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ON THE ANALYTIC REFLECTION OF A MINIMAL SURFACE

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For a long time it has been known that in a Euclidean space one can reflect a minimal surface across a part of its boundary if the boundary contains a line segment, or if the minimal surface meets a plane orthogonally along the boundary. The proof of this fact makes use of H. A. Schwarz's reflection principle for harmonic functions.

In this paper we show that a minimal surface, as a conformal and harmonic map from a Riemann surface into \mathbf{R}^3 , can also be reflected analytically if it meets a plane at a constant angle.

THEOREM 1. *Let $\Sigma \subset \mathbf{R}^3$ be a minimal surface with nonempty boundary $\partial\Sigma$ and let Π be a plane. Suppose that $L \subset \Sigma \cap \Pi$ is a C^1 curve, Σ is C^1 along L , and at all points of L the tangent plane to Σ makes a fixed angle $0 < \theta < 90^\circ$ with Π . Then Σ can be analytically extended across L to a minimal surface $\bar{\Sigma}$ satisfying the following properties:*

(i) $\bar{\Sigma} = \Sigma \cup \Sigma^*$, where Σ^* is the set of all images p^* of $p \in \Sigma$ under an analytic reflection map $*$.

(ii) p and p^* are separated by Π in such a way that

$$\text{dist}(p, \Pi) = \text{dist}(p^*, \Pi).$$

(iii) The Gauss map $g: \bar{\Sigma} \rightarrow \mathbf{C}$ satisfies

$$\overline{g(p)} \cdot g(p^*) = \left(\tan \frac{\theta}{2} \right)^{-2}.$$

(iv) $p^* \in \Sigma^*$ is a branch point (geometric) if and only if $p \in \Sigma$ is.

(v) The map $*$ is a single-valued immersion if Σ is simply connected and L is connected, or Σ is doubly connected and L is closed.

(vi) If $*$ is single-valued, then Σ^* has finite total curvature if and only if Σ does.

(vii) If $\partial\Sigma = L$, then $\bar{\Sigma}$ is complete.

Proof. Let x, y, z be coordinates of \mathbf{R}^3 such that $\Pi = \{(x, y, z): z = 0\}$. Since x, y, z are harmonic functions on the minimal surface Σ , one can find conjugate harmonic (possibly multiple-valued)

functions $\bar{x}, \bar{y}, \bar{z}$ to x, y, z respectively on Σ . Then

$$u = x + i\bar{x}, \quad v = y + i\bar{y}, \quad w = z + i\bar{z}$$

are holomorphic (possibly multiple-valued) functions on Σ , and

$$du = dx + id\bar{x}, \quad dv = dy + id\bar{y}, \quad dw = dz + id\bar{z}$$

are holomorphic 1-forms on Σ . Introduce z, \bar{z} as conformal parameters on Σ . Then Σ can be recaptured by setting

$$x = \operatorname{Re} \int^w du, \quad y = \operatorname{Re} \int^w dv, \quad z = \operatorname{Re} \int^w dw.$$

From the conjugacy of $\bar{x}, \bar{y}, \bar{z}$ to x, y, z , it follows that

$$du^2 + dv^2 + dw^2 = 0.$$

Define a holomorphic differential ω and a meromorphic function g on Σ by

$$\omega = du - idv, \quad g = \frac{dw}{du - idv}.$$

Then we have

$$(1) \quad \begin{aligned} x &= \operatorname{Re} \int^w \frac{1}{2} \left(-g + \frac{1}{g} \right) dw, \\ y &= \operatorname{Re} \int^w \frac{i}{2} \left(g + \frac{1}{g} \right) dw, \\ z &= \operatorname{Re} \int^w dw. \end{aligned}$$

It is well known that g is exactly the Gauss map of the surface Σ .

