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CONFORMAL DEFORMATIONS PRESERVING THE GAUSS
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ENALDO VERGASTA

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CONFORMAL DEFORMATIONS PRESERVING THE GAUSS MAP

ENALDO SILVA VERGASTA

In this work, given a conformal immersion $f: M^n \rightarrow \mathbb{R}^N$ of a Riemannian manifold M^n into a euclidean space \mathbb{R}^N , we establish conditions for the existence of another conformal immersion $\bar{f}: M^n \rightarrow \mathbb{R}^N$ with the same Gauss map as f . In particular, for $n = 2$ and $N = 3$, these conditions are described by means of a partial differential equation on the principal curvatures of f .

0. Introduction. Let M^n be a connected n -dimensional Riemannian manifold and let $f: M^n \rightarrow \mathbb{R}^N$ be a conformal immersion. We denote by $F: M^n \rightarrow G_{n,N}$ the *Gauss map* of f , which assigns to each point $p \in M^n$ the n -dimensional tangent space $f_*(T_p M)$ in the Grassmannian $G_{n,N}$. We consider the following problem: Under what conditions does there exist another conformal immersion $\bar{f}: M^n \rightarrow \mathbb{R}^N$ such that the Gauss map of \bar{f} coincides with the Gauss map of f , up to a congruence in $G_{n,N}$ induced by a congruence in \mathbb{R}^N ? When this occurs we say that \bar{f} is a *G-deformation* of f . This situation is equivalent to considering conformal immersions f and \bar{f} with parallel tangent spaces $f_*(T_p M)$ and $\bar{f}_*(T_p M)$ in \mathbb{R}^N , which we will always assume. The analogous problem for isometric immersions f and \bar{f} was considered by Dajczer and Gromoll [D&G].

In §1 we characterize our situation by means of a tensor field and a differentiable function satisfying certain conditions (see Proposition 1.5). This result will be used in §2, where we treat the above problem for $n = 2$.

For surfaces, we also consider the *oriented Gauss map* $F^*: M^2 \rightarrow G_{2,N}^*$, where now $f_*(T_p M)$ is seen as an oriented 2-plane in the oriented Grassmannian $G_{2,N}^*$. In regard to the above problem we have two different situations. The first one is when f and \bar{f} have the same oriented Gauss map. In this case, it was shown by Hoffman and Osserman [H&O-2] that either f and \bar{f} are minimal surfaces or \bar{f} coincides with f up to homothety and translation in \mathbb{R}^N . The other situation is when, for any local orientation in M^2 , the oriented Gauss maps of f and \bar{f} differ by the orientation-reversing congruence in $G_{2,N}^*$. In this case we call \bar{f} a *G*-deformation* and say that

f is G^* -deformable. If f is not totally umbilic, we show that a G^* -deformation is unique up to homothety and translation (Theorem 2.1). When $N = 4$, we also prove that G^* -deformable immersions must have flat normal bundle. For $N = 3$, Theorem 2.4 characterizes G^* -deformable immersions by means of a condition on their principal curvatures. We apply Theorem 2.4 to obtain \bar{f} when f is a rotation surface, a cyclid of Dupin or a surface with constant mean curvature. A similar result is obtained for constant mean curvature surfaces in the euclidean sphere S^3 .

For hypersurfaces in \mathbb{R}^{n+1} , $n \geq 3$, the problem considered here will be treated in a forthcoming paper. Most of the results contained in these two works were announced in [Ve] and were obtained in my doctoral thesis. I wish to express my deep gratitude to Professor M. Dajczer for valuable advice and constant encouragement. I also thank the referee for many helpful suggestions.

