MINIMAL ORBITS AT INFINITY IN HOMOGENEOUS SPACES OF NONPOSITIVE CURVATURE

María J. Druetta
Let $M$ denote a simply connected, homogeneous space of nonpositive curvature and let $G$ be the connected component of the identity of the isometry group of $M$.

In this paper we study the geometric consequences on $M$ if $M(\infty)$, the boundary sphere of $M$, admits a $G$-orbit whose closure is a minimal set for $G$. A characterization of symmetric spaces of noncompact type in terms of the action of $G$ in $M(\infty)$, is obtained. As an application we give some conditions, in terms of the Lie algebra of a simply transitive and solvable subgroup of $G$ that is in standard position, which are equivalent to the fact that $M$ is a symmetric space.

**Introduction.** Let $M$ denote a simply connected, homogeneous space of nonpositive curvature ($K \leq 0$) and let $G$ be the connected component of the identity in $I(M)$, the isometry group of $M$.

In this paper we study the geometric consequences on $M$ if $M(\infty)$, the boundary sphere of $M$, admits a $G$-orbit whose closure is a minimal set for $G$. In particular, we obtain a characterization of symmetric spaces of noncompact type in terms of the action of $G$ in $M(\infty)$. As an application, some conditions in terms of properties of the Lie algebra of a simply transitive, solvable subgroup of $G$ that is in standard position, which are equivalent to the fact that $M$ is a symmetric space, are obtained.

In §1 we give a characterization of symmetric spaces in terms of the $G$-minimality of the closure of some orbits of $G$ in $M(\infty)$, or equivalently in terms of $K$, the stability subgroup of $G$ at any point in $M$, we obtain that $M$ is a symmetric space of noncompact type if and only if $G(x) = K(x)$ for a particular $x$ in $M(\infty)$ (Theorem 1).

In §2 we get a decomposition of $\mathfrak{g}$, the Lie algebra of $G$, that coincides with the canonical one when $M$ is symmetric. It is used to give, as an application of Theorem 1, a characterization of symmetric spaces of noncompact type in terms of properties of the Lie algebra of a simply transitive, solvable group of isometries of $M$ that is in standard position (Theorem 2).
The author thanks the referee for his valuable comments, and in particular for shortening the proof of the main result. She is also grateful to IMPA (Rio de Janeiro, Brazil) for its support during her visit in the summer of 1990 when this paper was completed.

Preliminaries. Let $M$ be a complete, simply connected Riemannian manifold with nonpositive sectional curvature ($K \leq 0$). Let $I(M)$ and $I_0(M)$ denote the group of isometries of $M$ and the connected component of the identity respectively. All geodesics of $M$ will be assumed to have unit speed. Geodesics $\alpha$ and $\beta$ of $M$ are asymptotic if $d(\alpha(t), \beta(t)) \leq c$ for all $t \geq 0$ and some $c > 0$. $M(\infty)$ will denote the set of equivalence classes of asymptotic geodesics. $\overline{M} = M \cup M(\infty)$ equipped with the cone topology is a compactification of $M$ and $M(\infty)$, with the induced topology from $\overline{M}$, is homeomorphic to the $(n-1)$-sphere, where $n = \dim M$. For a geodesic $\gamma$ of $M$ we let $\gamma(\infty)$, $\gamma(-\infty)$ denote the asymptotic equivalence classes of $\gamma$ and $\gamma^{-1}(t \to \gamma(-t))$ respectively. Isometries of $M$ extend to homeomorphisms of $M(\infty)$ by defining $g(\gamma(\infty)) = (g \circ \gamma)(\infty)$. Moreover, the map $(g, x) \mapsto g(x)$ of $I(M) \times M(\infty)$ is continuous.