Put $-\Sigma = \{(x, y, -z) : (x, y, z) \in \Sigma\}$ and define a Riemann surface $\tilde{\Sigma}$ by $\tilde{\Sigma} = \Sigma \cup (-\Sigma)$. For any $p = (x, y, z) \in \Sigma$, let $-p = (x, y, -z) \in -\Sigma$. Since $z = 0$ on $\Sigma \cap (-\Sigma)$ ($\supset L$), we can extend the conformal parameters z, \bar{z} over to $\tilde{\Sigma}$ (across L) by the usual reflection with respect to Π , that is,

$$z(-p) = -z(p) \quad \text{and} \quad \bar{z}(-p) = \bar{z}(p) \quad \text{for any } -p \in -\Sigma.$$

Hence we see that dw is a well-defined holomorphic 1-form on the Riemann surface $\tilde{\Sigma}$.

Now note that the constant angle hypothesis implies

$$|g(p)| = \left(\tan \frac{\theta}{2} \right)^{-1} \quad \text{for all } p \in L.$$

In other words, g maps L into a circle in \mathbb{C} . Since Σ is C^1 along L and L plays the same role in the Riemann surface $\tilde{\Sigma}$ as a line does

in \mathbf{C} , we can extend the Gauss map g holomorphically over to $\tilde{\Sigma}$ (across L) as follows. Define an extension of g , still called g , by

$$(2) \quad g(-p) = \left(\tan^2 \frac{\theta}{2} \cdot \overline{g(p)} \right)^{-1}, \quad -p \in -\Sigma.$$

Clearly g is holomorphic on $-\Sigma$ and continuous on $\tilde{\Sigma}$. Let $h: \mathbf{C} \rightarrow \mathbf{C}$ be a linear transformation which maps the circle $|w| = (\tan \frac{\theta}{2})^{-1}$ onto the imaginary axis of \mathbf{C} . Then the real part of $h \circ g$ is continuous on $\tilde{\Sigma}$ and harmonic on Σ and $-\Sigma$. Moreover we have

$$\begin{aligned} \operatorname{Re}[h \circ g(-p)] &= \operatorname{Re}[h \circ g(p)] = 0 \quad \text{for } p \in L, \\ \operatorname{Re}[h \circ g(-p)] &= -\operatorname{Re}[h \circ g(p)] \quad \text{for } -p \in -\Sigma. \end{aligned}$$

Hence by the reflection principle we conclude that $h \circ g$ is holomorphic on $\tilde{\Sigma}$, and so is g .

Using this extended map g , the extended 1-form dw , and the Weierstrass representation formula (1), we can obtain the extended minimal surface $\tilde{\Sigma}$. Here, for any $p \in \Sigma$, p^* is determined by integrating (1) over a contour on $\tilde{\Sigma}$ from a fixed point to $-p$. In case Σ is multiply connected it may happen that the reflection map $*$ maps $p \in \Sigma$ to infinitely many points $p^* \in \Sigma^*$. Also we should discuss the case where $g(p) = 0$ or ∞ . At such a point p , w cannot be a parameter of Σ . However dw and $\frac{1}{g} \pm g$ have a zero and a pole of the same order respectively at $-p$ as well as p . Consequently du and dv are holomorphic at $-p$ and thus Σ^* is well defined in a neighborhood of p^* . This proves conclusion (i).

Conclusion (ii) follows from the symmetry of $-\Sigma$ to Σ and the formula for z in (1).

(2) implies (iii).

