} + \text{Sq}^2v_{14} + \text{Sq}^1v_{15}. \end{aligned}$$

Proof. Consider the following matrix of relations:

$$(1.2) \quad \begin{matrix} v_3 \\ v_7 \\ v_8 \\ v_{10} \\ v_{12} \\ v_{13} \\ v_{14} \\ v_{15} \end{matrix} \begin{pmatrix} \text{Sq}^2 & 0 & 0 & 0 \\ 0 & \text{Sq}^1 & 0 & 0 \\ 0 & \text{Sq}^2 & \text{Sq}^1 & 0 \\ \text{Sq}^9 & \text{Sq}^4 & \text{Sq}^3 & 0 \\ \text{Sq}^{8,2,1} & 0 & 0 & \text{Sq}^1 \\ \text{Sq}^{12} & 0 & \text{Sq}^{4,2} & 0 \\ \text{Sq}^{13} + \text{Sq}^{12,1} & \text{Sq}^8 & 0 & \text{Sq}^3 \\ \text{Sq}^{14} & 0 & \text{Sq}^8 & \text{Sq}^4 \end{pmatrix} \begin{pmatrix} \text{Sq}^2 \\ \text{Sq}^7 \\ \text{Sq}^8 \\ \text{Sq}^{8,4} \end{pmatrix} = 0.$$

Let

$$w: K(Z, n) \rightarrow K(Z_2; n + 2, n + 7, n + 8, n + 12) = K_0$$

be defined by

$$\begin{aligned} w^*(i_{n+2}) &= \text{Sq}^2 i_n; \quad w^*(i_{n+7}) = \text{Sq}^7 i_n; \\ w^*(i_{n+8}) &= \text{Sq}^8 i_n; \quad w^*(i_{n+12}) = \text{Sq}^{8,4} i_n. \end{aligned}$$

If E is the fiber of w , we have the following diagram

$$(1.3) \quad \begin{array}{ccc} & \Omega K_0 & \\ & \downarrow j & \\ & E & \\ & \downarrow & \\ K(Z, n) & \xrightarrow{w} & K_0 \end{array}$$

and there exist elements $v_j \in PH^{n+j}(E; Z_2)$ defined by the relations (1.2).

A calculation shows that the element

$$\begin{aligned} z &= \text{Sq}^{11,2} v_3 + (\text{Sq}^{7,2} + \text{Sq}^{6,3}) v_7 + (\text{Sq}^8 + \text{Sq}^{6,2}) v_8 \\ &\quad + \text{Sq}^{4,2} v_{10} + \text{Sq}^4 v_{12} + \text{Sq}^3 v_{13} + \text{Sq}^2 v_{14} + \text{Sq}^1 v_{15} \end{aligned}$$

lies in $PH^{16+n}(E) \cap \ker(j^*) = p^*PH^{16+n}(K(Z_2, n))$. It follows that $z = cp^*(\text{Sq}^{16} i_n)$, where $c \in Z_2$. For $n = 16$ there is a commutative

diagram

$$\begin{array}{ccccc}
 & & \Omega K_0 & & \\
 & & \downarrow j & & \\
 & & E & & \\
 & \nearrow \tilde{f} & \downarrow & & \\
 K(Z, 2) & \xrightarrow{f} & K(Z, n) & \xrightarrow{w} & K_0
 \end{array}$$

where $f^*(i_{16}) = i_2^8$. Now consider $\bar{w}: K(Z_2, 16) \rightarrow K_0 \times K(Z_2, 17)$ given by the same formulas as w on the fundamental classes in K_0 and such that $\bar{w}^*(i_{17}) = \text{Sq}^1 i_{16}$. Let \bar{E} be the fiber of \bar{w} .

There exists a commutative diagram

$$\begin{array}{ccccc}
 \Omega K_0 & \longrightarrow & \Omega K_0 \times K(Z_2, 16) & & \\
 j \downarrow & & \downarrow \bar{j} & & \\
 & & E & \xrightarrow{\bar{h}} & \bar{E} \\
 & \nearrow \tilde{f} & \downarrow & & \downarrow \\
 K(Z, 2) & \xrightarrow{f} & K(Z, 16) & \xrightarrow{h} & K(Z_2, 16) \\
 w \downarrow & & \downarrow & & \downarrow \bar{w} \\
 K_0 & \longrightarrow & K_0 \times K(Z_2, 17) & &
 \end{array}
 \tag{1.4}$$

Further, there is another lifting $\tilde{h}: K(Z, 2) \rightarrow \bar{E}$ of hf that has its H -deviation

$$D\tilde{h}: K(Z, 2) \times K(Z, 2) \rightarrow K(Z_2, 16)$$

given by $[D\tilde{h}] = i_2^4 \otimes i_2^4$. This holds because

$$B(hf)^* B\bar{w}^*(i_{18}) = \text{Sq}^{9,4,2} i_3 = (\text{Sq}^{4,2} i_3)^2$$

and because $B(hf)^* B\bar{w}^*$ is zero on the fundamental classes in K_0 .