We say that distinct points $x$ and $y$ in $M(\infty)$ can be joined by a geodesic of $M$ if there exists a geodesic $\gamma$ of $M$ such that $\gamma(\infty) = x$ and $\gamma(-\infty) = y$. For each point $p$ in $M$ the geodesic symmetry $s_p : M \to M$ is defined by $s_p(\gamma(t)) = \gamma(-t)$ for all geodesics $\gamma$ of $M$ with $\gamma(0) = p$ and for all $t$ in $\mathbb{R}$. The map $s_p$ fixes $p$ and is a diffeomorphism of $M$ ($s_p = \exp_p \circ S \circ \exp_p^{-1}$, where $S(v) = -v$ for all $v$ in $T_p M$). Let $G^*$ denote the subgroup of diffeomorphisms of $M$ generated by the geodesic symmetries $\{s_p : p \in M\}$. It is called the symmetry diffeomorphism group of $M$. The group $G^*$ acts on $M(\infty)$ by homeomorphisms setting for each $p \in M$ and $x \in M(\infty)$, $s_p(x) = \gamma_{px}(-\infty)$ (where $\gamma_{px}$ denotes the unique geodesic such that $\gamma_{px}(0) = p$ and $\gamma_{px}(\infty) = x$).

Let $\Gamma$ denote any subgroup of $I(M)$. Two points $x$ and $y$ in $M(\infty)$, not necessarily distinct, are said to be $\Gamma$-dual if there exists a sequence $\{g_n\} \subset \Gamma$ such that $g_n(p) \to x$ and $g_n^{-1}(p) \to y$ as $n \to \infty$ for some (or any) point $p$ of $M$ . The set of points in $M(\infty)$ that are $\Gamma$-dual to a given point $x \in M(\infty)$ is closed in $M(\infty)$ and invariant under $\Gamma$. The limit set $L(\Gamma)$ is defined by $L(\Gamma) = \Gamma(p)^- \cap M(\infty)$ ($p \in M$) where $\Gamma(p)^-$ is the closure of the $\Gamma$-orbit of $p$ in $\overline{M}$. A closed subset $X \subseteq M(\infty)$ is said to be a minimal set for $\Gamma$ if $\Gamma(x)^-$ (the closure of the $\Gamma$-orbit of $x$ in $M(\infty)$) coincides with $X$ for every $x \in X$. 
Assume that $M$ is homogeneous. Then $M$ admits a solvable Lie group $S$ acting simply and transitively on $M$ (see [1, Proposition 2.5]). Let $\mathfrak{s}$ denote the Lie algebra of $S$. We know that $S$ with the left invariant metric associated to the $p$-inner product on $\mathfrak{s}$, which is induced by the action of $I(M)$ on $M$ ($g \to g(p)$, $p$ is any fixed point in $M$) is isometric to $M$. Moreover, $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}] \oplus \mathfrak{a}$ where $\mathfrak{a}$, the orthogonal complement of $[\mathfrak{s}, \mathfrak{s}]$ in $\mathfrak{s}$, is an abelian subalgebra of $\mathfrak{s}$ (see [1, Theorem 5.2]). For each $H \in \mathfrak{a}$, $\gamma_H(t) = \exp tH(p)$ is a geodesic of $M$ since $\exp tH$ is a geodesic of $S$ (see in [2, §3] the expression of the Riemannian connection associated to a left invariant metric). The connected Lie subgroup $A = \exp(\mathfrak{a})$ with Lie algebra $\mathfrak{a}$ is a flat totally geodesic submanifold of $S$.

Let $\lambda \in (\mathfrak{a}^e)^*$. $\lambda$ is said to be a root of $\mathfrak{a}$ in $\mathfrak{s}$ if $\mathfrak{a}_\lambda = \{U \in \mathfrak{s}^e: (\text{ad}_H - \lambda(H)I)^k U = 0 \text{ for some } k \geq 1 \text{ and all } H \in \mathfrak{a}\}$ is nonzero. Here, $\mathfrak{a}^e$ and $\mathfrak{s}^e$ denote the complexification of $\mathfrak{a}$ and $\mathfrak{s}$ respectively (see [2, §5]).

If $G = I_0(M)$ and $K$ is any maximal compact subgroup of $G$, by the maximality of $K$ and the Cartan fixed point theorem there exists a point $p \in M$ such that $K = G_p$, the stability subgroup of $G$ at $p$ ($G_p$ is compact by Theorem 2.5 (Ch.IV) of [8]). Hence for any $p \in M$, $G_p$ is a maximal compact subgroup of $G$ since the stability subgroups of $G$ are conjugate in $G$.