Suppose p is a regular point. If the tangent plane to Σ at p is parallel to Π , then $dw = 0$ at p . For this reason, w is not a good conformal parameter near the point p . However, for any conformal parametrization in a neighborhood of p , the metric of the corresponding immersion is, by [BC],

$$ds^2 = \frac{1}{2}(1 + |g|^2)|\omega|^2 = \frac{1}{2}(|g| + |g|^{-1})^2 |dw|^2.$$

Hence the ratio between the metrics at p and $-p$ is given by

$$\begin{aligned} \frac{ds^2(-p)}{ds^2(p)} &= \frac{\frac{1}{2}(\tan^{-2} \frac{\theta}{2} \cdot |g|^{-1} + \tan^2 \frac{\theta}{2} \cdot |g|)^2 |dw|^2}{\frac{1}{2}(|g| + |g|^{-1})^2 |dw|^2} \\ &= \left(\frac{\tan^2 \frac{\theta}{2} \cdot |g| + \tan^{-2} \frac{\theta}{2} \cdot |g|^{-1}}{|g| + |g|^{-1}} \right)^2. \end{aligned}$$

Note here that this ratio depends not on the parametrization of Σ but on the geometry of Σ . Furthermore one can easily show that

$$(3) \quad 0 < \min \left(\tan^2 \frac{\theta}{2}, \tan^{-2} \frac{\theta}{2} \right) \leq \frac{ds(-p)}{ds(p)} \leq \max \left(\tan^2 \frac{\theta}{2}, \tan^{-2} \frac{\theta}{2} \right) < \infty.$$

Therefore Σ^* is also regular at p^* . Since $\Sigma = (\Sigma^*)^*$ and $p = (p^*)^*$, we can obtain the converse similarly.

For (v), we note that in either case every contour in $\tilde{\Sigma}$ is nullhomotopic or homotopic to a contour in Σ and that no forms in formula (1) have real periods on Σ . Hence $*$ is single-valued and so, by (3), an immersion.

To prove (vi), we use a formula for the Gauss curvature of Σ [BC]:

$$K = - \left[\frac{4|g'|}{|f|(1+|g|^2)^2} \right]^2.$$

The curvature ratio between p and $-p$ is given by

$$\frac{K(-p)}{K(p)} = \frac{\left[\frac{4|g'|}{\tan^6 \frac{\theta}{2} \cdot |g|^3 (1 + \tan^{-4} \frac{\theta}{2} \cdot |g|^{-2})^2} \right]^2}{\left[\frac{4|g'|}{|g|^{-1} (1 + |g|^2)^2} \right]^2} = \frac{\tan^4 \frac{\theta}{2} \cdot (1 + |g|^2)^4}{(1 + \tan^4 \frac{\theta}{2} \cdot |g|^2)^4}.$$

Therefore

$$\begin{aligned} 0 &< \min \left(\tan^{12} \frac{\theta}{2}, \tan^{-4} \frac{\theta}{2} \right) \\ &\leq \frac{K(-p)}{K(p)} \leq \max \left(\tan^{12} \frac{\theta}{2}, \tan^{-4} \frac{\theta}{2} \right) < \infty, \end{aligned}$$

and the conclusion follows.

Finally it is not difficult to see that (vii) can be derived from (3). Thus the proof of the theorem is now complete.

COROLLARY. *Let Σ be a complete minimal surface of finite total curvature in \mathbf{R}^3 . If an end E of Σ meets a plane along ∂E at a constant angle, then Σ is the catenoid.*

Proof. From Theorem 1 it follows that $\bar{E} = E \cup E^*$ is a complete minimal surface of finite total curvature with two ends. \bar{E} must then be the catenoid [L]. Obviously, by the unique continuation property of a minimal surface, we have $\bar{E} = \Sigma$.

Let Σ be a minimal surface in \mathbf{R}^3 with Gauss map g . For any real number $0 < r < \infty$, let us denote by Σ_r the minimal immersion of Σ into \mathbf{R}^3 defined by the formula

$$\begin{aligned} x &= \operatorname{Re} \int^w \frac{1}{2} \left(-rg + \frac{1}{rg} \right) dw, \\ y &= \operatorname{Re} \int^w \frac{i}{2} \left(rg + \frac{1}{rg} \right) dw, \\ z &= \operatorname{Re} \int^w dw. \end{aligned}$$

Then we see that every minimal surface can be deformed into a 1-parameter family of minimal surfaces and that this deformation preserves the z -coordinate and multiplies the Gauss map by r .