1. Conformal deformation in \mathbb{R}^N preserving the Gauss map. Let us denote by $\langle \cdot, \cdot \rangle_0$ the Riemannian metric on M^n and by A_ξ the second fundamental form of the conformal immersion $f: M^n \rightarrow \mathbb{R}^N$ in the normal direction ξ , defined by

$$(1.1) \quad f_* A_\xi X = -(\tilde{\nabla}_X \xi)^t,$$

where $(\cdot)^t$ denotes the tangent projection along f and $\tilde{\nabla}$ is the Levi-Civita connection of the canonical metric $\langle \cdot, \cdot \rangle$ on the euclidean space \mathbb{R}^N . We denote also by $\langle \cdot, \cdot \rangle$ the metric on M^n induced by f , defined by $\langle \cdot, \cdot \rangle = e^{2\varphi_1} \langle \cdot, \cdot \rangle_0$, where $e^{2\varphi_1}: M^n \rightarrow \mathbb{R}$ is the conformal factor of f .

Let $\bar{f}: M^n \rightarrow \mathbb{R}^N$ be a G -deformation of f with conformal factor $e^{2\varphi_2}$. We define an orthogonal tensor field $T: TM \rightarrow TM$ by

$$(1.2) \quad T = e^{-\varphi} f_*^{-1} \circ P \circ \bar{f}_*,$$

where $\varphi = \varphi_2 - \varphi_1$ and, for each $q \in M^n$,

$$(1.3) \quad P_q: T_{\bar{f}(q)} \mathbb{R}^N \rightarrow T_{f(q)} \mathbb{R}^N$$

denotes the parallel transport in \mathbb{R}^N . For any vector field V along \bar{f} we have

$$(1.4) \quad \tilde{\nabla}_X P V = P \tilde{\nabla}_X V,$$

where X is any tangent field on M^n . We denote by ∇ the Levi-Civita connection on M^n relative to the metric $\langle \cdot, \cdot \rangle$ and by $\nabla \varphi$ the gradient of φ with respect to this metric. The following result gives

necessary and sufficient conditions on T and φ for the existence (at least locally) of a G -deformation of f .

PROPOSITION 1.5. *Let $f: M^n \rightarrow \mathbb{R}^N$ be a conformal immersion.*

(i) *If \bar{f} is a G -deformation of f , then*

$$(1.6) \quad \nabla_X T = T \circ (X \wedge \nabla \varphi)$$

and

$$(1.7) \quad A_\xi \circ T = T^{-1} \circ A_\xi$$

for any tangent field X and normal field ξ . Moreover the second fundamental form \bar{A} of \bar{f} is given by

$$(1.8) \quad \bar{A}_\xi = e^{-\varphi} T^{-1} \circ A_{P\xi}.$$

(ii) *If M^n is simply connected and there exist an orthogonal tensor field T and a differentiable function φ satisfying (1.6) and (1.7), then for any $q_0 \in M^n$,*

$$\bar{f}(q) = \int_{q_0}^q e^\varphi f_* T$$

defines a G -deformation of f .

Proof. We will make use of the Gauss formula

$$(1.9) \quad \tilde{\nabla}_X f_* Y = f_* \nabla_X Y + \alpha(X, Y),$$

where $X, Y \in TM$ and $\alpha(X, Y)$ denotes the normal component of $\tilde{\nabla}_X f_* Y$. Recall the relationship between α and A_ξ , given by

$$\langle \alpha(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

The Levi-Civita connection $\bar{\nabla}$ of the metric on M^n induced by \bar{f} (see [Ku], p. 316) is given by

$$\bar{\nabla}_X Y = \nabla_X Y + X(\varphi)Y + Y(\varphi)X - \langle X, Y \rangle \nabla \varphi.$$

Thus we can write

$$(1.10) \quad (\tilde{\nabla}_X \bar{f}_* Y)^t = \bar{f}_* (\nabla_X Y + X(\varphi)Y + Y(\varphi)X - \langle X, Y \rangle \nabla \varphi).$$

From (1.4), (1.9) and (1.10) we obtain

$$\begin{aligned} f_* \nabla_X T Y &= f_* (-X(\varphi) e^{-\varphi} f_*^{-1} P \bar{f}_* Y + e^{-\varphi} \nabla_X f_*^{-1} P \bar{f}_* Y) \\ &= -X(\varphi) e^{-\varphi} P \bar{f}_* Y + e^{-\varphi} (\tilde{\nabla}_X P \bar{f}_* Y)^t \\ &= -X(\varphi) e^{-\varphi} P \bar{f}_* Y \\ &\quad + e^{-\varphi} P \bar{f}_* (\nabla_X Y + X(\varphi)Y + Y(\varphi)X - \langle X, Y \rangle \nabla \varphi) \\ &= f_* (T \nabla_X Y + Y(\varphi) T X - \langle X, Y \rangle T \nabla \varphi), \end{aligned}$$

and this proves (1.6).