1. The orbits of $G = I_0(M)$ as minimal sets for $G$ in $M(\infty)$. Let $M$ be a simply connected, homogeneous space of nonpositive sectional curvature. In this section we give a characterization of symmetric spaces of noncompact type in terms of the $G$-minimality of the closure of some orbits of $G = I_0(M)$ in $M(\infty)$. For any $z \in M(\infty)$ let $G_z$ denote the subgroup of $G$ defined by $G_z = \{g \in G: g(z) = z\}$.

The proof of the following lemma can be found in [3, Lemma 2.4a]. We state it here because it will be used often.

**Lemma 1.1.** Let $\Gamma$ be any group of isometries of $M$. Let $x \in M(\infty)$ and let $\gamma$ be a geodesic in $M$ such that $x = \gamma(\infty)$. If $y = \gamma(-\infty)$ and $z$ is $\Gamma$-dual to $y$ then $z \in \Gamma(x)^\perp$.

**Proposition 1.2.** Let $\Gamma$ be a subgroup of $I(M)$ acting transitively on $M$. Assume that $\Gamma(y)^\perp$, the closure of the $\Gamma$-orbit of $y$ in $M(\infty)$, is a minimal set for $\Gamma$. If $x$ is a point in $M(\infty)$ which is joined to $y$ by a geodesic of $M$ then $x$ is $\Gamma$-dual to $y$. 
Proof. Let $\gamma$ be a geodesic of $M$ with end points $x = \gamma(\infty)$ and $y = \gamma(-\infty)$ and set $p = \gamma(0)$. Since $L(\Gamma) = M(\infty)$ ($\Gamma$ acts transitively on $M$) we can find a sequence $\{g_n\} \subset \Gamma$ such that $g_n(p) \to x$ as $n \to \infty$. Passing to a subsequence if necessary, $g_n^{-1}(p)$ converges to a point $z \in M(\infty)$ as $n \to \infty$. By Lemma 1.1, $z \in \Gamma(y)$ since $z$ is $\Gamma$-dual to $x$ and $y$ is joined to $x$ by $\gamma$. By hypothesis, $\Gamma(z) = \Gamma(y)$ and hence $y \in \Gamma(z)$. We note that $\Gamma(z)$ is contained in the set of points which are $\Gamma$-dual to $x$ since this set is closed, invariant under $\Gamma$ and $z$ is $\Gamma$-dual to $x$. Thus, $y$ is $\Gamma$-dual to $x$ or $x$ is $\Gamma$-dual to $y$.

THEOREM 1. Let $M$ be an irreducible, simply connected and nonflat homogeneous space of nonpositive sectional curvature. Set $G = I_0(M)$. Let $x \in M(\infty)$ be a point such that $G_x$ acts transitively on $M$. If $y \in M(\infty)$ is a point that can be joined to $x$ by a geodesic of $M$, then the following properties are equivalent:

1. $G(y)$ is a minimal set for $G$ in $M(\infty)$.
2. $G(y) = K(y)$ for any maximal compact subgroup $K$ of $G$.
3. $G(y)$ is a closed subset of $M(\infty)$.
4. $M$ is a symmetric space of noncompact type.
5. $G_y$ acts transitively on $M$.

REMARK. If $M$ is a simply connected, homogeneous space of nonpositive curvature then $M$ admits a simply transitive, solvable group $S$ of isometries that has a fixed point in $M(\infty)$ by Theorem 3.4 of [5] ($M$ has no flat de Rham factor). Moreover, if $S$ is a transitive group of isometries of $M$ that does not have a fixed point in $M(\infty)$, then $M$ must be symmetric of noncompact type by [7, Proposition 4.4.7].