In $PH^*(\bar{E})$ there exist elements \bar{v}_j such that $\bar{h}^*(\bar{v}_j) = v_j$. The components of \bar{v}_j in $H^*(K(Z_2, 16))$ are:

$$\begin{aligned}
 (1.5) \quad & \bar{v}_3: \text{Sq}^3 i_{16}; \quad \bar{v}_7: 0; \quad \bar{v}_8: \text{Sq}^8 i_{16}; \quad \bar{v}_{10}: 0; \\
 & \bar{v}_{12}: \text{Sq}^{8,4} i_{16}; \quad \bar{v}_{13}: \text{Sq}^{4,9} i_{16}; \quad \bar{v}_{14}: 0; \quad \bar{v}_{15}: \text{Sq}^{15} i_{16}.
 \end{aligned}$$

It follows that $\tilde{h}^*(\bar{v}_j)$ is primitive except for $\tilde{h}^*(\bar{v}_8)$ which has

$$\bar{\Delta}\tilde{h}^*(\bar{v}_8) = i_2^8 \otimes i_2^4 + i_2^4 \otimes i_2^8 = \bar{\Delta}(i_2^{12}).$$

Because $H^*(K(Z, 2))$ is trivial in odd degrees and is Z_2 in even degrees, we conclude

$$\begin{aligned} \tilde{h}^*(\bar{v}_j) &= 0 \quad \text{if } j \neq 8, \quad \text{and} \\ \tilde{h}^*(\bar{v}_8) &= i_2^{12}. \end{aligned}$$

Therefore if

$$\begin{aligned} \bar{z} = & \text{Sq}^{11,2}\bar{v}_3 + (\text{Sq}^{7,2} + \text{Sq}^{6,3})\bar{v}_7 + (\text{Sq}^8 + \text{Sq}^{6,2})\bar{v}_8 + \text{Sq}^{4,2}\bar{v}_{10} \\ & + \text{Sq}^4\bar{v}_{12} + \text{Sq}^3\bar{v}_{13} + \text{Sq}^2\bar{v}_{14} + \text{Sq}^1\bar{v}_{15}, \end{aligned}$$

it follows that

$$(1.6) \quad \tilde{h}^*(\bar{z}) = \text{Sq}^8\tilde{h}^*(\bar{v}_8) = \text{Sq}^8(i_2^{12}) = i_2^{16} = \text{Sq}^{16}(i_2^8).$$

Now \tilde{h} and $\bar{h}\tilde{f}$ both lift hf , so

$$\bar{h}\tilde{f} = \tilde{h} + \bar{j}F,$$

for some

$$F = (F_1, F_2): K(Z, 2) \rightarrow \Omega K_0 \times K(Z_2, 16).$$

But ΩK_0 is odd-dimensional with the exception of $K(Z_2, 22)$. If \tilde{f} is altered by F_1 , then

$$\bar{h}\tilde{f} = \tilde{h} + \bar{j}F_2,$$

and $[F_2] = di_2^8, d \in Z_2$.

Using (1.5) we calculate that for all j

$$(1.7) \quad F_2^*\bar{j}^*(\bar{v}_j) = 0, \quad \text{and hence } \tilde{h}^*(\bar{v}_j) = \tilde{f}^*(v_j).$$

Therefore

$$\tilde{f}^*(z) = \tilde{h}^*(\bar{z}) = \text{Sq}^{16}(i_2^8).$$

It follows that $c = 1$. □

THEOREM 1.2. $QH^{23}(X) = 0$.