If ξ is a vector field normal to f , by (1.1) and (1.4) we have

$$\bar{f}_* \bar{A}_\xi X = -(\tilde{\nabla}_X \xi)^t = -(P^{-1} \tilde{\nabla}_X P \xi)^t = P^{-1} f_* A_{P\xi} X.$$

Thus $\bar{A}_\xi = e^{-\varphi} T^{-1} A_{P\xi}$. Now (1.7) follows from the fact that \bar{A}_ξ and $A_{P\xi}$ are self-adjoint.

In order to prove (ii), we compute the exterior differential of the 1-form $e^\varphi f_* T$ defined on M^n with values in \mathbb{R}^N :

$$\begin{aligned} d(e^\varphi f_* T)(X, Y) &= \tilde{\nabla}_X e^\varphi f_* T Y - \tilde{\nabla}_Y e^\varphi f_* T X - e^\varphi f_* T([X, Y]) \\ &= X(\varphi) e^\varphi f_* T Y + e^\varphi \tilde{\nabla}_X f_* T Y - Y(\varphi) e^\varphi f_* T X \\ &\quad - e^\varphi \tilde{\nabla}_Y f_* T X - e^\varphi f_* T(\nabla_X Y - \nabla_Y X). \end{aligned}$$

Now we use (1.9) to get

$$\begin{aligned} d(e^\varphi f_* T)(X, Y) &= e^\varphi (\alpha(X, T Y) - \alpha(Y, T X)) \\ &\quad + e^\varphi f_* (\nabla_X T Y - T \nabla_X Y - \nabla_Y T X + T \nabla_Y X \\ &\quad + X(\varphi) T Y - Y(\varphi) T X). \end{aligned}$$

By (1.6) the above equality becomes

$$d(e^\varphi f_* T)(X, Y) = e^\varphi (\alpha(X, T Y) - \alpha(Y, T X)).$$

But, for each vector ξ normal to f we have

$$\begin{aligned} \langle \alpha(X, T Y) - \alpha(Y, T X), \xi \rangle \\ = \langle A_\xi X, T Y \rangle - \langle A_\xi T X, Y \rangle = \langle (T^{-1} A_\xi - A_\xi T) X, Y \rangle \end{aligned}$$

and this vanishes by (1.7). Thus $e^\varphi f_* T$ is a closed 1-form on M^n . Since M^n is simply connected, we can define $\bar{f}: M^n \rightarrow \mathbb{R}^N$ by

$$\bar{f}(q) = \int_{q_0}^q e^\varphi f_* T.$$

Then $\bar{f}_* = e^\varphi f_* T$ and $\langle \bar{f}_* X, \bar{f}_* Y \rangle = e^{2\varphi} \langle X, Y \rangle$. So \bar{f} is a G -deformation of f . □

REMARK 1.11. As an immediate consequence of (1.6), we see that φ is constant along M^n if and only if T is a parallel tensor field with respect to the metric $\langle \cdot, \cdot \rangle$. When this occurs, f and \bar{f} induce the same metric on M^n , up to a constant factor. Thus, in this case the problem considered here is equivalent to considering isometric immersions f and \bar{f} with the same Gauss map. This was done by Dajczer and Gromoll in [D&G], where the orthogonal tensor field (1.2) becomes $T = f_*^{-1} P \bar{f}_*$, is parallel and satisfies (1.7).

2. Conformal deformations of surfaces preserving the Gauss map. In this section we study conformal surfaces in \mathbb{R}^N that are G^* -deformable. In this case, the above tensor field T must satisfy the additional condition $\det T = -1$ on M^2 . We also obtain a result for surfaces with constant mean curvature in the euclidean sphere S^3 . We begin with a uniqueness result.