Proof of Theorem 1. (1) $\Rightarrow$ (2) Let $K \subseteq G$ be any maximal compact subgroup. Then there exists a point $p \in M$ such that $K = G_p$, and hence $G = K \cdot G_x$ since $G_x$ acts transitively on $M$. Let $y \in M(\infty)$ be a point that can be joined to $x$ by a geodesic of $M$. Then,

(i) $G(x) = K(x)$.

Let $p \in M$ be the point above, and let $x^* = s_p(y) = \gamma_{py}(-\infty)$. Then $x^*$ is $G$-dual to $y$ by Proposition 1.2 and the fact that $G(y)$ is a minimal set for $G$ in $M(\infty)$. Hence $x^* \in G(x) = K(x)$ by (i) and Lemma 1.1, and we obtain

(ii) $x = k^*(x^*)$ for some $k^* \in K$.

Let $g \in G$ be given, and let $z = s_p(g(y)) = \gamma_{pg(y)}(-\infty)$. Then $y$ can be joined to $g^{-1}(z)$ by a geodesic of $M$, and hence $y$ and $g^{-1}(z)$ are $G$-dual by Proposition 1.2. Therefore $y$ and $z$ are $G$-dual, and
it follows that \( z \in G(x)^- = K(x) \) by (i) and Lemma 1.1. From (ii) we obtain

(iii) \( z = k(x^*) \) for some \( k \in K \).

Finally, \( y = \gamma_{p x^*}(-\infty) \) and therefore \( k(y) = \gamma_{k(p) k(x^*)}(-\infty) = \gamma_{p x}(-\infty) = g(y) \) by (iii) and the definitions of \( p \) and \( z \). Hence \( G(y) = K(y) \).

(2) \( \Rightarrow \) (3) This is obvious since \( K \) is compact.

(3) \( \Rightarrow \) (4) We set \( K = G_p \) where \( p = \gamma(0) \) and \( \gamma \) is the geodesic in \( M \) such that \( \gamma(-\infty) = y \) and \( \gamma(\infty) = x \). If \( G(y) \) is closed then \( G(y)^- = G(y) \) is a minimal set for \( G \) in \( M(\infty) \), and hence \( G(y) = K(y) \) by (1) \( \Rightarrow \) (2) since \( K \) is a maximal compact subgroup of \( G \).

If \( X = K(x) \cup K(y) \) then \( X = G(x) \cup G(y) \) is a closed, \( G \)-invariant subset of \( M(\infty) \). It then follows that \( X \) is invariant under the symmetry diffeomorphism group \( G^* \) since \( s_p(k(x)) = k(y) \), \( s_p(k(y)) = k(x) \) for any \( k \in K \) (\( k \circ \gamma \) joins \( k(x) \) and \( k(y) \) through \( p \)) and \( s_{g(p)} = g \circ s_p \circ g^{-1} \) (\( M = G(p) \)).

Suppose that \( X = M(\infty) \). Since \( M(\infty) \) is homeomorphic to the \((n - 1)\)-sphere, it follows from Baire’s Theorem that \( G(x) \) (or \( G(y) \)) has interior nonempty. Then \( G(x) \) (or \( G(y) \)) is an open set in \( M(\infty) \) which is also closed, and consequently \( G(x) = M(\infty) \). In this case, by applying Proposition 4.12 of [3], \( M \) is a symmetric space of rank one.

If \( X \subsetneq M(\infty) \), it follows from Theorem 3.2 of [6] that \( M \) is a symmetric space of noncompact type of rank \( \geq 2 \) since it is irreducible.

We remark that in the proof above we only needed a geodesic \( \gamma \) of \( M \) satisfying \( \gamma(0) = p \) and \( G(\gamma(\pm \infty)) = K(\gamma(\pm \infty)) \).

(4) \( \Rightarrow \) (5) Note that if \( K = G_p \ (p \in M) \), \( G(y) = K(y) \) by Theorem 4.5 of [3], and it follows immediately that \( G = K \cdot G_y = G_y \cdot K \). Hence \( M = G(p) = G_y(p) \) since \( G \) acts transitively on \( M \).