Proof. By the restrictions on the degrees of generators of $H^*(X)$, $H^i(X \wedge X) = 0$ for $i = 7, 15$, and 31 . So by the Steenrod connections, all generators in degrees less than 63 may be chosen to be primitive. Further, in degrees < 40 , $H^*(X)$ is an exterior algebra in which

$$(1.8) \quad QH^k(X) = 0, \quad k \neq 7, 15, 23, 27, 29, 31, 39.$$

The lowest-dimensional possible non-trivial Steenrod operation is Sq^8 acting on $H^{15}(X)$. So let $x_{23} = \text{Sq}^8 x_{15} \neq 0$. By (1.8), Sq^2 , Sq^8 , and $\text{Sq}^{8,4}$ are all zero on x_{23} , and $\text{Sq}^7 x_{23} = \text{Sq}^{15} x_{15}$, which is zero by hypothesis. Thus the factorization (1.1) applies to $\text{Sq}^{16} x_{23}$. We now construct the universal example.

Let $p_0: E_0 \rightarrow K(Z, 23)$ be the fiber of the map

$$g: K(Z, 23) \rightarrow K(Z_2; 25, 30, 31, 35)$$

given by

$$\begin{aligned} g^*(i_{25}) &= \text{Sq}^2(i_{23}), \\ g^*(i_{30}) &= \text{Sq}^7(i_{23}), \\ g^*(i_{31}) &= \text{Sq}^8(i_{23}), \quad \text{and} \\ g^*(i_{35}) &= \text{Sq}^{8,4}(i_{23}). \end{aligned}$$

Next, define $p_1: E_1 \rightarrow E_0$ to be the fiber of the map

$$g_0: E_0 \rightarrow K_0 = K(Z_2; 26, 30, 33, 35, 36, 37, 38; \overline{32}, \overline{33}, \overline{35})$$

given by

$$g_0^*(i_{23+m}) = v_m \quad (m \neq 8),$$

and

$$g_0^*(\bar{i}_{31+k}) = \text{Sq}^k v_8 \quad (k = 1, 2, 4).$$

Consider the element in $H^{47}(K_0)$:

$$\begin{aligned} \chi &= \text{Sq}^8[\text{Sq}^{11,2} i_{26} + (\text{Sq}^{7,2} + \text{Sq}^{6,3}) i_{30} + \text{Sq}^{4,2} i_{33} + \text{Sq}^4 i_{35} \\ &\quad + \text{Sq}^3 i_{36} + \text{Sq}^2 i_{37} + \text{Sq}^1 i_{38}] \\ &\quad + \text{Sq}^{15} \bar{i}_{32} + (\text{Sq}^{14} + \text{Sq}^{10,4}) \bar{i}_{33} + \text{Sq}^{12} \bar{i}_{37}. \end{aligned}$$

Applying g_0^* to χ , we get

$$\begin{aligned} g_0^*(\chi) &= \text{Sq}^8[\text{Sq}^{11,2} v_3 + (\text{Sq}^{7,2} + \text{Sq}^{6,3}) v_7 + \text{Sq}^{4,2} v_{10} + \text{Sq}^4 v_{12} \\ &\quad + \text{Sq}^3 v_{13} + \text{Sq}^2 v_{14} + \text{Sq}^1 v_{15}] \\ &\quad + (\text{Sq}^{15,1} + \text{Sq}^{14,2} + \text{Sq}^{12,4} + \text{Sq}^{10,4,2}) v_8 \\ &= \text{Sq}^8[\text{Sq}^{11,2} v_3 + (\text{Sq}^{7,2} + \text{Sq}^{6,3}) v_7 + \text{Sq}^{4,2} v_{10} + \text{Sq}^4 v_{12} \\ &\quad + \text{Sq}^3 v_{13} + \text{Sq}^2 v_{14} + \text{Sq}^1 v_{15} + (\text{Sq}^8 + \text{Sq}^{6,2}) v_8] \\ &= \text{Sq}^8 \text{Sq}^{16} p_0^*(i_{23}) \\ &= (\text{Sq}^{24} + \text{Sq}^{23,1} + \text{Sq}^{22,2} + \text{Sq}^{20,4}) p_0^*(i_{23}). \end{aligned}$$

The values of the last three operations on i_{23} are in the kernel of p_0^* . So

$$g_0^*(\chi) = \text{Sq}^{24} p_0^*(i_{23}).$$

Hence there exists an element $v \in H^{46}(E_1)$ such that $\bar{\Delta}(v) = p_1^* p_0^*(l_{23}) \otimes p_1^* p_0^*(l_{23})$ and $j_1^*(v) = \sigma^*(\chi)$, where j_1 is the fiber of p_1 .