THEOREM 2. *Assume $\Sigma \subset \mathbf{R}^3$ is a minimal surface with nonempty boundary $\partial\Sigma$ which makes a constant angle θ with a plane Π along $\partial\Sigma \cap \Pi$.*

(i) *For any real number $0 < r < \infty$, the minimal surface Σ_r makes a constant angle $\phi = 2 \tan^{-1}(\frac{1}{r} \tan \frac{\theta}{2})$ with Π along $\partial\Sigma_r \cap \Pi$.*

(ii) *There exists a positive real number s such that the minimal surface Σ_s meets Π orthogonally along $\partial\Sigma_s \cap \Pi$, and the analytic extension $\bar{\Sigma}$ of Σ is the same as $(\Sigma_s \cup (\Sigma_s)^*)_{1/s}$, where $(\Sigma_s)^*$ is the usual reflection (mirror image) of Σ_s with respect to Π .*

Proof. (i) By hypothesis, $|g(p)| = (\tan \frac{\theta}{2})^{-1}$ for all $p \in \partial\Sigma \cap \Pi$. Then

$$|rg(p)| = r \left(\tan \frac{\theta}{2} \right)^{-1} = \left(\tan \frac{\phi}{2} \right)^{-1},$$

where $\phi = 2 \tan^{-1}(\frac{1}{r} \tan \frac{\theta}{2})$. Since the deformation of Σ into Σ_r

preserves the z -coordinate and multiplies the Gauss map by r , Σ_r meets Π along $\partial\Sigma_r \cap \Pi$ at the constant angle ϕ .

(ii) Let s be the positive real number satisfying

$$2 \tan^{-1} \left(\frac{1}{s} \tan \frac{\theta}{2} \right) = 90^\circ.$$

Then Σ_s meets Π orthogonally. Clearly we have

$$(\overline{\Sigma})_s = \overline{(\Sigma_s)}.$$

Since $\overline{(\Sigma_s)}$ is the union of Σ_s and its mirror image $(\Sigma_s)^*$ with respect to Π , we conclude that

$$\overline{\Sigma} = ((\overline{\Sigma})_s)_{1/s} = ((\overline{\Sigma_s}))_{1/s} = (\Sigma_s \cup (\Sigma_s)^*)_{1/s}.$$

REMARKS. 1. A nice example of the analytic reflection can be seen in the catenoid. Let Π_1, Π_2 , and Π_3 be the parallel planes with $\text{dist}(\Pi_1, \Pi_2) = \text{dist}(\Pi_2, \Pi_3)$. Let Σ be the catenoid whose ends are parallel to the Π_i . Then Σ intersects the Π_i along circles at constant angles α_i . Assume $\alpha_2 \neq 90^\circ$ and define D_1, D_3 to be the two bounded components of $\Sigma \sim (\Pi_1 \cup \Pi_2 \cup \Pi_3)$. Then D_3 is the analytic reflection of D_1 with respect to Π_2 and D_1 is that of D_3 . If we define D_+, D_- to be the components of $\Sigma \sim \Pi_2$, then $D_+ = (D_-)^*$ and $D_- = (D_+)^*$.

2. Embeddedness of Σ does not necessarily imply that of Σ^* .

3. If the tangent plane to Σ at p is parallel to Π , so is the tangent plane to Σ^* at p^* . This is clear in view of Theorem 1(iii).