(5) \( \Rightarrow \) (1) If \( G_y \) acts transitively on \( M \) we have that \( G = K \cdot G_y = G_y \cdot K \), where \( K = G_p \ (p \in M) \). Thus \( G(y) = K(y) \) is a closed subset of \( M(\infty) \), and hence \( G(y)^- = G(y) \) is a minimal set for \( G \) in \( M(\infty) \).

This completes the proof of Theorem 1.

Note that Theorem 5.4 of [3] and Proposition 4.7.1 of [7] show that if \( M \) is simply connected and homogeneous with sectional curvature \( K \leq 0 \), then \( M \) is symmetric of noncompact type if and only if \( G(y)^- \) is a minimal set for \( G \) for every \( y \in M(\infty) \). Thus, by the remark above, Theorem 1 gives us a strengthened version of this result.
2. A canonical decomposition of the Lie algebra of $I_0(M)$. Let $M$ be a simply connected, homogeneous space of nonpositive sectional curvature. We assume that $M$ has no flat de Rham factor. We denote by $B$ the Killing form on $\mathfrak{g}$, the Lie algebra of $G = I_0(M)$.

In this section, a decomposition of $\mathfrak{g}$ that coincides with the canonical one when $M$ is symmetric of noncompact type, is obtained. As an application of Theorem 1, we get some algebraic conditions in terms of $\mathfrak{g}$ and the data $\mathfrak{o}$, the Lie algebra of a subgroup $S$ of $G$ that acts simply transitively on $M$ and is in standard position, in order to ensure that $M$ is a symmetric space of noncompact type.

A closed subgroup $S$ of $G$ is said to be in standard position if

(i) $S$ acts simply transitively on $M$.

(ii) For some point $p \in M$, $B(H, U) = 0$ for all $H \in \mathfrak{a}$ and $U \in \mathfrak{k}$, where $\mathfrak{a}$ is the orthogonal complement of $[\mathfrak{o}, \mathfrak{o}]$ relative to the $p$-inner product on $\mathfrak{o}$ and $\mathfrak{k}$ is the Lie algebra of $K$, the stability subgroup of $G$ at $p$.

We remark on the following facts about groups that are in standard position:

(1) If $B(\mathfrak{a}, \mathfrak{k}) = 0$ for one point $p \in M$ then $B(\mathfrak{a}, \mathfrak{k}) = 0$ for every $p \in M$.

(2) There is a simply transitive, solvable group of isometries of $M$ that is in standard position. If a simply transitive, solvable group $S$ is in standard position, then $gSg^{-1}$ is also in standard position for any $g \in G$.

(3) If $S_1$ and $S_2$ are two simply transitive, solvable groups of isometries on $M$ in standard position, then they are conjugate by an element of $G$.

(4) If $M$ is a symmetric space of noncompact type and $G = K \cdot A \cdot N$ is an Iwasawa decomposition of $G$, then $S = A \cdot N$ is a simply transitive, solvable group of isometries of $M$ in standard position.

We refer the reader to §6 (pages 45–57) of [2] for a more complete discussion. The definition and facts mentioned above are explicitly stated there (6.4, 6.5-(a), 6.5-(c), Theorem 6.7 and Corollary 6.10).

Let $S$ be a solvable Lie subgroup of $G$ that acts simply-transitively on $M$ and is in standard position. Let $K$ be the stability subgroup of $G$ at $p$, a point in $M$ chosen arbitrarily, and let $\mathfrak{p} = \{X_t \in \mathfrak{g} : B(X_t, U) = 0 \text{ for every } U \in \mathfrak{k}\}$.

**Proposition 2.1.** $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a direct sum decomposition of $\mathfrak{g}$ such that $\text{Ad}(k)(\mathfrak{p}) \subseteq \mathfrak{p}$. Moreover, $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$.
Proof. We first show that $B$ restricted to $\mathfrak{h} \times \mathfrak{h}$ is negative definite. Although the proof of this fact is the same as that in the symmetric case, we include it for the sake of completeness.