We now need to map X into E_1 . Let $f: X \rightarrow K(Z, 23)$ be such that $f^*(l_{23}) = x_{23}$. We remark that f can be chosen to be an H -map, since $H^{23}(X \wedge X; Z) = 0$. Since the composition $g \circ f$ is nullhomotopic, there exists a lifting $f_0: X \rightarrow E_0$ of f . The H -deviation of f_0 factors through j_0 , the fiber of p_0 , say $Df_0 = j_0 \circ \tilde{D}_0$. The map \tilde{D}_0 corresponds to a set of classes in $H^k(X \wedge X)$, $k = 24, 29, 30$, and 34 .

We shall work in P_2X , the projection plane of X . Recall that there is an exact triangle [3]

$$(1.9) \quad \begin{array}{ccc} H^*(P_2X) & \longrightarrow & IH^*(X) \\ & \swarrow & \searrow \\ & IH^*(X) \otimes IH^*(X) & \end{array}$$

that relates P_2X to X . This implies that

$$(1.10) \quad H^k(P_2X) = 0 \quad (17 \leq k \leq 22).$$

Let $u_{16} \in H^{16}(P_2X)$ correspond to x_{15} and set $u_{24} = \text{Sq}^8 u_{16}$. By (1.10) and the Adem relations, Sq^2 , Sq^8 , and $\text{Sq}^{8,4}$ are all zero on u_{24} . So by [3], the components of \tilde{D}_0 in degrees 24, 30, and 34 are all zero. Thus $\tilde{D}_0 \in H^{29}(X \wedge X)$, so it is a sum of terms of the form $x_7 \otimes x'_7 x_{15}$, $x_7 x'_7 \otimes x_{15}$, and twists of these terms. Consider the elements $f_0^* \circ g_0^*(l)$, where l is one of the fundamental classes of K_0 . We have

$$\bar{\Delta}(f_0^* \circ g_0^*(l)) = (Df_0)^* g_0^*(l) = \tilde{D}_0^* \circ j_0^* \circ g_0^*(l).$$

Referring to the matrix relation (1.2), we see that the only possible non-zero values can be when $l = l_{37}$, when

$$\tilde{D}_0^* \circ j_0^* \circ g_0^*(l) = \text{Sq}^8 \tilde{D}_0^*(l_{29}).$$

Hence the images under $f_0^* \circ g_0^*$ of all the fundamental classes of K_0 , with the possible exception of l_{37} , are primitive, so for degree reasons they must be zero. We might possibly have

$$f_0^* \circ g_0^*(l_{37}) = \sum x_{i,7} x'_{i,7} x_{i,23}.$$

But since $\text{Sq}^8: H^{15}(X) \rightarrow H^{23}(X)$ is onto, we may alter the lift f_0 by the action of the fiber on the map $\tilde{f}: X \rightarrow K(Z_2, 29)$ given by

$$\tilde{f}^*(l_{29}) = \sum x_{i,7} x'_{i,7} x_{i,15}$$

so as to make, for the altered f_0 , $f_0^* \circ g_0^*(t_{37}) = 0$. Thus there exists a lifting $f_1: X \rightarrow E_1$.

We now consider the element $f_1^*(v) \in H^{46}(X)$. We have

$$\bar{\Delta}(f_1^*(v)) = (f_1^* \otimes f_1^*)(\bar{\Delta}v) + (Df_1)^*(v) = x_{23} \otimes x_{23} + (Df_1)^*(v).$$

There is no term in $H^*(X)$ whose coproduct has $x_{23} \otimes x_{23}$ as a summand. Now

$$Df_1 = \theta + j_1 \circ \tilde{D}_1,$$

where $\theta: X \wedge X \rightarrow E_1$ is a map given by applying the Cartan formula, Theorem 3.1 of [7], to Df_0 . The map θ factors through cohomology classes in $H^*(X \wedge X)$ of which one factor is a primary or secondary operation applied to a decomposable element, and, by the Cartan formulae for primary and secondary operations, such operations cannot hit x_{23} . Also, $(j_1 \circ \tilde{D}_1)^*(v)$ lies in the image of Steenrod operations applied to elements of degrees $\neq 30$ or 38 , so $x_{23} \otimes x_{23}$ cannot be in this image. Thus Sq^8 is identically zero on $H^{15}(X)$ and hence $QH^{23}(X) = 0$. \square

COROLLARY 1.3. *$H^*(X)$ is an exterior algebra on generators concentrated in degrees of the form $2^d - 1$ for $d \geq 3$. Further, the action of the Steenrod algebra on $H^*(X)$ is trivial.*