THEOREM 2.1. *Let $f: M^2 \rightarrow \mathbb{R}^N$ be a conformal immersion which is not totally umbilic. If there exists a G^* -deformation \bar{f} , then \bar{f} is unique up to homothety and translation in \mathbb{R}^N .*

Proof. Let \bar{M}^2 denote M^2 with the opposite orientation. Denote the Gauss maps of $f: M^2 \rightarrow \mathbb{R}^N$ and $\bar{f}: \bar{M}^2 \rightarrow \mathbb{R}^N$ by F and \bar{F} , respectively. Then $F = \bar{F}$ as maps of M^2 (without orientation) into $G_{2,N}^*$. Now apply Theorem 1.1 of [H&O-1] and the basic uniqueness result in [H&O-2]. □

Using some results of [We-1] and [We-2] we prove the following two theorems.

THEOREM 2.2. *Let $f: M^2 \rightarrow \mathbb{R}^4$ be a G^* -deformable conformal immersion. Then the normal bundle of f is flat.*

Proof. We may assume the Gauss map $F: M^2 \rightarrow G_{2,4}^*$ is an immersion since the curvature of the normal bundle is zero anywhere F fails to be regular. Then as in the proof of the previous theorem $F: M^2 \rightarrow G_{2,4}^*$ and $\bar{F}: \bar{M}^2 \rightarrow G_{2,4}^*$ are equal. Using Corollary 3 on p. 464 of [We-2], and the notation there, the existence of $f: M^2 \rightarrow \mathbb{R}^4$ implies

$$\varepsilon_1(g) + \rho_1(g) = \varepsilon_2(g) + \rho_2(g);$$

from the existence of $\bar{f}: \bar{M}^2 \rightarrow \mathbb{R}^4$, it follows that

$$\varepsilon_1(g) - \rho_1(g) = \varepsilon_2(g) - \rho_2(g),$$

where g is the metric induced on M^2 by f . Thus $\varepsilon_1(g) = \varepsilon_2(g)$ and by Corollary 2 on p. 464 of [We-2] it follows that the normal bundle is flat. □

THEOREM 2.3. *Let $f: M^2 \rightarrow \mathbb{R}^N$ with $N \geq 5$ be a conformal immersion. If there exists a point of M^2 which is not an inflection point of f , then f is not G^* -deformable.*

Proof. This follows immediately from Proposition 5 of [We-1] and the observation that if \bar{f} existed then as above $f: M^2 \rightarrow \mathbb{R}^N$ and $\bar{f}: \bar{M}^2 \rightarrow \mathbb{R}^N$ would have the same Gauss map. \square

The main result of this section is the following.

THEOREM 2.4. *Let $f: M^2 \rightarrow \mathbb{R}^3$ be a conformal immersion without umbilic points and let ν and ω be unit principal vector fields of f , with eigenvalues λ and μ respectively.*

(i) *If f is G^* -deformable, then*

$$(2.5) \quad (\lambda - \mu)(\nu(\omega(\lambda)) + \omega(\nu(\mu))) + \nu(\mu)\omega(\mu) - \nu(\lambda)\omega(\lambda) = 0.$$

(ii) *If M^2 is simply connected and (2.5) is satisfied, then f is G^* -deformable.*

Proof. Let p be any point M^2 and ξ be a unit normal field defined on a neighborhood of p . Since f has no umbilic points, there exist differentiable functions λ and μ , and orthonormal tangent fields ν and ω , defined on a neighborhood of p , such that

$$(2.6) \quad A_\xi \nu = \lambda \nu, \quad A_\xi \omega = \mu \omega.$$

From Codazzi equation

$$\nabla_\nu(\mu \omega) - A_\xi \nabla_\nu \omega = \nabla_\omega(\lambda \nu) - A_\xi \nabla_\omega \nu$$

we get

$$(2.7) \quad \nu(\mu) = (\lambda - \mu)\langle \nabla_\omega \nu, \omega \rangle, \quad \omega(\lambda) = (\mu - \lambda)\langle \nabla_\nu \omega, \nu \rangle.$$