Since $K$ is compact and acts on $\mathfrak{g}$ by the adjoint representation $\text{Ad}(K) \subset \text{Gl}(\mathfrak{g})$, $\mathfrak{g}$ admits an inner product $(\ , \ )$ such that $\text{Ad}(k)$ are isometries for all $k \in K$. Thus, $\text{ad}_X$ is skew symmetric with respect to $(\ , \ )$ for every $X \in \mathfrak{h}$. Let $\{X_i\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to $(\ , \ )$. For $X \in \mathfrak{g}$,

$$B(X, X) = \text{tr}(\text{ad}_X \circ \text{ad}_X) = \sum_i (\text{ad}_X^2 X_i, X_i)$$

and the equality holds if and only if $X \in \mathfrak{z}(\mathfrak{g})$, the center of $\mathfrak{g}$. By Theorem 2.1 and Proposition 2.3 of [3], $\mathfrak{z}(\mathfrak{g}) = 0$ since it is the Lie algebra of the center of $G$. Thus, $B|_{\mathfrak{h} \times \mathfrak{h}}$ is negative definite.

Next we will prove the proposition. It is clear that $\mathfrak{p}$ is a subspace of $\mathfrak{g}$ which is $\text{Ad}(K)$ invariant since $\mathfrak{h}$ and $B$ are both invariant under $\text{Ad}(K)$. From the assertion above, we have that $\mathfrak{h} \cap \mathfrak{p} = 0$. It remains to show that $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$. Let $\{X_i\}$ be a basis for $\mathfrak{h}$ so that $B(X_i, X_j) = -\delta_{ij}$ ($B|_{\mathfrak{h} \times \mathfrak{h}}$ is negative definite). If $X \in \mathfrak{g}$, we set $Y = X - \sum_i B(X, X_i)/B(X_i, X_i)X_i$. $Y \in \mathfrak{p}$, and hence $X = \sum_i B(X, X_i)/B(X_i, X_i)X_i + Y \in \mathfrak{h} + \mathfrak{p}$.

Since $B(a, \mathfrak{h}) = 0$, we have that $a \subset \mathfrak{p}$. The last assertion follows from Lemma 2.2 below since $C_\mathfrak{p}(a)$, the centralizer of $a$ in $\mathfrak{p}$, is $a$.

**Lemma 2.2.** $C_\mathfrak{p}(a) = \{X \in \mathfrak{p} : [X, H] = 0 \text{ for all } H \in a\} = a$.

Proof. Let $H$ be an element in $a$ satisfying $\alpha(H) > 0$ for all $\alpha \in a^*$ such that $\alpha + i\beta$ is a root of $a$ in $\mathfrak{g}' = [\mathfrak{h}, \mathfrak{h}]$. Such an $H$ exists since $M$ has no flat de Rham factor (see [1, Proposition 5.6]).

Let $X$ be a unit vector in $\mathfrak{p}$ such that $[X, H] = 0$. If $X(t)$ is the variation vector field $\partial f/\partial s (0, t)$ on $\gamma_H(t) = \exp tH(p)$, where $f: \mathbb{R} \times \mathbb{R} \to M$ is the geodesic variation of $\gamma_H$ given by $f(s, t) = \exp sX \exp tH(p)$, then $X$ is a Jacobi vector field on $\gamma_H$ with $X(0) = d\phi eX$ ($\phi: G \to M$ is defined by $\phi(g) = g(p)$). Moreover, $f(s, t) = (\exp tH \exp sX)(p)$ since $[H, X] = 0$. Therefore, $X(t) = d(\exp tH)p(d\phi eX)$ and $|X(t)| = |d\phi eX|$. Since $X$ is a Jacobi vector field on $\gamma_H$, it follows that it is also parallel on $\gamma_H$. 

**MINIMAL ORBITS AT INFINITY** 293
(the convex function $g(t) = |X(t)|^2$ is constant and hence, $g''(t) = 
abla_{\gamma_H} X|_{\gamma_H(t)}^2 - K(\gamma'_H(t), X(t)) = 0$). Hence, $X$ induces a parallel Jacobi vector field $J$ on the geodesic $exp tH$ in $S$ such that $p(J(0)) = X$, where $p$ denotes the projection from $\mathfrak{g}$ onto $\mathfrak{p}$ associated to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. By the same argument as in the proof of Theorem 1.3 of [4], it follows that $J(0) \in a$ and hence, $X \in a$ since $a \subset \mathfrak{p}$.