Proof. By the Steenrod connections, any element of $QH^*(X)$ not in a degree of the form $2^d - 1$ lies in the image of Steenrod operations applied to generators in degrees of the form $2^d - 1$. By Theorem 1.2 and the Steenrod Connections, it follows that

$$\text{Sq}^{2^i} QH^{2^d-1}(X) = 0 \quad \text{for } i = 0, 1, 2, 3.$$

By [1], Sq^{2^i} factors through secondary operations for $i \geq 4$ if x_{2^d-1} lies in the kernel of Sq^{2^j} for $0 \leq j \leq i - 1$.

So consider the first nontrivial Steenrod operation, say $\text{Sq}^{2^i} x_{2^d-1}$. By the Cartan formula, $\text{Sq}^{2^i} x_{2^d-1}$ is primitive, so it must be a generator. By the Steenrod connections we must have $i = d - 1$. By Theorem 1.2 we must have $d \geq 5$, so $i \geq 4$. But this implies $\text{Sq}^{2^{d-1}} x_{2^d-1}$ is in the image of Steenrod operations of lower degree, which cannot happen. Thus the action of the Steenrod algebra on $H^*(X)$ is trivial. Hence $H^*(X)$ is an exterior algebra on generators in degrees of the form $2^d - 1$, $d \geq 3$. \square

2. $H^{15}(X)$.

THEOREM 2.1. $H^{15}(X) = 0$.

Proof. Let x_{15} be a nonzero element of $H^{15}(X)$. We define a cohomology operation as follows. Consider the diagram:

$$(2.1) \quad \begin{array}{ccccc} & & E_2 & & \\ & \nearrow f_2 & \downarrow p_2 & & \\ & & E_1 & \xrightarrow{g_2} & K(Z_2; 18, 19, 22, 23, 24, 30) \\ & \nearrow f_1 & \downarrow p_1 & & \\ X & \xrightarrow{f} & K(Z, 15) & \xrightarrow{g_1} & K(Z_2; 17, 19, 23) \end{array}$$

which is associated with a factorization of Sq^{16} as

$$Sq^{16} = \sum \alpha_{ij} \phi_{ij}$$

in which the α_{ij} are Steenrod operations and the ϕ_{ij} are the secondary operations of Adams, [1], and is constructed as follows.

The map g_1 is given by the formulas

$$g_1^*(t_{15+2^k}) = Sq^{2^k}(t_{15}), \quad k = 1, 2, 3.$$

The map g_2 is given by the formulas

$$g_2^*(t_{14+2^i+2^{i'}}) = v_{ij},$$

where v_{ij} is an element in $H^*(E_1)$ that represents the secondary operation ϕ_{ij} .

The map f represents the element x_{15} . The lift f_1 exists since all Steenrod operations are zero on x_{15} , by Corollary 1.3. Now the H -deviation of f_1 factors through the fiber of p_1 , namely $K(Z_2; 16, 18, 22)$. Hence the reduced coproducts of the $f_1^*(v_{ij})$ are in the image of Steenrod operations, which are all zero. Hence the $f_1^*(v_{ij})$ are primitive, so they are all zero. Therefore the lift f_2 exists.

In $H^{30}(E_2)$ there is an element v whose reduced coproduct is $p_2^*p_1^*(t_{15}) \otimes p_2^*p_1^*(t_{15})$. We shall show that the reduced coproduct of $f_2^*(v)$ contains a term $x_{15} \otimes x_{15}$, which will be a contradiction. Let us write the factorization of the H -deviation Df_1 of f_1 as

$$Df_1 = \tilde{D} \circ j_1.$$

The map \tilde{D} determines elements in degrees 16, 18, and 22 of $X \wedge X$. Checking possibilities, we see that the components in degrees 16 and

18 are zero, while we may express the component in degree 22 as

$$\tilde{D}^*(l_{22}) = \sum x_{7,i} \otimes x_{15,i}$$

for elements $x_{7,i}$ and $x_{15,i}$ in degrees 7 and 15 respectively. Use of the Cartan formula, [7], now enables us to express the H -deviation of f_2 as the sum of terms in the image of Steenrod operations (which are all zero) together with terms of the form

$$\psi_i(x_{7,i}) \otimes x_{15,i},$$

where the ψ_i are secondary operations. We need to check that it cannot happen for $x_{15,i}$ and $\psi_i(x_{7,i})$ both to be x_{15} . To determine the secondary operations involved here, we may consider the diagram

$$\begin{array}{ccccc} & & G & & \\ & \nearrow h_1 & \downarrow \pi & & \\ X & \xrightarrow{h} & K(Z, 7) & \xrightarrow{\text{Sq}^2, \text{Sq}^4} & K(Z_2; 9, 11). \end{array}$$