Let us suppose that f is G^* -deformable. Then there exist a differentiable function $\varphi: M^2 \rightarrow \mathbb{R}$ and an orthogonal tensor field T with $\det T = -1$, satisfying (1.6) and (1.7). Let ν_1 and ω_1 be orthonormal tangent fields on a neighborhood of p such that $T\nu_1 = \nu_1$ and $T\omega_1 = -\omega_1$. Then by (1.7) we have

$$A_\xi \nu_1 = A_\xi T^{-1} \nu_1 = T A_\xi \nu_1.$$

Thus $A_\xi \nu_1$ is parallel to ν_1 . So ν_1 and ω_1 are principal directions, and this determines T up to sign. From now on we will suppose that $T\nu = \nu$ and $T\omega = -\omega$. Now (1.6) is equivalent to

$$(2.8) \quad \nu(\varphi) = -2\langle \nabla_\omega \nu, \omega \rangle, \quad \omega(\varphi) = -2\langle \nabla_\nu \omega, \nu \rangle.$$

From (2.7) and (2.8) we have

$$(2.9) \quad \nu(\varphi) = -\frac{2\nu(\mu)}{\lambda - \mu}, \quad \omega(\varphi) = \frac{2\omega(\lambda)}{\lambda - \mu}.$$

By (2.8) we get

$$\begin{aligned} [\nu, \omega](\varphi) &= (\nabla_{\nu}\omega)(\varphi) - (\nabla_{\omega}\nu)(\varphi) \\ &= \langle \nabla_{\nu}\omega, \nu \rangle \nu(\varphi) - \langle \nabla_{\omega}\nu, \omega \rangle \omega(\varphi) \\ &= 0. \end{aligned}$$

Thus we must have

$$\nu(\omega(\varphi)) - \omega(\nu(\varphi)) = 0$$

or, using (2.9),

$$(2.10) \quad \nu \left(\frac{\omega(\lambda)}{\lambda - \mu} \right) + \omega \left(\frac{\nu(\mu)}{\lambda - \mu} \right) = 0,$$

which is equivalent to (2.5). Note that equation (2.5) is invariant by change of sign of the vector fields ξ , ν and ω . Thus it is valid everywhere on M^2 .

Suppose now that (2.5) is satisfied on M^2 . We define the tangent vector field

$$\delta = -2\langle \nabla_{\omega}\nu, \omega \rangle \nu - 2\langle \nabla_{\nu}\omega, \nu \rangle \omega$$

and observe that δ does not depend on the unit vector fields ξ , ν and ω satisfying (2.6). Now we define in M^2 the 1-form γ given by

$$(2.11) \quad \gamma(X) = \langle \delta, X \rangle.$$

Using (2.7) we compute

$$\begin{aligned} d\gamma(\nu, \omega) &= \nu(\gamma(\omega)) - \omega(\gamma(\nu)) - \gamma(\nabla_{\nu}\omega - \nabla_{\omega}\nu) \\ &= \frac{2}{\lambda - \mu} (\nu(\omega(\lambda)) + \omega(\nu(\mu))) \\ &\quad + \frac{2}{(\lambda - \mu)^2} (\nu(\mu)\omega(\mu) - \nu(\lambda)\omega(\lambda)). \end{aligned}$$

By (2.5) the 1-form γ is closed. Since M^2 is simply connected, there exists $\varphi: M^2 \rightarrow \mathbb{R}$ such that $\nabla\varphi = \delta$, that is,

$$(2.12) \quad \nu(\varphi) = -2\langle \nabla_{\omega}\nu, \omega \rangle, \quad \omega(\varphi) = -2\langle \nabla_{\nu}\omega, \nu \rangle.$$

We define the tensor field T by $T\nu = \nu$ and $T\omega = -\omega$. Then T is orthogonal, $\det T = -1$ and (1.7) is satisfied. From (2.12) it is easy to show that (1.6) is satisfied. By Proposition 1.5, f is G^* -deformable. \square

The next result is needed for the proof of some of the corollaries to Theorem 2.4.