We observe that we have actually shown that $C_\mathfrak{p}(H)$, the centralizer of $H$ in $\mathfrak{p}$, is $a$. Here, $H$ is chosen as in the beginning of the proof of Lemma 2.2.

**Theorem 2.** Let $M$ be a simply connected, homogeneous space of nonpositive curvature with no flat de Rham factor. Let $\mathfrak{g}$ be the Lie algebra of $G = \text{I}_0(M)$ and let $S$ be a subgroup of $G$ that acts simply transitively on $M$ and is in standard position. Let $\mathfrak{s}$ and $\mathfrak{k}$ be the Lie algebra of $S$ and $K$ respectively, where $K$ is the stability subgroup of $G$ at a point $p$ in $M$ chosen arbitrarily. If $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form $B$ on $\mathfrak{g}$ and $\mathfrak{a}$ is the orthogonal complement of $[\mathfrak{s}, \mathfrak{s}]$ in $\mathfrak{s}$, relative to the inner product on $\mathfrak{s}$ induced by $p$, then the following properties are equivalent.

1. $[\mathfrak{a}, \mathfrak{p}] \subset \mathfrak{k}$.
2. $\mathfrak{p} = \bigcup \{\text{Ad}(k)(a): k \in K\}$.
3. The geodesics through the point $p$ are orbits $\text{exp } tX(p)$ for every $X \in \mathfrak{p}$.
4. $M$ is a symmetric space of noncompact type.

**Proof.** $(1) \Rightarrow (2)$ (See [8, Lemma 6.3 (iii), Ch. V].) Let $H$ be an element in $\mathfrak{a}$ such that $C_\mathfrak{p}(H) = a$ (see the remark at the end of Lemma 2.2). Let $X \in \mathfrak{p}$ be fixed and let $f: K \rightarrow \mathbb{R}$ be the map defined by $f(k) = B(H, \text{Ad}(k)X)$. We will show that $\text{Ad}(k_0)X \in a$ whenever $k_0$ is a critical point of $f$. In fact, for such a $k_0$ (it exists since $K$ is compact) and any $U \in \mathfrak{k}$ the function of $t \in \mathbb{R}$, $f_U(t) = f(\exp tUk_0)$ has a critical point at $t = 0$. Hence, $0 = f'_U(t) = B(H, [U, \text{Ad}(k_0)X]) = B([H, U], \text{Ad}(k_0)X)$

$= -B(U, [H, \text{Ad}(k_0)X])$

(for any $Z \in \mathfrak{g}$, $\text{ad}_Z$ is skew symmetric relative to $B$). Note that $[H, \text{Ad}(k_0)X] \in \mathfrak{k}$ since $[\mathfrak{a}, \mathfrak{p}] \subset \mathfrak{k}$ and $\mathfrak{p}$ is $\text{Ad}(K)$-invariant. Moreover, the result above is true for all $U \in \mathfrak{k}$. Now, from the fact that $B$ is negative definite on $\mathfrak{k}$, it follows that $[H, \text{Ad}(k_0)X] = 0$. Hence, $\text{Ad}(k_0)X \in a$ or $X \in \text{Ad}(k_0^{-1})(a)$. 


(2) \(\Rightarrow\) (3) Note that under our hypothesis \(K \neq \text{id};\) otherwise, \(a = \varphi\) and \(M\) is Euclidean \((K(X, Y) = 0 \text{ for all } X \text{ and } Y \in a).\) Given \(X \in \rho\) choose \(k \in K\) and \(H \in a\) so that \(X = \text{Ad}(k)H.\) Then \(\exp tH(p)\) is a geodesic of \(M\) and hence so is \(k\gamma(t) = k(\exp tHk^{-1})(p) = \exp tX(p).\)