Using either the Serre or the Eilenberg-Moore spectral sequence we see that a basis for $H^{15}(G)$ is given by elements in the image of Steenrod operations together with an element $\tilde{w}_{0,3}$ that restricts to the fiber of π to be $(\text{Sq}^5 + \text{Sq}^{4,1})l_{10}$. So we need to determine whether $h_1^*(\tilde{w}_{0,3})$ can be x_{15} .

For dimensional reasons, h_1 is an H -map. Hence it determines a map $\hat{h}_1: P_2X \rightarrow BG$, where BG denotes the classifying space of G . If $h_1^*(\tilde{w}_{0,3}) = x_{15}$, then $y_{16} = \hat{h}_1^*(B\tilde{w}_{0,3})$ is a representative in $H^*(P_2X)$ of the primitive class x_{15} . In [10], Corollary 1.3, we derived the formula (in the cohomology of BG)

$$(B\pi^*(l_8))^3 \equiv \text{Sq}^8(B\tilde{w}_{0,3}), \quad \text{modulo } \text{Im}(\text{Sq}^{12}, \text{Sq}^{6,3}, \text{Sq}^{4,2,1}).$$

In general, three-fold cup products in $H^*(P_2X)$ are all zero. By the hypotheses on X and (1.9), $H^*(P_2X) = 0$ in degrees 12, 15, and 17. So $\text{Sq}^8 \hat{h}_1^*(B\tilde{w}_{0,3}) = 0$. By [13], $\text{Sq}^8(y_{16}) = \sum y_{8,i} y_{16,i}$, where the $y_{8,i}$ and $y_{16,i}$ correspond to $x_{7,i}$ and $x_{15,i}$, respectively. So we obtain that $\psi_i(x_{7,i})$ cannot contain x_{15} as a summand; hence the reduced coproduct

$$\bar{\Delta}h_2^*(v) = x_{15} \otimes x_{15},$$

which, as stated above, is a contradiction. □

3. $QH^{2^k-1}(X)$. By Corollary 1.3 and Theorem 2.1, $H^*(X)$ is an exterior algebra on generators in degrees 7 and $2^d - 1$, for $d \geq 5$, and has trivial action of the Steenrod algebra.

THEOREM 3.1. $QH^*(X)$ is concentrated in degree 7.

Proof. Let $x = x_{2^k-1}$, $k \geq 5$, be a generator of lowest degree greater than seven. Let $\xi H^*(X)$ denote the image of the cup-squaring map $\xi(x) = x^2$. Since $H_*(X)$ is associative, we may assume by [8] that $\bar{\Delta}x \in \xi H^*(X) \otimes H^*(X)$, which is trivial since $\xi H^*(X) = 0$. Hence x may be chosen to be primitive. We shall construct an operation similar to that in the proof of Theorem 1.4. Consider the following diagram:

$$(3.1) \quad \begin{array}{ccccc} & & E_2 & & \\ & \nearrow f_2 & \downarrow p_2 & & \\ & & E_1 & \xrightarrow{g_2} & K_2 \\ & \nearrow f_1 & \downarrow p_1 & & \\ X & \xrightarrow{f} & K(Z, 2_1^k) & \xrightarrow{g_1} & K_1, \end{array}$$

in which $K_1 = \prod_i K(Z_2; 2^k - 1 + 2^n)$, $1 \leq n \leq k - 1$, and

$$g_1^*(i_{2^k-1+2^n}) = \text{Sq}^{2^n} i_{2^k-1},$$

and in which $K_2 = \prod_{i,j} K(Z_2; 2^k - 2 + 2^i + 2^j)$, and g_2 represents the secondary operations φ_{ij} associated with a factorization of Sq^{2^k} .

By Corollary 1.3, all Steenrod equations vanish on x , so $g_1 f \simeq *$ and the lift f_1 exists.