4. Given an angle $0 < \theta < 90^\circ$, two points p_1, p_2 on Π , and a curve $\Gamma \subset \mathbf{R}^3$ from p_1 to p_2 , one can construct an area minimizing surface Σ with the fixed boundary Γ and a free boundary $L \subset \Pi$ along which Σ meets Π at the angle θ as follows. Let Γ_1 be the line segment on Π from p_2 to p_1 . We regard Γ, Γ_1 as 1-dimensional sets with orientation, i.e., 1-currents. Let S be a surface with $\partial S = \Gamma \cup \Gamma_S, \Gamma_S \subset \Pi$. Give S and Γ_S orientations, S is then called a 2-current, in such a way that $\partial S = \Gamma - \Gamma_S$. As sets, Γ_1 and Γ_S bound a planar domain $D \subset \Pi$ with $\partial D = \Gamma_1 \cup \Gamma_S$. Giving suitable orientations to each component of D , we can make D into a 2-current such that $\partial D = \Gamma_1 + \Gamma_S$. Let us fix an orientation of the plane Π . Then D , as a set, is divided into two disjoint domains D_1, D_2 such that D_1 and D_2 with the orientation inherited from Π can be thought of as 2-currents, and

$$D = D_1 - D_2.$$

Now we define $\tilde{A}(S)$, the *modified area* of S , by

$$\tilde{A}(S) = \text{Area}(S) + \cos \theta [\text{Area}(D_1) - \text{Area}(D_2)].$$

Let \mathcal{F} be the family of all 2-currents S such that $\partial S - \Gamma$ is a 1-current on Π . Then it is not difficult to see that $-\infty < \inf\{\tilde{A}(S) : S \in \mathcal{F}\}$ and therefore we can find a modified area minimizing current Σ . Σ , as a set, is a desired minimal surface, and by [T] it is Hölder continuously differentiable up to its free boundary. Thus we can analytically extend Σ across its free boundary $\partial\Sigma \sim \Gamma$ to obtain the *θ -reflection* Σ^* of Σ with respect to Π .

Open problems. 1. Is it possible to extend Theorem 1 to the case of a constant mean curvature surface in \mathbf{R}^3 or a minimal hypersurface in \mathbf{R}^n ? It is well known that the answer is yes if a constant mean curvature surface (a minimal hypersurface respectively) meets a plane (a hyperplane respectively) orthogonally.

2. As a generalization of Corollary, is it true that if a complete constant mean curvature surface Σ of finite topological type intersects a plane at a constant angle $\neq 90^\circ$, then Σ is a Delaunay's surface?

3. Given a compact convex body U in \mathbf{R}^3 , one can construct a minimal disk D in U which makes a constant contact angle θ with the convex boundary ∂U ? Grüter and Jost [GJ] solved the problem affirmatively when $\theta = 90^\circ$.

4. Most complete minimal surfaces are known to have at least one plane of symmetry. However, some complete immersed minimal surfaces of genus zero constructed by H. Karcher do not have a plane of symmetry. Nevertheless, given a complete minimal surface in \mathbf{R}^3 , can one find a plane which intersects the minimal surface at a constant angle?

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PACIFIC JOURNAL OF MATHEMATICS

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Permutation enumeration symmetric functions, and unimodality	1
FRANCESCO BRENTI	
On the analytic reflection of a minimal surface	29
JAIGYOUNG CHOE	
Contractive zero-divisors in Bergman spaces	37
PETER LARKIN DUREN, DMITRY KHAVINSON, HAROLD SEYMOUR SHAPIRO and CARL SUNDBERG	
On the ideal structure of positive, eventually compact linear operators on Banach lattices	57
RUEY-JEN JANG and HAROLD DEAN VICTORY, JR.	
A note on the set of periods for Klein bottle maps	87
JAUME LLIBRE	
Asymptotic expansion at a corner for the capillary problem: the singular case	95
ERICH MIERSEMANN	
A state model for the multivariable Alexander polynomial	109
JUN MURAKAMI	
Free Banach-Lie algebras, couniversal Banach-Lie groups, and more	137
VLADIMIR G. PESTOV	
Four manifold topology and groups of polynomial growth	145
RICHARD ANDREW STONG	
A remark on Leray's inequality	151
AKIRA TAKESHITA	
A_∞ and the Green function	159
JANG-MEI GLORIA WU	
Integral spinor norms in dyadic local fields. I	179
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