PROPOSITION 2.13. *If (x, y) are principal coordinates on an open set $U \subset M^2$, then on U (2.5) is equivalent to*

$$(2.14) \quad \left(\frac{(\lambda + \mu)_x}{\lambda - \mu} \right)_y + \left(\frac{(\lambda + \mu)_y}{\lambda - \mu} \right)_x = 0.$$

Proof. Let E and G be positive functions such that

$$(2.15) \quad \frac{\partial}{\partial x} = E\nu \quad \text{and} \quad \frac{\partial}{\partial y} = G\omega.$$

Since $[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = 0$ we obtain

$$\begin{aligned} E\nu(G) - EG\langle \nabla_{\omega}\nu, \omega \rangle &= 0, \\ -G\omega(E) + EG\langle \nabla_{\nu}\omega, \nu \rangle &= 0, \end{aligned}$$

and then by (2.7) we have

$$(2.16) \quad G_x = \frac{G\mu_x}{\lambda - \mu}, \quad E_y = -\frac{E\lambda_y}{\lambda - \mu}.$$

From (2.15) and (2.16) it follows that

$$\begin{aligned} \omega(\nu(\mu)) &= \frac{\mu_{xy}}{EG} + \frac{\mu_x\lambda_y}{(\lambda - \mu)EG}, \\ \nu(\omega(\lambda)) &= \frac{\lambda_{xy}}{EG} - \frac{\mu_x\lambda_y}{(\lambda - \mu)EG}, \end{aligned}$$

and then (2.5) and (2.14) are equivalent. \square

In the next two corollaries, we will consider f as the inclusion map of an open subset of M^2 in \mathbb{R}^3 .

COROLLARY 2.17. *Let M^2 be a rotation surface which does not meet its axis of symmetry. Then f is G^* -deformable. If M^2 is not totally umbilic then \bar{f} is unique up to homothety and translation; also, $\bar{f}(M)$ is again a rotation surface. If M^2 is part of a sphere and $\bar{f}(M)$ is a rotation surface, then $\bar{f}(M)$ is part of a catenoid.*

Proof. We take on $f(M^2)$ the parametrization $\phi: (0, 2\pi) \times I \rightarrow \mathbb{R}^3$ given by

$$\phi(x, y) = (\alpha(y) \cos x, \alpha(y) \sin x, \beta(y)),$$

where $(\alpha(y), \beta(y))$ is a plane curve defined on an open interval $I \subset \mathbb{R}$ and satisfying $\alpha(y) \neq 0$ for any $y \in I$.

Now define $\varphi: M^2 \rightarrow \mathbb{R}$ by $e^\varphi = 1/\alpha^2(y)$. One can immediately verify that

$$(2.18) \quad \bar{\phi}(x, y) = \left(\frac{1}{\alpha(y)} \cos x, \frac{1}{\alpha(y)} \sin x, \bar{\beta}(y) \right),$$

where

$$\bar{\beta}(y) = \int_{y_0}^y \frac{\beta'(t)}{\alpha^2(y)} dt,$$

satisfy $\bar{\phi}_x = e^\varphi \phi_x$ and $\bar{\phi}_y = -e^\varphi \phi_y$. Thus \bar{f} defined by $\bar{f}(\phi(x, y)) = \bar{\phi}(x, y)$ is a G^* -deformation. Now, if M^2 is part of a sphere, then up to homothety and translation f is the normal Gauss map of f into the unit sphere S^2 . So $\bar{f}(M^2)$ must be a minimal surface, hence it is part of a catenoid. \square

A surface in \mathbb{R}^3 that is the envelope of a family of spheres tangent to three fixed spheres in \mathbb{R}^3 is called a cyclid of Dupin. These surfaces can be characterized by the fact that they are the surfaces without umbilic points whose principal curvatures are constant along the respective curvature lines (see [C&R], pp. 151–166).