We note that in degrees below $2^k - 1$, $H^*(X)$ is concentrated in degrees divisible by seven. Since x is primitive, f is an H -map. Therefore $D_{g_2 f_1}$ factors through the fiber of p_1 . Hence the formula for the H -deviation of a composition yields that $D_{g_2 f_1}$ is in the image of primary operations in $H^*(X \wedge X)$, so it is zero by Corollary 1.3. Hence $g_2 f_1$ is represented by primitive elements of $H^*(X)$ in degrees not of the form $2^d - 1$. Since all primitives are concentrated in degrees of the form $2^d - 1$, $g_2 f_1$ is nullhomotopic, and the lift f_2 exists.

To simplify the situation, we loop the entire diagram to obtain

$$(3.2) \quad \begin{array}{ccccc} & & \Omega E_2 & & \\ & \nearrow^{\Omega f_1} & \downarrow^{\Omega p_2} & & \\ & & \Omega E_1 & \xrightarrow{\Omega g_2} & \Omega K_2 \\ & \nearrow^{\Omega f_2} & \downarrow^{\Omega p_1} & & \\ \Omega X & \xrightarrow{\Omega f} & K(Z, 2^k - 2) & \xrightarrow{\Omega g_1} & \Omega K_1. \end{array}$$

Note. The c -invariant was introduced in [14] as the obstruction to an H -map between two homotopy-commutative H -spaces preserving the homotopy-commutative structure. There are various choices for this invariant, which depend on the choice of homotopy realizing the H -map. It was observed that if Y and Z are H -spaces and $h: Y \rightarrow Z$ a map, then the composition

$$(3.3) \quad \sum \Omega Y \wedge \sum \Omega Y \xrightarrow{\varepsilon \wedge \varepsilon} Y \wedge Y \xrightarrow{Dh} Z$$

has as its double adjoint $\Omega Y \wedge \Omega Y \rightarrow \Omega Z$ a particular choice for the c -invariant $c(\Omega h)$. In the sequel we shall always make this choice for our c -invariants.

We have a suspension element v in $H^{2^{k+1}-1}(\Omega E_2)$ such that

$$c(v) = (\Omega(p_1 p_2))^* \iota_{2^k - 2} \otimes (\Omega(p_1 p_2))^* \iota_{2^k - 2}.$$

We shall consider the c -invariant of the element

$$(\Omega f_2)^*[v] \in H^{2^{k+1}-1}(\Omega X) = 0.$$

Let $u_{2^k - 2} = \sigma^*(x_{2^k - 1})$. Then, applying (3.3) to the formula for the H -deviation for a composition of maps, we obtain

$$0 = c((\Omega f_2)^*[v]) = u_{2^k - 2} \otimes u_{2^k - 2} + c(\Omega f_2)^*[v].$$

Since $x_{2^k - 1}$ is primitive, $u_{2^k - 2}$ is a c -class. Hence $c(\Omega f_1)$ factors as

$$\Omega X \wedge \Omega X \xrightarrow{\tilde{c}} \Omega^3 K_1 \rightarrow \Omega^2 E_1.$$

We have a commutative diagram

$$(3.4) \quad \begin{array}{ccc} & & \Omega^2 E_2 \\ & \nearrow^{c(\Omega f_2)} & \downarrow^{\Omega^2 p_2} \\ \Omega X \wedge \Omega X & \xrightarrow{c(\Omega f_1)} & \Omega^2 E_1 \end{array}$$

Now $c(\Omega f_1)$ is adjoint to

$$\begin{array}{ccccc} \Sigma \Omega X \wedge \Sigma \Omega X & \rightarrow & \Omega K_1 & \rightarrow & E_1, \\ & \searrow & \nearrow \tilde{D}_1 & & \\ & & X \wedge X & & \end{array}$$

hence $[c(\Omega f_1)] \in (PH^*(\Omega X) \otimes PH^*(\Omega X))^{2^k+2^n-4}$.

According to [6], there is an isomorphism of coalgebras

$$\text{Tor}_{H^*(X)}(Z_2, Z_2) \cong H^*(\Omega X).$$

It follows that $H^*(\Omega X)$ in degrees less than $2^k - 2$ is a divided polynomial coalgebra on primitive elements of degree 6. Therefore

$$(3.5) \quad [c\Omega f_1] \in PH^6(\Omega X) \otimes PH^{2^k-2}(\Omega X) + PH^{2^k-2}(\Omega X) \otimes PH^6(\Omega X).$$

Further, the indecomposables of $H^*(\Omega X)$ in degrees less than $2^k - 2$ are concentrated in degrees of the form $3 \cdot 2^r$. But if $k > 4$, no Steenrod operation on an element in one of these degrees can hit an indecomposable in degree $2^k - 2$, so u_{2^k-2} is not in the image of the Steenrod algebra.