(3) \(\Rightarrow\) (4) Assume first that \(M\) is irreducible. Let \(\gamma_H\) be the geodesic of \(M\) defined by \(\gamma_H(t) = \exp tH(p).\) We choose \(H\) a unit vector in \(a\) such that \(x = \gamma_H(\infty)\) is a fixed point of \(S\) (see Theorem 3.4 of [5]) and we will show that if \(y = \gamma_H(-\infty)\) then \(G(y) = K(y)\) for \(K = G_p.\) It will then follow from Theorem 1 that \(M\) is a symmetric space of noncompact type since \(G(y)\) is closed in \(M(\infty).\)

Let \(g\) be any element in \(G\) and set \(g_n = \exp nHg^{-1}.\) Since \(y = \lim \exp -nH(p)\) as \(n \to \infty,\) we have that \(g(y) = \lim g \exp -nH(p) = \lim g_n^{-1}(p)\) as \(n \to \infty.\) Suppose that \(g_n(p) = \exp t_nX_n(p)\) with \(X_n\) a unit vector in \(\rho.\) Therefore, there exists \(\{k_n\} \subset K\) so that \(g_n^{-1}\exp t_nX_n = k_n\) and \(g_n^{-1} = k_n \exp -t_nX_n.\) By assuming that \(k_n \to k,\) choosing a subsequence if necessary, we get \(g(y) = \lim g_n^{-1}(p) = k(y)\) as \(n \to \infty\) since \(X_n \to H (g_n(p) \to x).\)

In the general case, assume that \(M = M_1 \times M_2\) where \(M_1\) and \(M_2\) are irreducible. Since \(G = G_1 \times G_2\) (direct product) with \(G_i = I_0(M_i),\) if \(p = (p_1, p_2)\) and \(K_i\) is the stability subgroup of \(G_i\) at \(p_i\) \((i = 1, 2),\) we have that \(K = K_1 \times K_2\) and hence \(\mathfrak{h} = \mathfrak{k}_1 \oplus \mathfrak{k}_2,\) where \(\mathfrak{h}_i\) is the Lie algebra of \(K_i\) \((i = 1, 2).\) Thus, if \(\rho_i\) is the orthogonal complement of \(\mathfrak{h}_i\) with respect to the Killing form \(B_i\) on \(\mathfrak{g}_i,\) the Lie algebra of \(G_i,\) it follows that \(\rho = \rho_1 \oplus \rho_2\) since \(B = B_1 \oplus B_2\) (note that \(\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2\) is a direct sum of ideals). Then the geodesics through the points \(p_i\) are orbits \(\exp tX_i\) with \(X_i \in \rho_i\) for \(i = 1, 2,\) and hence \(M_i\) is a symmetric space of noncompact type. Therefore \(M\) is symmetric.

(4) \(\Rightarrow\) (1) We note that \(\mathfrak{g}\) is semisimple \((M\) has no flat de Rham factor\) and \(\mathfrak{g} = \mathfrak{k} \oplus \rho\) is the canonical decomposition of \(\mathfrak{g}\) associated to \(M = G/K.\) Hence, \([\rho, \rho] \subseteq \mathfrak{k}\) and (1) follows since \(a \subset \rho.\)

References


Received January 1, 1991 and in revised form August 14, 1991.
Surfaces in the 3-dimensional Lorentz-Minkowski space satisfying $\Delta x = Ax + B$

Luis Alías, Angel Ferrandez and Pascual Lucas

Lie algebras of type $D_4$ over number fields

Bruce Allison

Subsemigroups of completely simple semigroups

Anne Antonippillai and Francis Pastijn

Studying links via closed braids. VI. A nonfiniteness theorem

Joan Birman and William W. Menasco

Minimal orbits at infinity in homogeneous spaces of nonpositive curvature

María J. Druetta

Generalized horseshoe maps and inverse limits

Sarah Elizabeth Holte

Determinantal criteria for transversality of morphisms

Dan Laksov and Robert Speiser

Four dodecahedral spaces

Peter Lorimer

Semifree actions on spheres

Monica Nicolau

Conformal deformations preserving the Gauss map

Enaldo Vergasta

Hecke eigenforms and representation numbers of arbitrary rank lattices

Lynne Walling