COROLLARY 2.19. *Let M^2 be a cyclid of Dupin and U an open simply connected subset of M^2 . Then f restricted to U is G^* -deformable.*

Proof. If (x, y) are principal coordinates, then the respective principal curvatures λ and μ satisfy $\lambda_x = \mu_y = 0$. Thus (2.14) is verified and we can apply Theorem 2.4. \square

REMARK 2.20. In the preceding corollary, we have by (2.9),

$$\varphi = \log(c(\lambda - \mu)^2)$$

for some positive constant $c \in \mathbb{R}$. By (1.8), the principal curvatures of \bar{f} are

$$\bar{\lambda} = \frac{c\lambda}{(\lambda - \mu)^2}, \quad \bar{\mu} = \frac{-c\mu}{(\lambda - \mu)^2}.$$

Thus, in general, the new surface $\bar{f}(U)$ is not a cyclid of Dupin.

COROLLARY 2.21. *Let $f: M^2 \rightarrow \mathbb{R}^3$ be an oriented minimal surface without umbilic points and let $N: M^2 \rightarrow S^2 \subset \mathbb{R}^3$ be the normal Gauss map. Then f is G^* -deformable and $\bar{f} = N$ up to homothety and translation.*

Proof. Taking principal coordinates (x, y) , we have

$$N_x = -\lambda \frac{\partial}{\partial x}, \quad N_y = \lambda \frac{\partial}{\partial y}.$$

Thus the corollary is a consequence of Theorem 2.1. \square

COROLLARY 2.22. *Let $f: M^2 \rightarrow \mathbb{R}^3$ be an oriented surface free of umbilic points, with constant mean curvature $H \neq 0$, and let $N: M^2 \rightarrow S^2 \subset \mathbb{R}^3$ be the normal Gauss map. Then f is G^* -deformable and \bar{f} is the parallel surface $g = f + \frac{1}{H}N$, up to homothety and translation.*

Proof. Taking principal coordinates (x, y) , we have

$$g_x = f_x + \frac{1}{H}N_x = \left(1 - \frac{\lambda}{H}\right) f_x = \left(\frac{\mu - \lambda}{\lambda + \mu}\right) f_x$$

and

$$g_y = \left(\frac{\lambda - \mu}{\lambda + \mu}\right) f_y.$$

Thus g is a G^* -deformation of f . We observe that, since

$$N_x = -\lambda f_x = \lambda \left(\frac{\lambda + \mu}{\lambda - \mu}\right) \bar{f}_x$$

and

$$N_y = -\mu f_y = -\mu \left(\frac{\lambda + \mu}{\lambda - \mu}\right) \bar{f}_y,$$

one sees that the mean curvature of \bar{f} is also H . \square

We conclude this work with a result analogous to the preceding corollary, for a constant mean curvature surface in S^3 .

PROPOSITION 2.23. *Let $f: M^2 \rightarrow S^3$ be an oriented surface free of umbilic points, with constant mean curvature H , and let $N: M^2 \rightarrow S^3$ be a vector field normal to f . Then f (seen as a surface in \mathbb{R}^4) is G^* -deformable and \bar{f} is the parallel surface*

$$\bar{f} = \frac{1}{(H^2 + 1)^{1/2}}(N + Hf): M^2 \rightarrow S^3,$$

up to homothety and translation.

Proof. It is analogous to the preceding proof. \square

REMARK 2.24. One can easily check that the above immersion \bar{f} has constant mean curvature in S^3 . When $H = 0$, $-\bar{f}$ is the polar map of the minimal immersion f , as defined by Lawson [La].

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University of California, San Diego
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NICHOLAS ERCOLANI
University of Arizona
Tucson, AZ 85721
ercolani@math.arizona.edu

R. FINN
Stanford University
Stanford, CA 94305
finn@gauss.stanford.edu

VAUGHAN F. R. JONES
University of California
Berkeley, CA 94720
vfr@math.berkeley.edu

STEVEN KERCKHOFF
Stanford University
Stanford, CA 94305
spk@gauss.stanford.edu

C. C. MOORE
University of California
Berkeley, CA 94720

MARTIN SCHARLEMANN
University of California
Santa Barbara, CA 93106
mgscharl@henri.ucsb.edu

HAROLD STARK
University of California, San Diego
La Jolla, CA 92093

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