An analysis of the Cartan formula [7] for secondary operations applied to diagrams 3.4 and 3.5 yields that $u_{2^k-2} = \psi(u_6)$, where ψ is a secondary operation defined on 6-dimensional primitives in the kernel of all Steenrod operations. We proceed to study all such operations. Note that ψ has degree $2^k - 8$. The possibilities come from the suspension elements in $H^{2^k-2}(G)$, where G is the space defined as follows. Let G' be defined to be the fiber of the horizontal map g' in the diagram

$$\begin{array}{ccc} G' & & \\ \downarrow & & \\ K(Z, 2^k - 1) & \xrightarrow[g']{\text{Sq}^2, \text{Sq}^4, \dots, \text{Sq}^{2^k-1}} & \Pi K(Z_2; 2^k - 1 + 2^n) \end{array}$$

Now set

$$G = \Omega^{2^k-7} G' \quad \text{and} \quad g = \Omega^{2^k-7} g'.$$

So G is fibered as $\pi: G \rightarrow K(Z, 6)$. We shall see that in $H^{2^k-2}(G)$, $\text{im}(\sigma^*) \subset \overline{A(2)} \cdot H^*(G)$. For, if an element ψ of $H^{2^k-2}(G)$ is a stable operation, then by [1] ψ can be expressed as a sum

$$\psi = \sum \alpha_{ij} v_{ij},$$

in which the v_{ij} represent the operations ψ_{ij} applied to $\pi^*(t_6)$. We note that none of the v_{ij} occurs in degree $2^k - 2$.

If $v \in H^{2^k-2}(G)$ represents an unstable operation, then it must be in the image of $(\sigma^*)^N$ but not in the image of $(\sigma^*)^{N+1}$, for some N . Write $v = (\sigma^*)^N[\hat{v}]$, $\hat{v} \in H^{2^k-2+N}(B^N G)$. Since \hat{v} is not a suspension, its a_m -obstruction [12] must be non-zero for some m . Such an obstruction must arise from having

$$i_{N+7}^m \in \text{Im}(B^{N+1}g)^*$$

for some m of the form $m = 2^r$.

If $r > 1$, then $i_{N+7}^m = \text{Sq}^{2^{r-1}(N+7)}\gamma i_{N+7}$, where

$$\gamma = \text{Sq}^{2^{r-2}(N+7)} \dots \text{Sq}^{N+7}.$$

If $r = 1$, then $N = 2^k - 14$, so that $i_{N+7}^2 = \text{Sq}^1\gamma i_{N+7}$, where $\gamma = \text{Sq}^{2^k-8}$. In either case there is a relation

$$\gamma = \sum \alpha_n \text{Sq}^{2^n}, \quad \alpha_n \in A(2),$$

so there exists an element $w \in H^{2^k-3}(G)$ that restricts to the fiber to be $\sum \alpha_n i_{2^n+5}$. Hence a representative of v is given by $\text{Sq}^1 w$ if $r = 1$ and by $\text{Sq}^{2^{r-1}(N+7)} w$ if $r > 1$.

Thus $\psi(u_6)$ must be in the image of the Steenrod operations. This implies that u_{2^k-2} lies in the image of Steenrod operations which is a contradiction. Since

$$\sigma^*: QH^{2^k-1}(X) \rightarrow PH^{2^k-2}(\Omega X)$$

is monic, we conclude that $QH^{2^k-1}(X) = 0$. □

Proof of the Main Theorem. We now know that $H^*(X)$ is an exterior algebra on seven-dimensional generators. If $H^*(X; Z)$ has odd torsion, then for some odd prime p , there is an even generator of the form $\beta_1 P^n x_{2n+1}$ by [9]. Applying the Bockstein spectral sequence, this yields an odd generator in the rational cohomology of degree $(2np + 2)p^d - 1$ for $d \geq 1$. But

$$(2np + 2)p^d - 1 > 7$$

so $H^*(X; Z)$ has no odd torsion. Hence it is torsion-free. Therefore

$$H^*(X; Z) \cong \Lambda(x_1, \dots, x_r)$$

where $\text{deg}(x_i) = 7$.

We now use the Hurewicz isomorphism to obtain our desired homotopy equivalence

$$S^7 \times \cdots \times S^7 \xrightarrow{f} X. \quad \square$$

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